Arithmetic Functions Over Rings with Zero Divisors

PATTIRA RUANGSINSAP, VICHIAN LAOHAKOSOL AND PATTANEE UDOMKAVANICH

1Department of Mathematics, Kasetsart University, Bangkok, Thailand
2Department of Mathematics, Chulalongkorn University, Bangkok, Thailand

Abstract. It is well known that the complex-valued arithmetic functions form a unique factorization domain under addition and convolution, and so do those functions with values in other suitable rings. Here we consider instead such functions with values in a unique factorization ring with zero divisors and prove that under certain yet similar conditions, they also form a unique factorization ring with zero divisors.

1. Introduction

The set of complex-valued arithmetic functions, denoted by \( \mathcal{A}_C \), is known to be an integral domain under addition and convolution (or Dirichlet multiplication), see e.g. Apostol [1]. Cashwell and Everett [5] proved in 1959 that \( \mathcal{A}_C \) is in fact a unique factorization domain or factorial ring. The proof of factoriality due to Cashwell and Everett runs briefly as follows: first, it is shown that \( \mathcal{A}_C \) is isomorphic to \( \mathbb{C}[x_1, x_2, \cdots] \), the ring of formal power series in countably many indeterminates. Since the rings of formal power series (over \( \mathbb{C} \)) in a finite number of indeterminates are factorial, and \( \mathbb{C} \) is a field, this induces \( \mathbb{C}[x_1, x_2, \cdots] \) to be factorial which in turn establishes factoriality for \( \mathcal{A}_C \). That \( \mathbb{C} \) is a field plays significant role in the proof, for changing it to other integral domains the situations become much more complicated. This sparked off, especially in the sixties, a number of investigations related to problems about unique factorization, see e.g. Auslander and Buchsbaum [2], Buchsbaum [4], Cashwell and Everett [6], Cohn [7], Lu [12], Samuel [13], [14]. All afore-mentioned investigations are done over rings without zero divisors. In this work, we consider instead the case where the complex field is replaced by a unique factorization ring with zero divisors. We take essentially the same definition of unique factorization rings with zero divisors as in Galovich [9]. Our proof of the main theorem is a combination of the ones due to Cashwell and Everett [5] and Lu [12]. There are, however, additional difficulties due firstly to the fact that the usual cancellation law in the case of domain must be replaced by a weak cancellation law, which amounts to multiplying by units, and secondly the presence of units makes it impossible to take appropriate limits. This is overcome by invoking upon the idea of compactness.
2. Unique factorization rings with zero divisors

Let \( R \) be a commutative ring with unity. An element \( u \in R \) is a unit if there is \( v \in R \) such that \( uv = 1 \). If \( r, s \in R \), then \( r \) divides \( s \), written \( r | s \), if there exists \( t \in R \) such that \( rt = s \). Two elements \( r \) and \( s \) are associates, written \( r \sim s \), if there exists a unit \( u \) such that \( su = rv \). An element \( r \in R - \{0\} \) is a zero divisor if there exists \( s \in R - \{0\} \) such that \( rs = 0 \). Let \( r \in R - \{0\} \); \( r \) is prime if, whenever \( r | ab \), where \( a, b \in R - \{0\} \), then \( r | a \) or \( r | b \); \( r \) is reducible if there exist non-units \( a, b \) such that \( r = ab \); \( r \) is irreducible if \( r \) is not reducible. It is easily shown that if \( r \) and \( s \) are irreducible, then \( r \) divides \( s \) if and only if \( r \sim s \). An element \( d \in R - \{0\} \) is called a greatest common divisor of \( a, b \in R \), not both 0, if \( d \) divides \( a \) and \( b \), and if \( c \in R - \{0\} \) is such that \( c | a \) and \( c | b \), then \( c | d \).

A commutative ring \( R \), with unity and zero divisors, is a unique factorization ring, UFR for short, if for each non-zero non-unit \( r \in R \),

(i) there exist irreducible elements \( r_1, \ldots, r_n \) such that \( r = r_1 \cdots r_n \) and
(ii) whenever \( r = r_1 \cdots r_n = s_1 \cdots s_m \) where \( r_1, \ldots, r_n, s_1, \ldots, s_m \) are irreducible, then \( n = m \) and the \( s_j \) can be renumbered so that \( r_i \sim s_j \) (\( i = 1, \ldots, n \)).

A typical example of UFR is \( \mathbb{Z}/p^n \mathbb{Z} \), where \( p \) is a rational prime and \( m \) is a positive integer \( \geq 2 \), see Billis [3].

