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Normal Families and Shared Values of Meromorphic Functions

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Abstract. Some criteria for determining the normality of the family F of meromorphic functions in the unit disc, which share values depending on $f \in F$ with their derivatives is obtained. The new results in this paper improve some earlier related results given by Pang and Zalcman, Fang and Zalcman, A. P. Singh and A. Singh.

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1. Introduction and main results

Let D be a domain in C. For f meromorphic on D and $a \in C$, set

$$\overline{E}_f(a) = f^{-1}(\{a\}) \cap D = \{z \in D : f(z) = a\}.$$

Two meromorphic functions f and g on D are said to share the value a if $\overline{E}_f(a) = \overline{E}_g(a)$. Let b be a complex number. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Rightarrow g(z) = b.$$

If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Leftrightarrow g(z) = b.$$

Let S be a set and

$$\overline{E}_f(S) = f^{-1}(S) \cap D = \{z \in D : f(z) \in S\}.$$

Two meromorphic functions f and g on D are said to share the set S if $\overline{E}_f(S) = \overline{E}_g(S)$. In this paper, we use $\sigma(x, y)$ to denote the spherical distance between x and y and the definiton of the spherical distance can be found in [1].

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Schwick [6] was probably the first to find a connection between the normality criterion and shared values of meromorphic functions. He proved the following theorem.

Theorem 1.1. [6] *Let F be a family of meromorphic functions in the unit disc Δ, and let a_1, a_2, a_3 be distinct complex numbers. If f and f' share a_1, a_2 and a_3 for every $f \in F$, then F is normal in Δ.*

Pang and Zalcman [4] extended the above result as follows.

Theorem 1.2. [4] *Let F be a family of meromorphic functions in the unit disc Δ and let a and b be distinct complex numbers and c be a nonzero complex number. If for every $f \in F$, $f(z) = 0 \Leftrightarrow f'(z) = a$ and $f(z) = c \Leftrightarrow f'(z) = b$, then F is normal in Δ.*

In Theorem 1.1 and Theorem 1.2, the constants are the same for each $f \in F$. In 2004, A. P. Singh and A. Singh [7] proved that the condition for the constants to be the same can be relaxed to some extent. More precisely, they proved the following theorem.

Theorem 1.3. [7] *Let F be a family of meromorphic functions in the unit disc Δ. For each $f \in F$, suppose there exist nonzero complex numbers b_f, c_f satisfying*

(i) $\dfrac{b_f}{c_f}$ *is a constant;*
(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;
(iii) $f(z) = 0 \Leftrightarrow f'(z) = 0$ *and* $f(z) = c_f \Leftrightarrow f'(z) = b_f$.

Then F is normal in Δ.

Regarding to Theorem 1.3, it is natural to ask the following question:
Question 1. What can be said if we relax in any way the condition (iii) of Theorem 1.3?

In this paper, we can prove the following theorem, which deals with Question 1 and improves Theorem 1.3.

Theorem 1.4. *Let F be a family of meromorphic functions in the unit disc Δ. For each $f \in F$, all zeros of f have multiplicity at least 2 and there exist nonzero complex numbers b_f, c_f depending on f satisfying*

(i) $\dfrac{b_f}{c_f}$ *is a constant;*
(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;
(iii) f *and* f' *share* $S_f = \{b_f, c_f\}$ *in* Δ.

Then F is normal in Δ.

Corollary 1.1. *Let F be a family of meromorphic functions in the unit disc Δ. For each $f \in F$ all zeros of f have multiplicity at least 2 and there exist nonzero complex numbers b_f, c_f depending on f satisfying*

(i) $\dfrac{b_f}{c_f}$ *is a constant;*
(ii) $\min\{\sigma(0, b_f), \sigma(0, c_f), \sigma(b_f, c_f)\} \geq m$ *for some* $m > 0$;

(iii) $f(z) = c_f \Leftrightarrow f'(z) = b_f$.

Then F is normal in Δ.

2. Some lemmas

In order to prove our theorem, we need the following preliminary results.

