Eigenproblem of the Generalized Neumann Kernel

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To the memory of Mohamad Rashidi Md. Razali,
a friend, a colleague and source of inspiration

Abstract. Recently, the Riemann problem in the interior domain of a smooth Jordan curve was solved by transforming its boundary condition to a Fredholm integral equation of the second kind with the generalized Neumann kernel. The eigenvalues $\lambda = \pm 1$ play an important role in the solvability of these integral equations. In this paper, the necessary and sufficient conditions for $\lambda = \pm 1$ to be eigenvalues of the generalized Neumann kernel are given and the corresponding eigenfunctions are derived. Some examples are presented.

1. Introduction

Suppose that $\Gamma : t = t(s)$, $0 \leq s \leq \beta$ be a smooth Jordan curve, $\Omega^+$ and $\Omega^-$ its interior and exterior respectively such that the origin of the coordinate system belongs to $\Omega^+$ and $\infty$ belongs to $\Omega^-$. The unit tangent to $\Gamma$ at the point $t$ will be denoted by $T(t) = t'(s) / |t'(s)|$. Let $f^+(t)$ and $f^-(t)$ denote the limiting values of the analytic function $f(z)$ when the point $z$ tends to the point $t \in \Gamma$ from inside and outside of $\Gamma$ respectively. Assume that $a$, $b$ and $\gamma$ be three real functions of the point $t \in \Gamma$ all satisfying the Hölder condition and $a^2(t) + b^2(t) \neq 0$ for all $t \in \Gamma$. The Riemann Problem in $\Omega^+$ consists of finding all functions $f = u + iv$ that are analytic in $\Omega^+$, continuous on $\Omega^+ \cup \Gamma$ with limiting values of the real and imaginary parts on $\Gamma$ satisfying the linear relation

$$a(t) u^+(t) - b(t) v^+(t) = \gamma(t), \quad t \in \Gamma. \quad (1.1)$$
Let \( c(t) = a(t) + ib(t), \ t \in \Gamma, \) the boundary condition (1.1) may be rewritten as
\[
\text{Re}\left[ c(t) f^+(t) \right] = \gamma(t), \ t \in \Gamma. \tag{1.2}
\]

We assume that \( |c(t)| = 1 \) on \( \Gamma \) which is no loss of generality as can seen by divided (1.2) by \( |c(t)| \). When \( \gamma(t) = 0 \) we are faced with the homogeneous Riemann problem
\[
\text{Re}\left[ c(t) f^+(t) \right] = 0, \ t \in \Gamma. \tag{1.3}
\]

Similarly, the Riemann problem for exterior domain \( \Omega^- \) consists of finding all functions \( f = u + iv \) that are analytic in \( \Omega^- \) (including at \( \infty \)), continuous on \( \overline{\Omega^-} = \Omega^- \cup \Gamma \), and satisfy the boundary condition
\[
a(t)u^-(t) - b(t)v^-(t) = \gamma(t), \ t \in \Gamma \tag{1.4}
\]
which is equivalent to
\[
\text{Re}\left[ c(t) f^-(t) \right] = \gamma(t), \ t \in \Gamma. \tag{1.5}
\]
The homogeneous problem of the exterior domain \( \Omega^- \) is given by
\[
\text{Re}\left[ c(t) f^-(t) \right] = 0, \ t \in \Gamma. \tag{1.6}
\]

Recently, when \( a(t) \) and \( b(t) \) have continuous first order derivatives, Murid et al. [7] solved the Riemann problem using Fredholm integral equations of the second kind. The kernel of these integral equations is a generalization to the familiar Neumann kernel so it will be call called generalized Neumann kernel.

This kernel is very important in solving Dirichlet problem [3] and Riemann problem [7] using Fredholm integral equations. The solvability of the Riemann problem (1.2) and (1.5) depends on the index of the function \( c(t) \) with respect to the curve \( \Gamma \), for definition of the index see [2, pp. 85–89]. However, the solvability of the related integral equations depend on the eigenvalues \( \lambda = \pm1 \). It is found that the possibility of \( \lambda = \pm1 \) to be an eigenvalue of the generalized Neumann kernel depends on the index of \( c(t) \).

The organization of this paper is as follow. In Section 2, we give a brief derivation of the integral equations with generalized Neumann kernel related to the Riemann problem in the interior and exterior domains. Section 3 contains the proof of the continuity of the generalized Neumann kernel and some of its properties. In Section 4, the necessary and sufficient condition for which \( \lambda = \pm1 \) is an eigenvalue of the generalized Neumann kernel is given. Section 5 contained two examples. The conclusions are given in Section 6.
2. Fredholm integral equations related to Riemann problem

With the help of the Sokhotsky formula [2, p.25] we are able to extend the results obtained in [6] and [7] which will play a key role in deriving an integral equation related to (1.5). Suppose that $\gamma$ is a real function defined on $\Gamma$ and satisfies the Hölder condition. Suppose also that $c(t)$ is a complex valued function defined on $\Gamma$ such that $c'(t)$ is continuous on $\Gamma$ and $c(t) \neq 0$ for all $t \in \Gamma$ and define the function $L(z)$ in $\mathbb{C} \setminus \Gamma$ by

$$L(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{2\gamma(t)}{c(t)(t-z)} \, dt.$$  \hspace{1cm} (2.1)

Theorem 2.1 derives integral equation related to Riemann problem in the interior domain $\Omega^+$ of a smooth Jordan curve $\Gamma$. The proof can be found in [7].

**Theorem 2.1.** [7] Let $\Gamma$ be a smooth Jordan curve and $\Omega^+$ be its interior. Suppose that $f(z)$ is an analytic solution of the Riemann problem (1.2) in the interior domain $\Omega^+$. Define $g(t) = c(t) f^+(t), \, t \in \Gamma$, then $g(t)$ satisfies the integral equation

$$g(t) - \text{PV} \int_{\Gamma} N(c)(t, w) \frac{g(w)}{c(w)} \, dw = -\overline{c(t)} \, L^+(t).$$  \hspace{1cm} (2.2)

where

$$N(c)(t, w) = \frac{1}{\pi} \text{Im} \left[ \frac{c(t)}{c(w)} \frac{T(w)}{w-t} \right], \quad t, \ w \in \Gamma \text{ and } t \neq w.$$  \hspace{1cm} (2.3)

Similarly we can obtain an integral equation related to Riemann problem (1.5) in the exterior domain $\Omega^-$ with an additional condition $f(\infty) = 0$.

