Normal Functions and Normal Families

YAN XU
Department of Mathematics, Nanjing Normal University, Nanjing 210097
e-mail: xuyan@njnu.edu.cn

Abstract. In this paper, we prove the following theorem: Let $F$ be a family of holomorphic functions in the unit disc $D$ and let $a$ be a nonzero complex number. If, for any $f \in F$, $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = a$, then $F$ is uniformly normal in $D$, that is, there exists a positive constant $M$ such that $Mzf(z) \leq 1$ for each $f \in F$ and $z \in D$, where $M$ is independently of $f$. This result improves related results due to [2], [8], and [3].

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1. Introduction

Let $f$ and $g$ be two meromorphic functions, and let $a$ be a complex number. If $g(z) = a$ whenever $f(z) = a$, we denote it by $f = a \Rightarrow g = a$, and $f = a \Leftrightarrow g = a$ means $f(z) = a$ if and if only if $g(z) = a$.

Let $D$ denote the unit disk in the complex plane $C$. A function $f$ meromorphic in $D$ is called a normal function, in the sense of [6], if there exist a constant $M(f)$ such that

$$(1 - |z|^2) f^#(z) \leq M(f) \quad \text{for each} \quad z \in D,$$

where

$$f^#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative.

Let $F$ be a family of meromorphic functions defined in $D$. $F$ is said to be normal in $D$ (see [9]), in the sense of Montel, if for any sequence $f_n \in F$ there exists a subsequence $f_{n_j}$, such that $f_{n_j}$ converges spherically, locally and uniformly in $D$, to a meromorphic function or $\infty$. 
Suppose that \( F \) is a family of functions meromorphic in \( D \) such that each function of \( F \) is a normal function, then, for each function \( f \in F \), there exists a constant \( M(f) \) such that

\[
\left(1 - |z|^2\right)f'\sigma(z) \leq M(f)
\]

for each \( z \in D \). In general, \( M(f) \) is a constant dependent on \( f \), and we can not conclude that \( \{M(f), f \in F\} \) is bounded. If \( \{M(f), f \in F\} \) is bounded, we give the definition as follows

**Definition.** Let \( F \) be a family of meromorphic functions in the unit disc \( D \). If there exists a positive constant \( M \) such that

\[
\sup \left\{ \left(1 - |z|^2\right)f'\sigma(z) : z \in D, f \in F \right\} < M,
\]

we call the family \( F \) a uniformly normal family in \( D \).

**Remark 1.** The idea of this definition is suggested by Pang (see [7]).

**Remark 2.** A well-known result due to Marty (see [4], [9] and [11]) says that a family \( F \) of functions meromorphic in \( D \) is a normal family if and only if for each compact subset \( K \) of \( D \) there exists a constant \( M_K \) such that \( f'\sigma(z) \leq M_K \) for each \( f \in F \) and for each \( z \in K \). Clearly, by Marty’s criterion if \( F \) is a uniformly normal family in \( D \), then \( F \) must be normal in \( D \). However, it is obvious that the converse is not always true.

It is natural to ask: *When is a normal family \( F \) in \( D \) also uniformly normal in \( D \)?* (The question is first introduced by Bergweiler and Pang (see [7]).)

Schwick [10] discovered a connection between normality criteria and sharing values. He proved

**Theorem A.** Let \( F \) be a family of meromorphic functions in the unit disc \( D \) and let \( a_1, a_2 \) and \( a_3 \) be distinct complex numbers. If, for any \( f \in F \),

\[
f(z) = a_i \iff f'(z) = a_i \quad (i = 1, 2, 3),
\]

then \( F \) is normal in \( D \).

Pang [7] proved that the family \( F \) in Theorem A is also uniformly normal, as follows

**Theorem B.** Let \( F \) be a family of meromorphic functions in the unit disc \( D \) and let \( a_1, a_2 \) and \( a_3 \) be distinct complex numbers. If, for any \( f \in F \),

\[
f(z) = a_i \iff f'(z) = a_i \quad (i = 1, 2, 3),
\]

then \( F \) is uniformly normal in \( D \).
f(z) = a_i \iff f'(z) = a_i \ (i = 1, 2, 3), \text{ then } F \text{ is uniformly normal in } D, \text{ that is, there exists a positive constant } M \text{ such that }
\left(1 - |z|^2\right)f''(z) \leq M

for each } f \in F \text{ and } z \in D, \text{ where } M \text{ is independent of } f.