Lemma 2.1. [3] *Let F be a family of meromorphic functions in a domain D, and let a, b be nonzero finite values. If for every $f \in F$, all zeros of f have multiplicity at least 2, and $f(z) = a \Leftrightarrow f'(z) = b$, then F is normal on D.*

Lemma 2.2. [1] *Let m be any positive number. Then, the Möbius transformation g which satisfies $\sigma(g(a), g(b)) \geq m$, $\sigma(g(b), g(c)) \geq m$, $\sigma(g(c), g(a)) \geq m$ for some constant a, b and c, also satisfies the uniform Lipschitz condition*

$$\sigma(g(z), g(w)) \leq k_m \sigma(z, w),$$

where k_m is a constant depending only on m.

Lemma 2.3. [4] *Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions in a domain D, such that each function $f \in \mathcal{F}$ has only zeros of multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^{\alpha}} \to g(\zeta),$$

locally uniformly in C, where g is a nonconstant meromorphic function, all of whose zeros have multiplicity at least k. Moreover, $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1$. Here, as usual, $g^\sharp(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

Lemma 2.4. [2] *Let f be a transcendental meromorphic function of finite order, let a be a nonzero complex number, and let k be a positive integer. If all zeros of f have multiplicity at least $k+1$, then $f^{(k)} - a$ has infinitely many zeros.*

Lemma 2.5. [8] *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 + \dfrac{q(z)}{p(z)}$ where a_0, a_1,..., a_n are constants with $a_n \neq 0$, and q and p are two co-prime polynomials neither of which vanishes identically, with deg(q) <deg(p), and let k be a positive integer. If the zeros of f have multiplicity at least $k+1$, and $f^{(k)} \neq 1$, then*

$$f(z) = \frac{z^k}{k!} + ... + a_0 + \frac{1}{az + b},$$

where $a(\neq 0)$, b, a_0,... are constants.

3. Proof of Theorem 1.4

Proof. Let $M = b_f/c_f$. We can find nonzero constant c satisfying $M = \frac{1}{c}$. For each $f \in F$, define a Möbius map g_f by $g_f = c_f z/c$. Thus $g_f^{-1} = \frac{cz}{c_f}$ and $g_f^{-1}(c_f) = c$. Since f and f' share $S_f = \{b_f, c_f\}$ in Δ, we shall show that $g_f^{-1} \circ f$ and $(g_f^{-1} \circ f)'$ share the set $S_c = \{1, c\}$.

Suppose $(g_f^{-1} \circ f)(z_0) = 1$. Since g_f^{-1} is one to one, then $f(z_0) = b_f$, thus $f'(z_0) = b_f$ or c_f. If $f'(z_0) = b_f$, then $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = 1$. If $f'(z_0) = c_f$, then $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = c$.

Suppose $(g_f^{-1} \circ f)(z_0) = c$. Since g_f^{-1} is one to one, then $f(z_0) = c_f$, thus $f'(z_0) = b_f$ or c_f. If $f'(z_0) = b_f$, then $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = 1$. If $f'(z_0) = c_f$, then $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = c$.

Suppose $(g_f^{-1} \circ f)'(z_0) = 1$, then $(g_f^{-1})'(f(z_0))f'(z_0) = 1$. That is $f'(z_0) = b_f$. Thus $f(z_0) = b_f$ or c_f. If $f(z_0) = b_f$, then $(g_f^{-1} \circ f)(z_0) = 1$. If $f(z_0) = c_f$, then $(g_f^{-1} \circ f)(z_0) = c$.

Suppose $(g_f^{-1} \circ f)'(z_0) = c$, then $(g_f^{-1})'(f(z_0))f'(z_0) = c$. That is $f'(z_0) = c_f$. Thus $f(z_0) = b_f$ or c_f. If $f(z_0) = b_f$, then $(g_f^{-1} \circ f)(z_0) = 1$. If $f(z_0) = c_f$, then $(g_f^{-1} \circ f)(z_0) = c$.

We have shown that $g_f^{-1} \circ f$ and $(g_f^{-1} \circ f)'$ share the set $S_c = \{1, c\}$.

Next we shall show $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal in Δ. Suppose to the contrary, G is not normal in Δ. Then by Lemma 2.3, we can find $g_n \in G$, $z_n \in \Delta$ and $\rho_n \to 0^+$ such that $T_n = \rho_n^{-1} g_n(z_n + \rho_n \xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function T, all of whose zeros have multiplicity at least 2.