**Theorem 2.2.** Let $\Gamma$ be a smooth Jordan curve and $\Omega^-$ be its exterior. Suppose that $f(z)$ is an analytic solution of the Riemann problem (1.4) in the exterior domain $\Omega^-$ with the condition $f(\infty) = 0$. Define $g(t) = c(t) f^-(t)$, then $g(t)$ satisfies the integral equation

$$g(t) + \text{PV} \int_{\Gamma} N(c)(t, w) g(w) \, dw = -\overline{c(t)} \, L^-(t).$$  \hspace{1cm} (2.4)
Proof: Let \( f(z) \) be a solution of (1.4) in \( \Omega^- \) then \( f(z) \) is analytic in \( \Omega^- \) and continuous on \( \overline{\Omega^-} \), hence \( f^-(t) = f(t), \ t \in \Gamma \). From (1.5) we have \( c(t)f(t) + \overline{c(t)f(t)} = 2\gamma(t) \) which leads to

\[
f(t) = -\frac{\overline{c(t)}}{c(t)} \overline{f(t)} + \frac{2\gamma(t)}{c(t)}, \ t \in \Gamma.
\]

(2.5)

According to [2, p. 2], \( f(z) \) satisfies

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w-z} dw = 0, \ z \in \Omega^+.
\]

(2.6)

Taking the limit \( \Omega^+ \ni z \to t \in \Gamma \) and applying Sokhotskyi formula to (2.6), we get

\[
\frac{1}{2} f(t) + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w-t} dw = 0.
\]

(2.7)

Conjugating both side of (2.7) the using (2.5), we get

\[
-\frac{1}{2} \frac{c(t)}{c(t)} \left[ f(t) - \frac{2\gamma(t)}{c(t)} \right] + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{c(w)}{c(w)(w-t)} \left[ f(w) - \frac{2\gamma(w)}{c(w)} \right] dw = 0
\]

which leads to

\[
-\frac{1}{2} \frac{c(t)}{c(t)} f(t) + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{c(w)}{c(w)} \frac{f(w)}{w-t} dw
= \left( \frac{1}{2} \frac{2\gamma(t)}{c(t)} + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{2\gamma(w)}{c(w)(w-t)} dw \right)^{-1}.
\]

(2.8)

Using the fact that \( T(w)\sqrt{dw} = dw \) and from the definition of \( L(z) \), (2.8) becomes

\[
\frac{1}{2} \frac{c(t)}{c(t)} f(t) - \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{c(w)}{c(w)} \frac{T(w)}{w-t} f(w) \left| dw \right| = L^*(t)
\]

(2.9)

Equation (2.6) may be written in the form

\[
\frac{1}{2} f(t) + \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{T(w)}{w-t} f(w) \left| dw \right| = 0, \ t \in \Gamma
\]

(2.10)
Multiply (2.9) by $\overline{c(t)}$ and add the result to (2.10) multiplied by $c(t)$, and using the definitions of $g(t)$ and $N(c)$ we get (2.4).

3. Generalized Neumann Kernel

The kernel $N(c)\,(t,w)$ defined by (2.3) is continuous at all points $(t,w)\in \Gamma \times \Gamma$ except for $t = w$ where it is undefined. In Theorem 3.2, it will be shown that if $\Gamma$ is sufficiently smooth then $N(c)\,(t,w)$ is continuous even at $t = w$ where $N(c)\,(w,w) = \lim_{t \to w} N(c)\,(t,w)$. Then, the symbol PV appears in (2.2) and (2.4) will be dropped and the equations (2.2) and (2.4) are Fredholm integral equations of the second kind.

**Theorem 3.1.** Let the smooth Jordan curve $\Gamma : t = t(s), \ 0 \leq s \leq \beta$, be such that $c'(t(s))$ and $t''(s)$ exist and are continuous on $[0, \beta]$. Then the limit of $N(c)(\tau,w)$ as $\tau \to w$ exists for every $w = t(s) \in \Gamma$, and

$$
\lim_{\tau \to w} N(c)(w,w) = \kappa(w)
$$

(3.1)

where

$$
\kappa(w) = \frac{1}{2\pi |t'(s)|} \text{Im} \left[ \frac{t''(s)}{t'(s)} - \frac{2c'(t(s))t'(s)}{c(t(s))} \right].
$$

(3.2)

Moreover this limit exists uniformly.

**Proof:** Let $\tau = t(\sigma), \ w = t(s)$. Then

$$
N(c)(\tau,w) = \frac{1}{\pi} \text{Im} \left[ \frac{c(\tau)}{c(w)} \frac{T(w)}{w - \tau} \right] = \frac{1}{\pi |t'(s)|} \text{Im} \left[ \frac{c(t(\sigma))}{c(t(s))} \frac{t'(s)}{t(s) - t(\sigma)} \right].
$$

(3.3)

Using

$$
t(\sigma) - t(s) = t'(s)(\sigma - s) + \frac{1}{2}t''(s)(\sigma - s)^2 + o((\sigma - s)^2),
$$

$$
c(t(\sigma)) = c(t(s)) + c'(t(s))r'(s)(\sigma - s) + o((\sigma - s))
$$

and

$$
\frac{1}{1 + \theta} = 1 - \theta + o(\theta)
$$
for \( \theta \) near to 0, we have for \( s \) near enough to \( \sigma \),

\[
\frac{t'(s)}{t(s) - t(\sigma)} = -\frac{1}{\sigma - s} \left\{ 1 + \frac{1}{2} \frac{t''(s)}{t'(s)} (\sigma - s) + o(\sigma - s) \right\}^{-1}
\]

\[
= -\frac{1}{\sigma - s} \left\{ 1 - \frac{1}{2} \frac{t''(s)}{t'(s)} (\sigma - s) + o(\sigma - s) \right\}
\]

\[
= -\frac{1}{\sigma - s} + \frac{1}{2} \frac{t''(s)}{t'(s)} + o(1)
\]

and

\[
\frac{c(t(\sigma))}{c(t(s))} = 1 + \frac{c'(t(s))t'(s)}{c(t(s))} (\sigma - s) + o(\sigma - s)
\]

implies

\[
\frac{c'(t(\sigma))}{c'(t(s))} \frac{t'(s)}{t(\sigma) - t(s)} = \frac{1}{\sigma - s} + \frac{1}{2} \frac{t''(s)}{t'(s)} - \frac{c'(t(s))t'(s)}{c(t(s))} + o(1).
\]