**Remark 3.** In fact, from the proof in [7], we see that Theorem B still remains true if
f(z) = a_i \iff f'(z) = a_i \ (i = 1, 2, 3) \text{ for any } f \in F.

Chen and Hua [2], Pang and Zalcman [8] proved the following normality criterion.

**Theorem C.** Let } F \text{ be a family of holomorphic functions in the unit disc } D \text{ and let } a \text{ be a nonzero complex number. If, for any } f \in F, f(z) = a \iff f'(z) = a, f'(z) = a \Rightarrow f''(z) = a, \text{ then } F \text{ is normal in } D.

In [3], Fang improved Theorem C as follows

**Theorem D.** Let } F \text{ be a family of holomorphic functions in the unit disc } D \text{ and let } a \text{ be a nonzero complex number. If, for any } f \in F, f(z) = a \Rightarrow f'(z) = a, f'(z) = a \Rightarrow f''(z) = a, \text{ then } F \text{ is normal in } D.

In this paper, by using a method different from that used in [3], we obtain the following stronger result.

**Theorem 1.** Let } F \text{ be a family of holomorphic functions in the unit disc } D \text{ and let } a \text{ be a nonzero complex number. If, for any } f \in F, f(z) = a \Rightarrow f'(z) = a, f'(z) = a \Rightarrow f''(z) = a, \text{ then } F \text{ is uniformly normal in } D, \text{ that is, there exists a positive constant } M \text{ such that }
\left(1 - |z|^2\right)f''(z) \leq M

for each } f \in F \text{ and } z \in D, \text{ where } M \text{ is independent of } f.

**Remark 4.** The following example (see [2] and [3]) shows that } a \neq 0 \text{ cannot be omitted in Theorem 1.

Let } F = \{ f_n(z) = e^{az} : n = 1, 2, 3 \cdots \}, \ D = \{ z : |z| < 1 \}. \text{ Then, for every } f_n \in F, \text{ it is easy to see that } f_n(z) = 0 \Rightarrow f'_n(z) = 0 \Rightarrow f''_n(z) = 0. \text{ However, } f''_n(0) = n/2 \to \infty \text{ as } n \to \infty, \text{ thus } F \text{ is not uniformly normal in } D.

We shall use the standard notations in Nevanlinna theory (see [4], [11]).
2. Lemmas

For convenience, we define

\[
LD(r, f) := c_1 m \left( r, \frac{f'}{f} \right) + c_2 m \left( r, \frac{f''}{f'} \right) + c_3 m \left( r, \frac{f'}{f - a} \right) + c_4 m \left( r, \frac{f''}{f - a} \right), \quad (a \in \mathbb{C})
\]

where \(c_1, c_2, c_3, c_4, c_5\) are constants, which may have different values at different occurrences.

Lemma 1. Let \(f\) be a non-constant holomorphic functions on the unit disc \(D\), and \(a\) be a nonzero complex number. Let

\[
\psi(z) := \psi(f(z)) = \frac{f'(z) + f''(z)}{f(z) - a} - \frac{2 f'''(z)}{f'(z) - a}.
\]

If \(f' = a \Rightarrow f'' = a\) on \(D\), and \(f(0) \neq a, f'(0) \neq a, f''(0) \neq 0, f'(0) \neq f''(0)\) and \(\psi(0) \neq 0\), then

\[
T(r, f) \leq LD(r, f) + O(1) + 3 \log \left| \frac{f(0) - a}{f''(0) - f'(0)} \right| + \log \left| \frac{(f(0) - a)(f''(0) - a)}{f''(0)} \right| + 2 \log \frac{1}{|\psi(0)|}.
\]

