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On the Normal Meromorphic Functions

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Abstract. Let $\mathcal{F}$ be a family of functions meromorphic in D such that all the zeros of $f \in \mathcal{F}$ are of multiplicity at least k (a positive integer), and let E be a set containing $k+4$ points of the extended complex plane. If, for each function $f \in \mathcal{F}$, there exists a constant M and such that $(1-|z|^2)^k|f^{(k)}(z)|/(1+|f(z)|^{k+1}) \le M$ whenever $z \in \{f(z) \in E, z \in D\}$, then $\mathcal{F}$ is a uniformly normal family in D, that is, $\sup\{(1-|z|^2)f^{\#}(z) : z \in D, f \in \mathcal{F}\} < \infty$.

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1. Introduction

Let D denote the unit disk in the complex plane $\mathbb{C}$. A function f meromorphic in D is called a normal function [4], in the sense of Lehto and Virtanen, if there exist a constant $M(f)$ such that

$$(1-|z|^2)f^{\#}(z) \le M(f),$$

for each $z \in D$, where $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is called the spherical derivative of f.

Suppose that $\mathcal{F}$ is a family of functions meromorphic in D such that each function of $\mathcal{F}$ is a normal function, then, for each function $f \in \mathcal{F}$, there exists a constant $M(f)$ such that

$$(1-|z|^2)f^{\#}(z) \le M(f),$$

for each $z \in D$. In general, $M(f)$ is a constant dependent on f, and we can not conclude that $\{M(f), f \in \mathcal{F}\}$ is bounded. If $\{M(f), f \in \mathcal{F}\}$ is bounded, we give the definition as follows.

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Definition 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc D. If*

$$\sup\{(1-|z|^2)f^{\#}(z) : z \in D, f \in \mathcal{F}\} < \infty,$$

we call the family $\mathcal{F}$ as a uniformly normal family in D.

Remark 1.1. The idea of this definition is suggested by Pang [5], and the concept of uniformly normal family seems to be connected to normal invariant families as defined by Hayman [2, p.163].

Remark 1.2. Clearly, if $\mathcal{F}$ is a uniformly normal family in D, then each function $f \in \mathcal{F}$ must be a normal function. However, the following example shows that the converse is not valid in general.

Example 1.1. Let $\mathcal{F} = \{nz : n = 1, 2, 3, \ldots\}$. Obviously, each $f \in \mathcal{F}$ is a normal function in D. But $\mathcal{F}$ is not uniformly normal in D. In fact, let $z_n = \frac{1}{n} \in D (n \geq 2), f_n(z) = nz$,

$$(1-|z_n|^2)f_n^{\#}(z_n) = \left(1 - \frac{1}{n^2}\right)\frac{n}{2} \to +\infty, \ (n \to \infty).$$

For a meromorphic function f in D and a positive integer n, the expression

$$\frac{|f^{(n)}(z)|}{1+|f(z)|^{n+1}}$$

represents an extension of the spherical derivative of f. This expression is meaningful when related to normal functions (for details, see [3]). In Xu [6], the author proved the following result, which gives a partial answer to the question due to Lappan (see [3]).

Theorem 1.1. *Let f be a function meromorphic in D such that all the zeros of f are of multiplicity at least n_0(a positive integer). If there exists a constant M such that*

$$(1-|z|^2)^{n_0}\frac{|f^{(n_0)}(z)|}{1+|f(z)|^{n_0+1}} \leq M$$

for each $z \in D$, then f is a normal function.

In this paper, we prove the following theorem.

Theorem 1.2. *Let $\mathcal{F}$ be a family of functions meromorphic in D such that all the zeros of $f \in \mathcal{F}$ are of multiplicity at least k (a positive integer), and let E be a set containing $k+4$ points of the extended complex plane. If there exists a constant M such that, for each function $f \in \mathcal{F}$,*

$$(1-|z|^2)^k\frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} \leq M$$

whenever $z \in D$ and $f(z) \in E$, then $\mathcal{F}$ is a uniformly normal family in D.

2. Lemmas

To prove our result, we need some lemmas. Here we shall use the following standard notation of value distribution theory (see [1,2,7])

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots.$$

We use $\overline{N}_{(2}(r,f)$ to denote the Nevanlinna counting function of the poles of f with multiplicity≥ 2. We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to \infty$, possibly outside a set with finite measure.

Lemma 2.1. [2, 7] *Let f be a nonconstant transcendental meromorphic function, and $a_1, a_2, \dots, a_q \in \mathbb{C} \bigcup \{\infty\} (q \geq 3)$ such that $a_i \neq a_j (i \neq j)$. Then*

$$(q-2)T(r,f) < \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r,f).$$

Lemma 2.2. [2, 7] *Let f be a nonconstant transcendental meromorphic function, and $k \in \mathbb{N}$. Then*

$$T(r, f^{(k)}) \leq (k+1)T(r,f) + S(r,f).$$

The following is the well-known Zalcman's lemma [8].