(3.4)

Hence

\[
\lim_{\sigma \to s} \text{Im} \left[ \frac{c(t(\sigma))}{c(t(s))} \right] \frac{t'(s)}{t(\sigma) - t(s)} = \text{Im} \left[ \frac{1}{\sigma} \frac{t''(s)}{t'(s)} - \frac{c'(t(s))t'(s)}{c(t(s))} \right]
\]

(3.5)

Thus from (3.3) and (3.5), we have

\[
\lim_{\sigma \to s} N(c)(t(\sigma), t(s)) = \frac{1}{\pi |t'(s)|} \text{Im} \left[ \frac{1}{\sigma} \frac{t''(s)}{t'(s)} - \frac{c'(t(s))t'(s)}{c(t(s))} \right]
\]

implying (3.1) and (3.2). To show that the limit (3.5), hence also (3.1), exists uniformly, for any \( \tau, \omega \in \Gamma, \; \tau = t(\sigma) \) and \( \omega = t(s) \), let \( \varepsilon \) be any given positive real number, we must find \( \delta(c) > 0 \) such that \( |\sigma - s| < \delta \) implies

\[
\left| \text{Im} \left[ \frac{c(t(\omega))}{c(t(s))} \right] \frac{t'(s)}{t(\sigma) - t(s)} - \text{Im} \left[ \frac{1}{\sigma} \frac{t''(s)}{t'(s)} - \frac{c'(t(s))t'(s)}{c(t(s))} \right] \right| < \varepsilon.
\]

(3.6)
From (3.4) and (3.6), we have
\[
\left| \text{Im} \left[ \frac{c(t(\sigma))}{c(t(s))} \frac{t'(s)}{t'(\sigma) - t'(s)} \right] - \text{Im} \left[ \frac{1}{2} \frac{t''(s)}{c(t(s))} - \frac{c'(t(s))t'(s)}{c(t(s))} \right] \right| = \left| \text{Im} \left[ -\frac{1}{\sigma - s} + o(1) \right] \right|
\]
\[
= \left| \text{Im} \left[ o(1) \right] \right| = o(1)
\]
(3.7)

Since (3.4) holds for all \( \tau, w \in \Gamma, \tau = t(\sigma) \) and \( w = t(s) \), such that \( \sigma \) near enough to \( s \), thus from (3.7) there exists \( \delta(\varepsilon) > 0 \) such that for all \( \tau, w \in \Gamma, \tau = t(\sigma) \) and \( w = t(s), |\sigma - s| < \delta \) implies (3.6).

**Theorem 3.2.** Under the hypotheses of Theorem 3.1 the kernel \( N(c)(t, w) \) defined by
\[
N(c)(t, w) = \begin{cases} 
\frac{1}{2\pi} \left| t'(s) \right| \text{Im} \left[ \frac{c(t)}{c(w)} \frac{T(w)}{w - t} \right], & w \neq t, w, t \in \Gamma, \\
\frac{1}{2\pi} \left| t'(s) \right| \text{Im} \left[ \frac{t''(s)}{t'(s)} - \frac{2c'(t(s))t'(s)}{c(t(s))} \right], & w = t \in \Gamma,
\end{cases}
\]
(3.8)
is continuous on \( \Gamma \times \Gamma \).

**Proof:** To prove the continuity of the kernel \( N(c)(t, w) \) at a point \( (t_0, t_0) \in \Gamma \times \Gamma \), it must be shown that for any given \( \varepsilon > 0 \), we must find \( \delta > 0 \) such that
\[
\max \left\{ |t - t_0|, |w - t_0| \right\} < \delta
\]
(3.9)
implies
\[
|N(c)(t, w) - N(c)(t_0, t_0)| < \varepsilon.
\]
(3.10)

By triangle inequality (3.10) will hold if both
\[
|N(c)(t, w) - N(c)(t, t)| < \frac{\varepsilon}{2}
\]
(3.11)
and
\[
|N(c)(t, t) - N(c)(t_0, t_0)| < \frac{\varepsilon}{2}.
\]
(3.12)
From Theorem 3.1 the limit \( \lim_{w \to t} N(c)(t, w) = N(c)(t, t) \) exists uniformly for all \( t \in \Gamma \) which implies that there exists \( \delta_1 > 0 \) such that \( |w - t| < \delta_1 \) implies that the inequality (3.11) is held. The function \( \kappa(t) \) defined by (3.2) is continuous on the compact set \( \Gamma \), hence uniform continuous. Therefore there exists \( \delta_2 > 0 \) such that for all \( t \in \Gamma \) satisfies \( |t - t_0| < \delta_1 \) implies \( |\kappa(t) - \kappa(t_0)| < \frac{\xi}{2} \), then the inequality (3.12) holds. Let

\[
\delta = \frac{1}{2} \min \{ \delta_1, \delta_2 \},
\]

therefore for all \( t, w \in \Gamma \), if \( \max \{ |t - t_0|, \ |w - t_0| \} < \delta \) then the inequality (3.10) holds, hence \( N(c)(t, w) \) is continuous at \( (t_0, t_0) \in \Gamma \times \Gamma \).

The Neumann kernel arises frequently in the integral equation of potential theory and conformal mapping. If \( c(t) = 1 \) the kernel (3.8) is identical with the Neumann kernel [3, pp. 282–286] so the kernel (3.8) will be called the generalized Neumann kernel and it will be denoted by \( N(c)(t, w) \) or only \( N(c) \) when there is no confusion. When \( c(t) = 1 \), we write \( N(t, w) \) or only \( N \).

Some of the properties of the generalized Neumann kernel are listed in the following remarks.