A very useful fact which will be repeatedly used is that any UFR satisfies a weak cancellation law, i.e. whenever \( ax = ay \neq 0 \), then \( x \sim y \). This is easily shown as follows: from the unique factorization into irreducible elements \( a = r_1 \cdots r_n, x = s_1 \cdots s_m, y = t_1 \cdots t_k \), the equation \( ax = ay \neq 0 \) together with uniqueness implies that \( m = k \) and after some renumbering \( s_j \sim t_j \) (\( j = 1, \ldots, m \)) and so \( x \sim y \).

Denote the set of arithmetic functions over the UFR \( R \) by \( A_R \), i.e.

\[
A_R := \{ f : \mathbb{N} \to R; \ R \text{ is a UFR} \}.
\]

It is easily checked that \( (A_R, +, *) \) is a ring with respect to addition \( (f + g)(n) = f(n) + g(n) \), and convolution \( f * g(n) := \sum_{ij=n} f(i)g(j) \).

That \( A_R \) has zero divisors is directly inherited from \( R \) as seen by the following example. Let \( x, y \in R - \{0\} \) be zero divisors in \( R \) such that \( xy \neq 0 \). Take \( f(1) = x \), \( f(n) = 0 \) for all \( n > 1 \), and \( g(1) = y \), \( g(n) = 0 \) for all \( n > 1 \). Then \( f * g = 0 \in A_R \), while \( f, g \in A_R - \{0\} \).
Our first theorem gives a characterization of UFR.

**Theorem 1.** Let \( R \) be a commutative ring with unity and with zero divisors. Assume that every non-zero non-unit element of \( R \) can be written as a product of finitely many irreducible elements of \( R \). Then the following assertions are equivalent:

(i) \( R \) is a UFR.

(ii) Any two elements of \( R \) have a greatest common divisor and \( R \) satisfies the weak cancellation law.

**Proof.** Assume the truth of assertion (i). That \( R \) satisfies the weak cancellation law has already been observed. Let \( a, b \) be two elements of \( R \), which we may assume to be non-zero and non-unit, for otherwise the proof is trivial. Since \( R \) is a UFR, we can write

\[
a = r_1^{n_1} \cdots r_k^{n_k}, \quad b = r_1^{m_1} \cdots r_k^{m_k},
\]

where \( r_i \) are distinct irreducible elements of \( R \) and \( n_i, m_i \) are non-negative integers. Taking \( d = r_1^{\max(n_i, m_i)} \cdots r_k^{\max(n_i, m_k)} \), we easily check that \( d \) is a greatest common divisor of \( a, b \).

Assume the truth of assertion (ii). It follows easily by induction that the existence of a greatest common divisor of any two elements of \( R \) induces the existence of a greatest common divisor of any finite number of elements of \( R \). Denote by \( (a, b) \) a greatest common divisor of \( a, b \in R \). Then, with the aid of the weak cancellation law, we easily deduce the following properties (see e.g. pp.139-140 of Jacobson [10]):

\[
((a, b), c) \sim (a, (b, c)), \quad c(a, b) = (ca, cb), \quad (a, b) \sim 1 \text{ and } (a, c) \sim 1 \Rightarrow (a, bc) \sim 1.
\]

Let \( p \) be an irreducible element of \( R \) such that \( p \mid ab \) with \( a, b \in R \). If \( p \) does not divide \( a \) and \( b \), then by the existence of greatest common divisors, \( (p, a) = 1 \). For otherwise there would be a non-zero non-unit \( c \in R \) such that \( c = (p, a) \) and so \( cd = p \) for some \( d \in R \). Since \( p \) is irreducible, \( d \) must be a unit, yielding \( p \sim c \), and so \( p \mid a \), a contradiction. Similarly \( (p, b) = 1 \). Now by the third property above, \( (p, ab) = 1 \), contradicting \( p \mid ab \). Therefore, \( p \mid a \) or \( p \mid b \), indicating that an irreducible element must be a prime in \( R \). In order to conclude that \( R \) is a UFR, we must show the uniqueness of factorization. Let \( a \) be a non-zero non-unit element of \( R \) having two factorizations into irreducible elements \( a = r_1 \cdots r_k = s_1 \cdots s_n \) where \( r_j \) and \( s_j \) are irreducible. Since irreducible elements are primes, \( r_1 \mid \text{ some } s_j \), say \( s_1 \).