Lemma 2.3. *Let $\mathcal{F}$ be a family of meromorphic functions in D. If $\mathcal{F}$ is not normal at a point $z_0 \in D$, then there exists a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$, such that $f_j(z_j + \rho_j\zeta)$ spherically and uniformly converges to a non-constant meromorphic function on each compact subset of $\mathbb{C}$.*

3. Proof of Theorem 1.2

Proof. Suppose that $\mathcal{F}$ is not a uniformly normal family in D. Then, we can find $f_n \in \mathcal{F}$, $z_n \in D$, such that

$$g_n(z) = f_n(z_n + (1-|z_n|^2)z)$$

satisfies

$$\lim_{n\to\infty} g_n^{\#}(0) = \lim_{n\to\infty} (1-|z_n|^2) f_n^{\#}(z_n) = \infty.$$

It follows that $\{g_n(z)\}$ is not normal at $z = 0$. Thus, by Lemma 2.3, there exist a subsequence of functions g_n(without loss generality, we may assume g_n), a sequence of points $\zeta_n \in D, \zeta_n \to 0$, and a sequence of positive numbers $\rho_n \to 0$ such that

$$G_n(\zeta) = g_n(\zeta_n + \rho_n\zeta) = f_n\left(z_n + (1-|z_n|^2)\zeta_n + (1-|z_n|^2)\rho_n\zeta\right)$$

converges spherically and uniformly to a non-constant meromorphic function $G(\zeta)$ on each compact subset of $\mathbb{C}$. Since each function f_n has only zeros of multiplicity at least k, then the limit function $G^{(k)}(\zeta) \not\equiv 0$.

Obviously, there exists a point ζ_0 such that $G(\zeta_0) \in E$ and $|\zeta_0| < R$, where R is a positive number (for otherwise G is a constant, a contradiction). By Hurwitz's theorem, there exists a sequence of points ζ_n', $\zeta_n' \to \zeta_0$ such that

$$f_n\left(z_n + (1-|z_n|^2)\zeta_n + (1-|z_n|^2)\rho_n\zeta_n'\right) \in E.$$

For brevity, we use the notation $\widehat{\zeta_n'} = z_n + (1-|z_n|^2)\zeta_n + (1-|z_n|^2)\rho_n\zeta_n'$. According to the assumptions and noting that $\widehat{\zeta_n'} \in D$ (for n sufficiently large), we have

$$\left(1 - \left|\widehat{\zeta_n'}\right|^2\right)^k \frac{|f_n^{(k)}(\widehat{\zeta_n'})|}{1+|f_n(\widehat{\zeta_n'})|^{k+1}} \le M.$$

It follows that

$$\frac{|G_n^{(k)}(\zeta_n')|}{1+|G_n(\zeta_n')|^{k+1}} = \rho_n^k(1-|z_n|^2)^k \frac{|f_n^{(k)}(\widehat{\zeta_n'})|}{1+|f_n(\widehat{\zeta_n'})|^{k+1}} \le \rho_n^k M \left(\frac{1-|z_n|^2}{1-|\widehat{\zeta_n'}|^2}\right)^k.$$

Since $(1-|z_n|^2)/(1-|\widehat{\zeta_n'}|^2) \to 1$ as $n \to \infty$, we have

$$\frac{|G^{(k)}(\zeta_0)|}{1+|G(\zeta_0)|^{k+1}} = 0.$$

From this, we know that: (a) ζ_0 is a multiple pole of $G(\zeta)$, or (b) $G^{(k)}(\zeta_0) = 0$.

Without loss of generality, we may assume $E = \{a_1, a_2, \ldots, a_{k+4}\}$. By Lemma 2.1, we have

$$(k+2)T(r,G) < \sum_{i=1}^{k+4} \overline{N}\left(r, \frac{1}{G-a_i}\right) + S(r,G), \tag{3.1}$$

where $a_i \in E(i = 1, 2, \ldots, a_{k+4})$. By the above discussion, for each $a_i(i = 1, 2, \ldots, k+4)$, if $G(\zeta_0) = a_i$, then either ζ_0 is a multiple pole of $G(\zeta)$ (in this case $a_i = \infty$) or $G^{(k)}(\zeta_0) = 0$. We distinguish two cases.

Case 1. $\infty \in E$. Without loss of generality, we assume $a_1 = \infty$. Then

$$\overline{N}\left(r, \frac{1}{G-a_1}\right) \le \overline{N}_{(2}(r,G),$$

and

$$\sum_{i=2}^{k+4} \overline{N}\left(r, \frac{1}{G-a_i}\right) \le \overline{N}\left(r, \frac{1}{G^{(k)}}\right).$$

From (3.1), and using Nevanlinna first fundamental theorem (see [2,7]) and Lemma 2.2, we have

$$\begin{aligned} (k+2)T(r,G) &< \overline{N}_{(2}(r,G) + \overline{N}\left(r, \tfrac{1}{G^{(k)}}\right) + S(r,G) \\ &\le \tfrac{1}{2}N(r,G) + T(r,G^{(k)}) + S(r,G) \\ &\le \tfrac{1}{2}N(r,G) + (k+1)T(r,G) + +S(r,G) \\ &\le (k+\tfrac{3}{2})T(r,G) + S(r,G), \end{aligned}$$

that is,

$$\frac{1}{2}T(r,G) < S(r,G).$$

This is impossible since $G(\zeta)$ is nonconstant.

Case 2. $\infty \notin E$. Similarly as in Case 1, we have

$$\begin{aligned} (k+2)T(r,G) &\le \overline{N}\left(r, \tfrac{1}{G^{(k)}}\right) + S(r,G) \\ &\le T(r,G^{(k)}) + S(r,G) \\ &\le (k+1)T(r,G) + S(r,G). \end{aligned}$$

Thus $T(r,G) \leq S(r,G)$, a contradiction. Theorem 1.2 is thus proved. ∎

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