**Remark 3.1.** It is clear that \( N(c)(t, w) \) is a real kernel thus its adjoint kernel \( N^*(c)(t, w) \) is given by \( N^*(c)(t, w) = \overline{N(c)(w, t)} = N(c)(w, t) \). Moreover, for all \( t \neq w \) and \( t, w \in \Gamma \), we have

\[
N(c)(w, t) = \frac{1}{\pi} \text{Im} \left[ \left( \frac{c(w)}{c(t)} \right) \frac{T(t)}{t - w} \right] = -\frac{1}{\pi} \text{Im} \left[ \frac{T(t) / c(t)}{T(w) / c(w)} \frac{T(w)}{w - t} \right] = -N(\hat{c})(t, w). \tag{3.13}
\]

Therefore \( N^*(c) = -N(\hat{c}) \) where \( \hat{c}(t) = T(t) / (t) = T(t) c(t) \), \( t \in \Gamma \). Since \( \Gamma \) is a smooth Jordan curve, \( T(t) \) is the unit tangent vector at \( t \in \Gamma \), therefore \( T(t) = e^{i\theta(t)} \) where \( \theta(t) \) is the angle between the tangent to the contour \( \Gamma \) and the real axis. In going round the contour \( \Gamma \) in anticlockwise direction \( \theta(t) \) acquires the increment \( 2\pi \). Therefore \( \text{ind}_\Gamma(T) = 1 \) and hence

\[
\text{ind}_\Gamma(\hat{c}) = \text{ind}_\Gamma(T) - \text{ind}_\Gamma(c) = 1 - \text{ind}_\Gamma(c). \tag{3.14}
\]
Remark 3.2. If \( \Gamma \) is the unit circle then
\[
N(c)(t, w) = \frac{1}{\pi} \text{Im} \left[ \frac{c(t) - c(w) + c(w) T(w)}{c(w)} \right] = -\frac{1}{\pi} \text{Im} \left[ \frac{c(w) - c(t)}{w - t} \right] + \frac{1}{\pi} \text{Im} \left[ \frac{iw}{w - t} \right]
\]
which implies that \( N(c)(t, w) \) is symmetric when \( \Gamma \) is the unit circle.

4. Eigenvalue of the generalized Neumann kernel \( N(c)(t, w) \)

In this section we give the main results of this paper. First we need the following theorems from [2, pp. 221 & 226] with slight modifications. The first theorem discusses the solvability of the Riemann problem in the interior domain \( \Omega^+ \) and the second theorem discusses the solvability in the exterior domain \( \Omega^- \).

Theorem 4.1. [2, p. 222] Let \( x = \text{ind}_c(c) \), in the case \( x \leq 0 \) the homogeneous Riemann problem (1.3) has \(-2x+1\) linearly independent solutions and the non-homogeneous problem (1.2) is absolutely soluble and its solution depends linearly on \(-2x+1\) arbitrary real constants. In the case \( x > 0 \) the homogeneous Riemann problem (1.3) has only the trivial solution and the non-homogeneous problem (1.2) is soluble only if \(-2x-1\) conditions are satisfied. If the latter conditions are satisfied the non-homogeneous problem has a unique solution.

Theorem 4.2. [2, p. 226] Let \( x = \text{ind}_c(c) \), in the case \( x \geq 0 \) the homogeneous Riemann problem (1.6) in the exterior domain \( \Omega^- \) has \(2x+1\) linearly independent solutions and the non-homogeneous problem (1.4) is absolutely soluble and its solution depends linearly on \(2x+1\) arbitrary real constants. In the case \( x < 0 \) the homogeneous Riemann problem (1.6) in the exterior domain \( \Omega^- \) has only the trivial solution and the non-homogeneous problem is soluble only if \(-2x-1\) conditions are satisfied. If the latter conditions are satisfied the non-homogeneous problem has a unique solution.

Remark 4.1. If \( f(z) \) is any analytic function in \( \Omega^- \) and vanishes at infinity then its Taylor series in the vicinity of \( z = \infty \) is given by \( f(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots \). Suppose that the function \( f_1(z) \) is defined by \( f_1(z) = z f(z) \) then its Taylor series in the vicinity
of \( z = \infty \) is given \( f_1(z) = a_1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots \) and hence \( f_1(z) \) is analytic in \( \Omega^- \) and bounded at infinity.

**Corollary 4.1.** Let \( x = \text{ind}_\Gamma (c) \), if \( x > 0 \) then the homogeneous Riemann problem (1.6) in the exterior domain \( \Omega^- \) with the condition \( f(\infty) = 0 \) has \( 2x - 1 \) linearly independent solutions. If \( x \leq 0 \) then the homogeneous Riemann problem (1.6) in the exterior domain \( \Omega^- \) with the condition \( f(\infty) = 0 \) has only the trivial solution.

**Proof:** To prove this Corollary, we shall prove that the homogeneous Riemann problem (1.6) with the condition \( f(\infty) = 0 \) is equivalent to the homogeneous Riemann problem

\[
\text{Re} \left[ c_1(t) f_1^-(t) \right] = 0
\]

(4.1)

where \( c_1(t) = c(t) / t \), \( t \in \Gamma \), and \( f_1 \) is merely analytic at infinity. Suppose that \( f(z) \) is a solution to (1.6) with \( f(\infty) = 0 \) and define \( f_1(z) = z f(z), \, z \in \Omega^- \). According to the Remark 3.1 \( f_1(z) \) is analytic in \( \Omega^- \) and bounded at infinity. Since \( f_1^-(t) = t f^-(t), \, t \in \Gamma \) we get \( f_1(z) \) is a solution to (4.1). Similarly let \( f_1(z) \) is a solution to (4.1) and define \( f(z) = f_1(z) / z, \, z \in \Omega^- \). Then \( f(z) \) is analytic in \( \Omega^- \), vanishes at infinity and \( f^-(t) = f_1^-(t) / t, \, t \in \Gamma \). Thus \( f(z) \) is a solution to (1.6) with \( f(\infty) = 0 \). Therefore the homogeneous Riemann problem (1.6) with the condition \( f(\infty) = 0 \) is equivalent to the homogeneous Riemann problem (4.1) in \( \Omega^- \). Let \( x_1 = \text{ind}_\Gamma (c_1) \) then \( x_1 = \text{ind}_\Gamma (c) - \text{ind}_\Gamma (t) = x - 1 \). If \( x > 0 \) then \( x_1 \geq 0 \), from Theorem 4.2 the homogeneous Riemann problem (4.1) and hence (1.6) with the condition \( f(\infty) = 0 \) has exactly \( 2x - 1 \) real linearly independent solution. If \( x \leq 0 \) then \( x_1 < 0 \), according to Theorem 4.2 the homogeneous Riemann problem (4.1) and hence (1.6) with the condition \( f(\infty) = 0 \) has only the trivial solution.