Proof. Let \(f(z_0) = a\). By the assumptions we may assume that, near \(z_0\)

\[
f(z) = a + a(z-z_0) + \frac{a}{2}(z-z_0)^2 + b(z-z_0)^3 + O((z-z_0)^4),
\]

where \(b = f^{(3)}(z_0)/6\) is a constant. Then

\[
f'(z) = a + a(z-z_0) + 3b(z-z_0)^2 + O((z-z_0)^3),
\]

\[
f''(z) = a + 3b(z-z_0) + O((z-z_0)^2)
\]
and thus
\[
\frac{f'(z) + f''(z)}{f(z) - a} = \frac{2}{z - z_0} + \frac{6b}{a} + O(z - z_0),
\]
\[
\frac{2f''(z)}{f'(z) - a} = \frac{2}{z - z_0} + \frac{6b}{a} + O(z - z_0).
\]

Hence \(\psi(z_0) = 0\), and
\[
N\left(r, \frac{1}{f - a}\right) \leq N\left(r, \frac{1}{\psi}\right) \leq T\left(r, \psi\right) + \log \frac{1}{|\psi(0)|}
\leq N\left(r, \psi\right) + LD\left(r, f\right) + \log \frac{1}{|\psi(0)|}
= N_0\left(r, \frac{1}{f' - a}\right) + LD\left(r, f\right) + \log \frac{1}{|\psi(0)|},
\]

where \(N_0(r, 1/(f' - a))\) is the counting function for the zeros of \(f' - a\) which are not zeros of \(f - a\). Since \(f = a \Rightarrow f' = a\), form (2.1) we get
\[
2N\left(r, \frac{1}{f - a}\right) \leq N\left(r, \frac{1}{f' - a}\right) + LD\left(r, f\right) + \log \frac{1}{|\psi(0)|}.
\]

On the other hand, by the assumptions we have
\[
N\left(r, \frac{1}{f' - a}\right) \leq N\left(r, \frac{1}{f'}\right) \leq T\left(r, \frac{f''}{f'}\right) + \log \frac{|f'(0)|}{|f''(0) - f'(0)|} + O(1)
= N\left(r, \frac{f'}{f'}\right) + \log \frac{|f'(0)|}{|f''(0) - f'(0)|} + LD\left(r, f\right) + O(1)
= N\left(r, \frac{f'}{f'}\right) + \log \frac{|f'(0)|}{|f''(0) - f'(0)|} + LD\left(r, f\right) + O(1).
\]
Next we need the estimate of $N(r, \frac{1}{f'})$. Since

\[
m\left( r, \frac{1}{f - a} \right) \leq m\left( r, \frac{1}{f} \right) + LD\left( r, f \right)
\]

\[
\leq T\left( r, f' \right) - N\left( r, \frac{1}{f} \right) + LD\left( r, f \right) + \log \frac{1}{f'(0)}
\]

\[
\leq T\left( r, \frac{1}{f - a} \right) - N\left( r, \frac{1}{f} \right) + LD\left( r, f \right) + O(1) + \log \frac{|f(0) - a|}{|f'(0)|}
\]

\[
= m\left( r, \frac{1}{f - a} \right) + N\left( r, \frac{1}{f - a} \right) - N\left( r, \frac{1}{f} \right) + LD\left( r, f \right) + O(1)
\]

\[
+ \log \frac{|f(0) - a|}{|f'(0)|},
\]

we obtain

\[
N\left( r, \frac{1}{f} \right) \leq N\left( r, \frac{1}{f - a} \right) + LD\left( r, f \right) + O(1) + \log \frac{|f(0) - a|}{|f'(0)|}. \quad (2.4)
\]

Thus, from (2.2), (2.3) and (2.4), we get

\[
N\left( r, \frac{1}{f - a} \right) \leq LD\left( r, f \right) + O(1) + \log \frac{|f(0) - a|}{|f'(0)|} + \log \frac{1}{|\psi(0)|}, \quad (2.5)
\]

\[
N\left( r, \frac{1}{f' - a} \right) \leq LD\left( r, f \right) + O(1) + 2 \log \frac{|f(0) - a|}{|f''(0) - f'(0)|} + \log \frac{1}{|\psi(0)|}. \quad (2.6)
\]