We claim that $T' \neq 1$ and $T' \neq c$. Suppose that there exists a value $\xi_0 \in C$ such that $T'(\xi_0) = 1$. Then $T' \not\equiv 1$. Otherwise we can deduce T is a polynomial of degree at most 1, this is impossible. Noting that $T_n'(\xi) \to T'(\xi)$ where $T(\xi)$ is of finite order, by Hurwitz's theorem we can deduce that there exist ξ_n, $\xi_n \to \xi_0$, such that

$$T_n'(\xi_n) = g_n'(z_n + \rho_n \xi_n) = T'(\xi_0) = 1\,,$$

and so it follows that $g_n(z_n + \rho_n \xi_n) = 1$ or c by the condition that $g_n(z_n + \rho_n \xi_n)$ and $g_n'(z_n + \rho_n \xi_n)$ share the set $S_c = \{1, c\}$. Thus $T(\xi_0) = \lim_{n \to \infty} T_n(\xi_n) = \infty$, which contradicts $T'(\xi_0) = 1$. Similarly, we can prove that $T' \neq c$.

By Lemma 2.4, we know that T is not a transcendental meromorphic function. Since T has zeros of multiplicity at least 2 and $T' \neq 1$, it follows that T is not a polynomial. Hence by Lemma 2.5, we obtain that

$$T(\xi) = z + a_0 + \frac{1}{A\xi + B}\,.$$

Thus

$$T'(\xi) = 1 - \frac{A}{(A\xi + B)^2}\,.$$

It follows that $T'(\xi) = c$ has solutions, which is a contradiction. Hence $G = \{(g_f^{-1} \circ f) \mid f \in F\}$ is normal in Δ and hence equicontinuous in Δ. Therefore given $\left(\frac{\varepsilon}{k_m}\right) > 0$, where k_m is the constant of Lemma 2.2, there exist $\delta > 0$ such that for the spherical distance $\sigma(x, y) < \delta$,

$$\sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \frac{\varepsilon}{k_m}\,,$$

for each $f \in F$. Hence by Lemma 2.2,

$$\begin{aligned}\sigma(f(x), f(y)) &= \sigma((g_f \circ g_f^{-1} \circ f)(x), (g_f \circ g_f^{-1} \circ f)(y)) \\ &\le k_m \sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \varepsilon\,.\end{aligned}$$

Thus, the family F is equicontinuous in Δ. This completes the proof of Theorem 1.4. ∎

4. Proof of Corollary 1.1

Proof. Let $M = \frac{b_f}{c_f}$. We can find nonzero constants b and c satisfying $M = \frac{b}{c}$. For each $f \in F$, define a Möbius map g_f by $g_f = c_f z/c$. We now show that

$$(g_f^{-1} \circ f)(z) = c \Leftrightarrow (g_f^{-1} \circ f)'(z) = b. \tag{4.1}$$

Let $(g_f^{-1} \circ f)(z_0) = c$. Since g_f^{-1} is one to one and $g_f^{-1}(c_f) = c$, we have $f(z_0) = c_f$, so $f'(z_0) = b_f$. Then

$$(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = \frac{c}{c_f} b_f = b\,.$$

Let $(g_f^{-1} \circ f)'(z_1) = b$, thus $(g_f^{-1})'(f(z_1))f'(z_1) = b$, and

$$f'(z_1) = \frac{b}{(g_f^{-1})'(f(z_1))} = b \cdot \frac{c_f}{c} = b_f\,,$$

hence $f(z_1) = c_f$ and $(g_f^{-1} \circ f)(z_1) = c$. By Lemma 2.1, we can see that the family $\{(g_f^{-1} \circ f) | f \in F\}$ is normal and hence equicontinuous in Δ. Therefore given $(\varepsilon/k_m) > 0$, where k_m is the constant of Lemma 2.2, there exist $\delta > 0$ such that for the spherical distance $\sigma(x, y) < \delta$,

$$\sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \frac{\varepsilon}{k_m}\,,$$

for each $f \in F$. Hence by Lemma 2.2,

$$\sigma(f(x), f(y)) = \sigma((g_f \circ g_f^{-1} \circ f)(x), (g_f \circ g_f^{-1} \circ f)(y))$$
$$\leq k_m \sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \varepsilon\,.$$

Thus, the family F is equicontinuous in Δ. This completes the proof of Corollary 1.1. ∎

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