The following theorems discuss the eigen problem of the generalized Neumann kernel. We shall discuss only the cases \( \lambda = \pm 1 \) which are very important in the discussion of the solvability of the Fredholm integral related to Riemann problem.

**Theorem 4.3.** If \( x = \text{ind}_\Gamma (c) > 0 \), then \( \lambda = -1 \) is an eigenvalue of the generalized Neumann kernel \( N(c)(t,w) \).

**Proof:** Since \( x > 0 \), according to Corollary 4.1, the homogeneous Riemann problem (1.6) with the condition \( f(\infty) = 0 \) has \( 2x - 1 \) linearly independent solutions in
Eigenproblem of the Generalized Neumann Kernel 23

From Theorem 2.2 $c(t) f_j^-(t)$ satisfy the homogeneous Fredholm integral equation

$$c(t) f_j^-(t) + \int_{\Gamma} N(c)(t, w) c(w) f_j^-(w) \, dw = 0.$$  \tag{4.2}

Hence $\lambda = -1$ is an eigenvalue of $N(c)$.

**Theorem 4.4.** If $x = \text{ind}_T(c) > 0$ and $\phi(t)$ is a real eigenfunction of the generalized Neumann kernel $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$ then the function $\Phi(z)$ defined by

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{i \phi(w)}{c(w)(w - z)} \, dw$$  \tag{4.3}

is a solution to the homogeneous Riemann problem (1.6) in $\Omega^-$.  

**Proof:** The function $\Phi(z)$ defined by (4.3) is analytic in $(\mathbb{C} \setminus \Gamma) \cup \{z\}$. Taking the limit $\Omega^+ \to t \in \Gamma$ and using the Sokhotsky formula to (4.3), we get

$$2c(t) \Phi^+(t) = i \phi(t) + \frac{1}{\pi} \int_{\Gamma} \frac{c(t) T(w)}{c(w)(w - t)} \phi(w) \, dw$$  \tag{4.4}

Taking the imaginary part of both sides of (4.4) we have

$$2 \text{Im} \left[ c(t) \Phi^+(t) \right] = \phi(t) + \int_{\Gamma} N(t, w) \phi(w) \, dw.$$  \tag{4.5}

Since $\phi$ is an eigenfunction of $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$, (4.5) reduces to the following homogeneous Riemann problem in $\Omega^+$

$$\text{Im} \left[ c(t) \Phi^+(t) \right] = 0.$$  \tag{4.6}

Since $x > 0$, according to Theorem 4.1 the problem (4.6) has only the trivial solution. Therefore $\Phi(z) = 0$ for all $z \in \Omega^-$. Consequently, according to [2, p. 25] the function $\Phi(z)$ defined by (4.3) satisfies

$$c(t) \Phi^-(t) = -i \phi(t).$$  \tag{4.7}
Taking the real part of both sides of (4.7), we find that $\Phi(z)$ is a solution of homogeneous Riemann problem (1.6).

**Corollary 4.2.** If $x = \text{ind}_\Gamma(c) > 0$ then the eigenfunctions of $N(c)$ corresponding to the eigenvalue $\lambda = -1$ are

$$
\phi_j(t) = i c(t) f_j^-(t), \quad t \in \Gamma, \quad j = 1, 2, \cdots, 2x - 1
$$

where $f_j$ are the linearly independent solution of the homogeneous Riemann problem (1.6) in the exterior domain $\Omega^-$ with the condition $f(\infty) = 0$.

**Proof:** From Theorem 4.3, $\lambda = -1$ is an eigenvalue of $N(c)$ and the functions $\phi_j(t) = i c(t) f_j^-(t), \quad j = 1, 2, \cdots, 2x + 1$ are linearly independent solutions of the homogeneous Fredholm integral equation

$$
\phi(t) + \int_\Gamma N(c)(t, w) \phi(w) \, dw = 0.
$$

Since $\text{Re} \left[ c(t) f_j^-(t) \right] = 0$, hence $\phi_j(t)$ are real eigenfunctions of $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$. We next show that these are the only independent eigenfunctions of $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$.

Let $\phi$ be any real eigenfunction of $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$. From Theorem 4.4 the function $G(z)$ defined by

$$
G(z) = \frac{1}{2\pi i} \int_\Gamma \frac{i \phi(t)}{c(t)(t - z)} \, dt,
$$

is a solution to the homogeneous Riemann problem (1.6) with $G(\infty) = 0$ and $c(t) G^- (t) = -i \phi(t)$. Since $f_j, \quad j = 1, 2, \cdots, 2x - 1$ are the linearly independent solutions of the homogeneous Riemann problem (1.6) with $f(\infty) = 0$, hence $G(z) = \sum_{k=1}^{2x-1} b_k f_k(z)$, Consequently,

$$
\phi(t) = i c(t) G^-(t) = \sum_{k=1}^{2x-1} i b_k c(t) f_k^-(t) = \sum_{k=1}^{2x-1} b_k \phi_k(t).
$$

Therefore $\phi_1, \phi_2, \cdots, \phi_{2x-1}$ are the only linearly independent real eigenfunctions of $N(c)(t, w)$ corresponding to the eigenvalue $\lambda = -1$. 


Theorem 4.5. If \( x = \text{ind}_\Gamma(c) \leq 0 \), then \( \lambda = 1 \) is an eigenvalue of the generalized Neumann kernel \( N(c) \).

Proof: Since \( x \leq 0 \), according to Theorem 4.1 the homogeneous Riemann problem (1.3) has \( -2x + 1 \) linearly independent solution \( f_j(z), \ j = 1, 2, \cdots, -2x + 1, \ z \in \Omega^+ \).

From theorem 2.1 \( c(t) f_j^+(t) \) satisfy the homogeneous Fredholm integral equation
\[
c(t) f_j^+(t) - \int_{\Gamma} N(c)(t, w) c(w) f_j^+(w) dw = 0,
\]
and hence \( \lambda = 1 \) is an eigenvalue of \( N(c) \).

Theorem 4.6. If \( x = \text{ind}_\Gamma(c) \leq 0 \) and \( \phi(t) \) is a real eigenfunction of the generalized Neumann kernel \( N(c)(t, w) \) corresponding to the eigenvalue \( \lambda = 1 \) then the function \( \Phi(z) \) defined by
\[
\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{i\phi(w)}{c(w)(w - z)} dw
\]
is a solution to the homogeneous Riemann problem (1.3) in \( \Omega^+ \).