Using Milloux’s inequality, we have

\[
T\left( r, f \right) \leq N\left( r, \frac{1}{f - a} \right) + N\left( r, \frac{1}{f' - a} \right) + LD\left( r, f \right) + O(1)
\]

\[
+ \log \frac{|f(0) - a| (f'(0) - a)}{|f''(0)|}.
\]

Substituting (2.5) and (2.6) in the above inequality yields the conclusion.
Lemma 2. (Bureau [1]) Let $b_1, b_2,$ and $b_3$ be positive numbers and $U(r)$ a nonnegative, increasing and continuous function on an interval $[r_0, R)$, $R < \infty$. If

$$U(r) \leq b_1 + b_2 \log^{+} \frac{1}{\rho - r} + b_3 \log^{+} U(\rho)$$

for any $r_0 < r < \rho < R$, then

$$U(r) \leq B_1 + B_2 \log^{+} \frac{1}{R - r}$$

for $r_0 < r < R$, where $B_1$ and $B_2$ depend on $b_i (i = 1, 2, 3)$ only.

Lemma 3. (see Hiong [4]) If $f(z)$ is meromorphic in a disk $|z| < R$ such that $f(0) \neq 0, \infty$, then, for $0 < r < \rho < R$,

$$m \left( r, \frac{f^{(k)}}{f} \right) \leq C_k \left\{ 1 + \log^{+} \log^{+} \frac{1}{|f(0)|} + \log^{+} \frac{1}{r} + \log^{+} \frac{1}{\rho - r} + \log^{+} \rho + \log^{+} T(\rho, f) \right\},$$

where $C_k$ is a constant depending only on $k$.

The following is the wellknown Zalcman’s lemma [12].

Lemma 4. Let $F$ be a family of functions meromorphic in a domain $D$. If $F$ is not normal at $z_0 \in D$, then there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in F$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where $g$ is a non-constant meromorphic function on $C$.

3. Proof of Theorem 1

Proof. Suppose that $F$ is not uniformly normal in $D$. Then, we can find $f_n \in F$, $z_n \in D$, such that
\[ g_n(z) = f_n z_n + (1 - |z_n|^2) z \]
satisfies
\[ g_n'(0) = \left(1 - |z_n|^2\right) f_n'(z_n) \to \infty \]
as \( n \to \infty \). It follows that \( \{ g_n(z) \} \) is not normal at \( z = 0 \). We distinguish two cases:

1. \( g_n = g_n' \) for every \( n \in \mathbb{N} \) then \( g_n(z) = C_n e^z \), which is normal at \( z = 0 \).

2. Consider the case that \( g_n \) and \( g_n' \) are not identical. By Lemma 4, there exist a subsequence \( g_n \) (without loss generality, we may assume \( g_n \)), a sequence \( \eta_n \in D, \eta_n \to 0 \), and a positive sequence \( \rho_n \to 0 \) such that
\[
G_n(\zeta) = g_n(\eta_n + \rho_n \zeta) = f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right)
\]
converges uniformly to a non-constant entire function \( G(\zeta) \) on each compact subset of \( C \). Thus, for any positive integer \( k \),
\[
G_n^{(k)}(\zeta) = \left(1 - |z_n|^2\right)^k \rho_n^k f_n^{(k)} \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right). \tag{3.1}
\]
We claim that \( G(\zeta) \) is not a polynomial of degree less than 3. Indeed, if \( G(\zeta) \) is a polynomial, then there exists a point \( \zeta_0 \) such that \( G(\zeta_0) = a \). By Hurwitz’ theorem, there is a sequence \( \zeta_n \to \zeta_0 \) such that
\[
G_n(\zeta_n) = g_n(\eta_n + \rho_n \zeta_n) = f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_n \right) = a
\]
for \( n \) sufficiently large. It follows from the hypotheses on \( F \) that
\[
f_n' \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_n \right) = f_n' \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_n \right) = a
\]
for \( n \) sufficiently large. On the other hand, by (3.1), we have
\[
G_n(\zeta_n) = \left(1 - |z_n|^2\right)^2 \rho_n^2 f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_n \right) \to G'(\zeta_0),
\]
\[
G_n''(\zeta_n) = \left(1 - |z_n|^2\right)^2 \rho_n^2 f_n' \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_n \right) \to G''(\zeta_0).
\]
Thus \( G'(\zeta_0) = G''(\zeta_0) = 0 \).