Proof: The function \( \Phi(z) \) defined by (4.12) is analytic in \( C \setminus \Gamma \cup \{x\} \) with \( \Phi(x) = 0 \). Taking the limit \( \Omega^- \ z \to t \in \Gamma \) and using the Sokhotskyi formula to (4.12), we get
\[
2c(t) \Phi^-(t) = -i\phi(t) + \frac{1}{\pi} \int_{\Gamma} \frac{c(t)T(w)}{c(w)(w - t)} \phi(w) dw
\]
Taking the imaginary part of both sides of (4.13) we have
\[
2 \text{Im} \left[ c(t) \Phi^-(t) \right] = -\phi(t) + \int_{\Gamma} N(t, w) \phi(w) dw.
\]
Since \( \phi \) is an eigenfunction of \( N(c)(t, w) \) corresponding to the eigenvalue \( \lambda = 1 \), (4.14) reduces to the following homogeneous Riemann problem in \( \Omega^- \),
\[
\text{Im} \left[ c(t) \Phi^-(t) \right] = 0.
\]
with the condition $\Phi(x) = 0$. Since $x \leq 0$, according to Corollary 4.1 the problem (4.15) has only the trivial solution. Therefore $\Phi(z) = 0$ for all $z \in \Omega^-$. Consequently, according to [2, p. 25] the function $\Phi(z)$ defined by (4.12) satisfies

$$c(t) \Phi^+(t) = i \phi(t). \quad (4.16)$$

Taking the real part of both sides of (4.16), we find that $\Phi(z)$ is a solution of (1.3).

**Corollary 4.3.** If $x = \text{ind}_\Gamma(c) \leq 0$ then the eigenfunctions of $N(c)$ corresponding to the eigenvalue $\lambda = 1$ are

$$\phi_j(t) = ic(t) f_j^+(t), \quad t \in \Gamma, \quad j = 1, 2, \ldots, -2x + 1 \quad (4.17)$$

where $f_j$ are the linearly independent solution of the homogeneous Riemann problem (1.3) in the interior domain $\Omega^+$.

**Proof:** From Theorem 4.5, $\lambda = 1$ is an eigenvalue of $N(c)$ and the functions $\phi_j(t) = ic(t) f_j^+(t), \quad j = 1, 2, \ldots, -2x + 1$ are linearly independent solutions of the homogeneous Fredholm integral equation

$$\varphi(t) - \int_{\Gamma} N(c)(t, w) \varphi(w) \, dw = 0. \quad (4.18)$$

Since $\text{Re}[c(t)f_j^+(t)] = 0$, hence $\phi_j(t)$ are real eigenfunctions of the $N(c)$ corresponding to the eigenvalue $\lambda = 1$. We next show that these are the only independent eigenfunctions of $N(c)$ corresponding to the eigenvalue $\lambda = 1$. Let $\varphi$ be any real eigenfunction of $N(c)$ corresponding to the eigenvalue $\lambda = 1$. From Theorem 4.6 the function $G(z)$ defined by

$$G(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{ic(t)}{c(t)(t - z)} \, dt. \quad (4.19)$$

is a solution to the homogeneous Riemann problem (1.3) and $c(t)G^+(t) = i \phi(t)$. Therefore $G(z) = \sum_{k=1}^{-2x+1} b_k f_k(z)$ and hence

$$\varphi(t) = -ic(t)G^-(t) = -\sum_{k=1}^{-2x+1} b_k ic(t)f_k(t) = -\sum_{k=1}^{-2x+1} b_k \phi_k(t). \quad (4.20)$$
Therefore \( \phi_1, \phi_2, \cdots, \phi_{-2x+1} \) are the only linearly independent real eigenfunctions of \( N(c) \) corresponding to the eigenvalue \( \lambda = 1 \).

**Corollary 4.4.** Suppose that \( x = \text{ind}_c(c) \leq 0 \). If \( \phi_j(t), t \in \Gamma, j = 1, 2, \cdots, -2x + 1 \) are linearly independent real eigenfunctions of \( N(c) \) corresponding to the eigenvalue \( \lambda = 1 \) then the general solution of the homogeneous Riemann problem (1.3) is given by

\[
 f_h(z) = \sum_{j=-2x+1}^{2} a_j \frac{1}{2\pi i} \int_{\Gamma} \frac{i\phi_j(t)}{c(t)(t-z)} \, dt
\]

where \( a_j, j = 1, 2, \cdots, -2x + 1 \) are arbitrary real constants.

**Proof.** From Theorem 4.6 the function \( f_h(z) \) is a solution to (1.3). Since \( \phi_j \) are linearly independent, then so are the integrals in (4.21). According to Theorem 4.1 and since \( f_h(z) \) contains \(-2x + 1\) arbitrary real constants, the functions \( f_h(z) \) is the general solution to (1.3).

The following two Theorems are proved only for the unit disk where the kernel \( N(c)(t, w) \) in this case is symmetric.

**Theorem 4.7.** Suppose that \( \Gamma \) is the unit disk. If \( x = \text{ind}_c(c) \leq 0 \) then \( \lambda = -1 \) is not an eigenvalue of the generalized Neumann kernel \( N(c) \).

**Proof:** In accordance with the Fredholm’s Alternative to prove that \( \lambda = -1 \) is not an eigenvalue to \( N(c) \) it is sufficient to prove that the homogeneous equation

\[
 \rho(t) + \int_{\Gamma} N(c)(t, w) \rho(w) \, dw = 0
\]

has only the trivial solution. Let \( \rho(t) \) be any solution of (4.22) and let us set

\[
 H(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\rho(w)}{c(w)(w-z)} \, dw.
\]

Taking the limit \( \Omega^+ \to t \in \Gamma \) and using the Sokhotskyi formula to (4.23), we get

\[
 2c(t)H^+(t) = \rho(t) + \text{PV} \frac{1}{\pi i} \int_{\Gamma} \frac{c(t)}{c(w)} \frac{\rho(w)}{w-t} \, dw
\]
Taking the real parts of both sides of (4.24) and using (4.22) we conclude that \( H(z) \) is a solution to the homogeneous Riemann problem (1.3). Since \( x \leq 0 \), according to Corollary 4.3 the kernel \( N(c) \) has an eigenvalue \( \lambda = 1 \) with the corresponding linearly independent real eigenfunctions \( \phi_1, \phi_2, \cdots, \phi_{2x+1} \). Since \( \Gamma \) is the unit circle then \( N(c)(t, w) \) is symmetric and the eigenfunctions \( \phi_j \) may be assumed to be orthonormal [9, p. 129], i.e.

\[
\left| \int_{\Gamma} \phi_i(t) \phi_j(t) \, dt \right| = \delta_{ij}, \quad i, j = 1, 2, \cdots, 2x+1, \tag{4.25}
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \). From Corollary 4.4 the general solution of the homogeneous Riemann problem (1.3) is given by

\[
f_h(z) = \sum_{i=1}^{2x+1} a_i \frac{1}{2\pi i} \int_{\Gamma} \frac{i\phi_i(t)}{c(t)(t-z)} \, dt, \quad z \in \Omega^+ \tag{4.26}
\]

where \( a_i, i = 1, 2, \cdots, 2x+1 \) are arbitrary real constants. Thus there exists \(-2x+1\) certain real constants \( \hat{a}_1, \hat{a}_2, \cdots, \hat{a}_{2x+1} \) such that

\[
H(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\rho(w)}{c(w)(w-z)} \, dw = \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{i=1}^{2x+1} \hat{a}_i \phi_i(w)}{c(w)(w-z)} \, dw, \quad z \in \Omega^+. \tag{4.27}
\]

Letting

\[
\phi(t) = \sum_{i=1}^{2x+1} \hat{a}_i \phi_i(t), \quad t \in \Gamma \tag{4.28}
\]

then we get from (4.27)

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\rho(w) - i\phi(t)}{c(w)(w-z)} \, dw = 0, \quad z \in \Omega^+ \tag{4.29}
\]

According to [2, p. 25] the function \( G(z) \) defined in \( \Omega^- \) by

\[
G(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\rho(w) - i\phi(t)}{c(w)(w-z)} \, dw = 0, \quad z \in \Omega^- \tag{4.30}
\]

is analytic in \( \Omega^- \), vanishes at infinity and satisfies

\[
c(t)G^-(t) = \rho(t) - i\phi(t), \quad t \in \Gamma. \tag{4.31}
\]
Letting $\tilde{G}(z) = iG(z)$, then from (4.31) include that $G(z)$ is a solution to the Riemann problem

$$\text{Re} \left[ c(t) \tilde{G}^- (t) \right] = \phi(t) \quad (4.32)$$

in $\Omega^-$ with the condition $\tilde{G}(\infty) = 0$. Since $x \leq 0$, according to [2, p. 300], there exists a real function $\eta(t), \ t \in \Gamma$ depends on $-2x + 1$ real arbitrary constants such that

$$\tilde{G}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\eta(t)}{c(t)(t - z)} \, dt. \quad (4.33)$$

Taking the limit $\Omega^- \ x \to t \in \Gamma$ and using the Sokhotsky formula to (4.33), we get

$$2c(t)\tilde{G}^- (t) = -\eta(t) + \text{PV} \frac{1}{\pi i} \int_{\Gamma} \frac{c(t)}{c(w)} \frac{\eta(w)}{w - t} \, dw \quad (4.34)$$

Taking the real parts of both sides of (4.24) and using (4.32) we conclude that $\eta$ is a solution to Fredholm integral equation

$$\eta(t) - \int_{\Gamma} N(c)(t, w) \eta(w) \, dw = -2\phi(t). \quad (4.35)$$

Since $\lambda = 1$ is an eigenvalue, in accordance with Fredholm alternative [4, p. 45] $\phi(t)$ must be orthogonal with all eigenfunctions of the adjoint kernel $N(c)(w, t)$. Since $N(c)(t, w)$ is symmetric, hence $\phi(t)$ must be orthogonal with $\phi_1, \phi_2, \cdots, \phi_{-2x+1}$. Therefore for all $j = 1, 2, \cdots, -2x + 1$ we must have $\int_{\Gamma} \phi(t) \phi_j(t) \, dt = 0$. Using (4.28) we get

$$\frac{1}{2\pi i} \sum_{j=1}^{-2x+1} \tilde{a}_j \int_{\Gamma} \phi(t) \phi_j(t) \, dt = 0, \ j = 1, 2, \cdots, -2x + 1. \quad (4.36)$$

Using (4.25) in (4.36) we conclude that $\tilde{a}_j = 0$ for all $j = 1, 2, \cdots, -2x + 1$. Consequently $\phi(t) = 0$. Substituting $\phi(t) = 0$ in (4.32) we have

$$\text{Re} \left[ c(t) \tilde{G}^- (t) \right] = 0, \ t \in \Gamma \ \text{with} \ \tilde{G}(\infty) = 0. \ \text{Since} \ x \leq 0, \ \text{then from Corollary 4.1 we get} \ \tilde{G}(z) = 0 \ \text{for all} \ z \in \Omega^- \ \text{and hence} \ \tilde{G}^- (t) = 0, \ t \in \Gamma \ \text{which implies that} \ G^- (t) = 0, \ t \in \Gamma. \ \text{Substituting} \ G^- (t) = 0 \ \text{and} \ \phi(t) = 0 \ \text{in (4.31) we get} \ \rho(t) = 0. \ \text{Our assertion is thereby proved.}
Theorem 4.8. Suppose that $\Gamma$ is the unit circle. If $x = \text{ind}_\Gamma(c) > 0$ then $\lambda = 1$ is not an eigenvalue of the generalized Neumann kernel $N(c)(t, w)$.

Proof. Since $\Gamma$ is the unit circle, from Remark 3.2 the kernel $N(c)(t, w)$ is symmetric. In accordance with the Fredholm’s Alternative to prove $\lambda = 1$ is not an eigenvalue to $N(c)$ it is sufficient to prove that the homogeneous equation

$$\mu(t) - \int_\Gamma N(c)(t, w)\mu(w)dw = 0 \quad (4.37)$$

or, since $N(c)(t, w)$ is symmetric, the associated homogenous equation

$$\mu(t) - \int_\Gamma N(c)(w, t)\mu(w)dw = 0 \quad (4.38)$$

has only the trivial solutions. Let $\mu(t)$ be any solution of (4.38) and $c_1(t) = T(t)/c(t)$, $t \in \Gamma$. From Remark 3.1, $N(c)(w, t) = -N(c_1)(t, w)$, $x = \text{ind}_\Gamma(c_1) = 1 - x \leq 0$ and (4.38) becomes

$$\mu(t) + \int_\Gamma N(c_1)(t, w)\mu(w)dw = 0. \quad (4.39)$$

In view of Theorem 4.7 $\lambda = -1$ is not an eigenvalue of $N(c_1)$ and hence $\mu(t) = 0$. Consequently $\lambda = 1$ is not an eigenvalue of $N(c)$.