Choose \( \zeta_1 \) with

\[
G(\zeta_1) \neq 0, a; \quad G'(\zeta_1) \neq 0; \quad G''(\zeta_1) \neq 0.
\]

Then

\[
\frac{1}{\rho_1^2 (1 - |z_1|^2)^2} \times \frac{f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 \right) - a}{f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 \right) - f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 \right) - a}
\]

\[
\rightarrow \frac{G(\zeta_1)}{G''(\zeta_1)}.
\]

On the other hand, we claim that there are only finitely many \( f_n \) such that \( \psi(f_n) = 0 \).

Indeed, suppose that there is a subsequence \( \{f_n\} \subset \{f_n\} \) such that \( \psi(f_n) = 0 \).

Then

\[
\frac{f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right) + f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right) - a}{f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right) - a}
\]

\[
= \frac{2f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right) - a}{f''_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta \right) - a}.
\]
and thus
\[
\frac{\rho_n \left(1 - |z_n|^2 \right)}{G_n'(\zeta)} + G_n''(\zeta) = \frac{2 \rho_n \left(1 - |z_n|^2 \right)}{G_n'(\zeta) - a \rho_n \left(1 - |z_n|^2 \right)}.
\]

Letting \( j \to \infty \), we get \( G''(\zeta) = 0 \), a contradiction. Then we may assume that \( \psi(f_n) \neq 0 \), for all \( n \). Thus
\[
\rho_n^2 \psi_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 \right) \to \frac{G''(\zeta)}{G(\zeta) - a},
\]
where \( \psi_n = \psi(f_n) \). So we have
\[
\log \left( \frac{f_n(z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1) - a}{f_n'(z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1) - f_n'(z_n) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1} \right) \to -\infty,
\]
and
\[
\log \left( \frac{1}{\psi_n(z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1)} \right) \to -\infty,
\]
as \( n \to \infty \). For \( n = 1, 2, \ldots \), set
\[
P_n(z) = f_n \left( z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 + z \right).
\]
Let \( n \) be sufficiently large. Then \( P_n \) is defined and holomorphic on the disk \( 0 < |z| < \frac{1}{2} \), since
\[z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1 \to 0.\]
By (3.2) and (3.3), we have

$$P_n(0) = G_n(\zeta_1) \rightarrow G(\zeta_1) \neq 0, \eta,$$  \hspace{1cm} (3.7)

$$P'_n(0) = \frac{1}{(1 - |z_n|^2) \rho_n} G'_n(\zeta_1) \rightarrow \infty,$$  \hspace{1cm} (3.8)

$$P''_n(0) = \frac{1}{(1 - |z_n|^2)^2 \rho_n^2} G''_n(\zeta_1) \rightarrow \infty,$$  \hspace{1cm} (3.9)

$$\psi(P_n(0)) = \psi_n\left(z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1\right) \rightarrow \infty.$$  \hspace{1cm} (3.10)

Therefore, by (3.7)–(3.10) we may apply Lemma 1 to $P_n(z)$, and using (3.4), (3.5) and (3.6) we obtain

$$T(r, P_n) \leq LD(r, P_n),$$

for sufficiently large $n$. Hence by Lemma 2 and Lemma 3, we get

$$T\left(\frac{1}{4}, P_n\right) \leq M,$$

where $M$ is a constant independent of $n$. It follows that $f_n(z)$ are bounded for sufficiently large $n$ and $|z| < \frac{1}{4}$. But, from

$$\left(1 - |z_n|^2\right)^2 \rho_n^2 f''_n\left(z_n + (1 - |z_n|^2) \eta_n + (1 - |z_n|^2) \rho_n \zeta_1\right) = G''_n(\zeta_1) \rightarrow G''(\zeta_1) \neq 0$$

we know that $f_n(z)$ cannot be bounded in $|z| < \frac{1}{8}$. We arrive at a contradiction. This completes the proof of Theorem 1.

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References


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