5. Examples

Example 1. Suppose that $c(t) = 1$ and $\Gamma$ is any smooth Jordan curve. Since $x = \text{Ind}_\Gamma c = 0$, according to Corollary 4.3 $N(c)$ has a simple eigenvalue $\lambda = 1$. Since the homogeneous Riemann problem $\text{Re}[f(t)] = 0$ has only one independent solution $f(z) = i\omega$, $\omega$ is arbitrary real number. Therefore the eigenfunction of $N(c)$ corresponding to the eigenvalue $\lambda = 1$ is $\theta(t) = 1$.

Example 2. Consider the case when $c(t) = t^n$, $n$ is an integer and $\Gamma$ is the unit circle. First we consider the case $n \leq 0$. From Corollary 4.3 $\lambda = 1$ is an eigenvalue to $N(c)$ with $-2n + 1$ linearly independent eigenfunctions. According to [2, p. 221] the homogeneous Riemann problem $\text{Re}[c(t)f(t)] = 0$ has the general solution

$$f_H(z) = \sum_{k=1}^{-2n+1} c_k f_k(z) \quad (4.40)$$
where 
\( c_1, c_2, \cdots, c_{-2n+1} \) are arbitrary real constants and for \( k = 1, 2, \cdots, -n \),
\[
\begin{align*}
f_1(z) &= iz^{-n}, \\
f_{2k}(z) &= z^{-n-k} - z^{-n+k}, \\
f_{2k+1}(z) &= i\left(z^{-n+k} + z^{-n-k}\right). \quad (4.41)
\end{align*}
\]
Therefore
\[
\begin{align*}
f_H(z) &= c_1(iz^{-n}) + \sum_{k=1}^{-n} c_{2k}\left(z^{-n+k} - z^{-n-k}\right) + \sum_{k=1}^{-n} c_{2k+1}i\left(z^{-n+k} + z^{-n-k}\right) \quad (4.42)
\end{align*}
\]
and
\[
\begin{align*}
&ic(t)f_H(t) = it^n f_H(t) \\
&= it^n \left[c_1(ut^{-n}) + \sum_{k=1}^{-n} c_{2k}\left(t^{-n+k} - t^{-n-k}\right) + \sum_{k=1}^{-n} c_{2k+1}i\left(t^{-n+k} + t^{-n-k}\right)\right] \\
&= -c_1 + 2it\sum_{k=1}^{-n} \left(c_{2k}\cos ks - c_{2k+1}\sin ks\right)
\end{align*}
\]
where \( c_1, c_2, \cdots, c_{-2n+1} \) are arbitrary real number. Therefore from theorem 4.2 if \( n \leq 0 \), the eigenfunctions of \( N(c) \) are given by
\[
\theta_i(s) = 1, \quad \theta_{2k}(s) = \cos ks, \quad \theta_{2k+1}(s) = \sin ks, \quad k = 1, 2, \cdots, -n \quad (4.43)
\]
Next we consider the case \( n > 0 \). From Corollary 4.2 \( \lambda = -1 \) is an eigenvalue to \( N(c) \) with \( 2n - 1 \) linearly independent eigenfunctions. Let \( c_1(t) = t^{-n-1} \), according to [2, p. 225] the homogeneous Riemann problem \( \text{Re}[c_1(t)f^{-}(t)] = 0 \) has the general solution
\[
\begin{align*}
f_H(z) &= \sum_{k=1}^{2n-1} c_k f_k(z) \quad (4.44)
\end{align*}
\]
where \( c_1, c_2, \cdots, c_{2n-1} \) are arbitrary real constants and for \( k = 1, 2, \cdots, n - 1 \),
\[
\begin{align*}
f_1(z) &= iz^{-n+1}, \\
f_{2k}(z) &= z^{-n+1+k} - z^{-n+1-k}, \\
f_{2k+1}(z) &= i\left(z^{-n+1+k} + z^{-n+1-k}\right). \quad (4.45)
\end{align*}
\]
Therefore
\[
\begin{align*}
f_H(z) &= c_1(iz^{-n+1}) + \sum_{k=1}^{n-1} c_{2k}\left(z^{-n+1+k} - z^{-n+1-k}\right) \\
&+ \sum_{k=1}^{n-1} c_{2k+1}i\left(z^{-n+1+k} + z^{-n+1-k}\right) \quad (4.46)
\end{align*}
\]
and

\[ ic_1(t)f_H(t) = it^{n-1}f_H(t) \]

\[ = it^{n-1} \left( c_1(t) - \sum_{k=1}^{n-1} c_{2k} \left( t^{n-k+1} - t^{-n-k+1} \right) + \sum_{k=1}^{n-1} c_{2k} \left( t^{n-k+1} + t^{-n-k+1} \right) \right) \]

\[ = -c_1 + 2i \sum_{k=1}^{n-1} (c_{2k} \cos ks - c_{2k+1} \sin ks) \]

where \( c_1, c_2, \cdots, c_{2n-1} \) are arbitrary real number. Therefore from theorem 4.2 if \( n > 0 \), the eigenfunctions of \( N(c) \) are given by

\[ \theta_i(s) = i, \theta_{2k}(s) = \cos ks, \theta_{2k+1}(s) = \sin ks, k = 1, 2, \cdots, n-1 \quad (4.47) \]

6. Conclusions

The solvability of the Riemann problem (1.2) depends on the index of the function \( c(t) \) on \( \Gamma \) and the solvability of the Fredholm integral equation (2.2) related to the Riemann problem (1.2) and the integral equation (2.4) related to the Riemann problem (1.5) depend on whether \( \lambda = 1 \) or \( \lambda = -1 \) is an eigenvalue of the generalized Neumann kernel. In this paper we presented the relation between the index of \( c(t) \) and the eigenvalues \( \lambda = \pm 1 \) of the generalized Neumann kernel \( N(c) \). Theorems 4.7 and 4.8 are proved only for the unit circle and these proofs need to be extended for an arbitrary smooth Jordan curve.

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References


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