Since \( s_1 \) is irreducible, then \( r_1 \sim s_1 \). Cancelling out the factor \( r_1 \), which is permissible by the weak cancellation law, and continue the arguments. Should \( k \neq n \), we would end up having a product of irreducible elements equal to a unit, which is impossible. Thus \( k = n \) and simultaneously, after appropriate renumbering \( r_j \sim s_j \) for all \( i \).
3. Power series

Denote by \( R_u := R[[x_1, x_2, \cdots]] \), \( R_m := R[[x_1, \cdots, x_m]] \) the rings of formal power series in countably many indeterminates \( x_1, x_2, \cdots \), respectively, finitely many indeterminates \( x_1, \cdots, x_m \), over a UFR \( R \). Since \( R \subset R_u \), it follows that \( (R_u, +, \cdot) \) and \( (R_m, +, \cdot) \) are rings with zero divisors, with respect to addition and multiplication of formal power series. As shown in Section 14 of Cashwell and Everett [5], for \( n = p_1^{e_1} p_2^{e_2} \cdots \), where \( p_1 < p_2 < \cdots \) denotes the (ascending) set of rational primes, the correspondence \( f \leftrightarrow \sum f(n) x_1^{e_1} x_2^{e_2} \cdots \), establishes the fact that \( (R_u, +, \cdot) \) is isomorphic to \( (R_m, +, \cdot) \).

Recall that a commutative ring with unity, \( R \), is said to satisfy the ascending chain condition for principal ideals, ACCP for short, provided that every strictly increasing sequence of principal ideals of \( R \) admits a maximal element. It is easy to check that any UFR satisfies the ACCP and for any two isomorphic commutative rings with unity \( R_1 \) and \( R_2 \), if \( R_1 \) satisfies the ACCP, so does \( R_2 \). Before we proceed to consider the power series ring, we recall another definition which will be used in the next lemma. The order of \( f \in A_R \) is defined to be

\[
v(f) := \begin{cases} 
\min\{n; f(n) \neq 0\} & \text{if } f \neq 0 \\
\infty & \text{if and only if } f = 0.
\end{cases}
\]

It is easily checked that \( v(f \ast g) \geq v(f)v(g) \), and \( v(f + g) \geq \min\{v(f), v(g)\} \).

Lemma 2.1. Let \( R \) be a commutative ring with unity satisfying the weak cancellation law. If \( R \) satisfies the ACCP, so does \( R_u \).

Proof. Since \( R_u \) is isomorphic to \( A_R \), it suffices to show that \( A_R \) satisfies the ACCP. Let \( (f_1) \subset (f_2) \subset \cdots \) be an ascending chain of principal ideals in \( A_R \). Without loss of generality, assume that \( f_1 \neq 0 \) and so \( f_i \neq 0 \) for all \( i > 1 \). Ideal inclusions imply that there exists \( g_i \in A_R - \{0\} \) such that \( f_i = f_{i+1} \ast g_i \). Considering orders, we get \( v(f_i) = v(f_{i+1} \ast g_i) \geq v(f_{i+1})v(g_i) \). This yields a non-increasing sequence of positive integers \( v(f_1) \geq v(f_2) \geq \cdots \) which must then terminate, i.e. there are two positive integers \( r \) and \( k \) for which \( v(f_{r+j}) = v(f_r) = k \), say, for each integer \( j \geq 0 \). Now \( 0 \neq f_j(k) = (f_{r+j} \ast g_j)(k) = f_{r+j}(k)g_j(1) \) and so we obtain an ascending chain of non-zero principal ideals in \( R \) of the form \( (f_r(k)) \subset (f_{r+j}(k)) \subset \cdots \). As \( R \) satisfies the ACCP, there exists an integer \( m \geq r \) such that \( f_m(k) = (f_{m+j}(k)) \) for all \( j \geq 0 \), which implies, using the weak cancellation law of \( R \), that \( f_m(k) = u_jf_{m+j}(k) \), where \( u_j \) is a...
unit in $R$. But the chain inclusions in $A_R$ give $f_m = h_j * f_{m+j}$ for some $h_j \in A_R - \{0\}$.

By the weak cancellation law of $R$, $u_j \sim h_j(1)$ and so $h_j$ is a unit in $A_R$, which means that the chain in $A_R$ terminates, and we are done.

**Lemma 2.2.** Let $R$ be a commutative ring with unity satisfying the weak cancellation law. If $R$ satisfies the ACCP, then every non-zero non-unit element of $R_w$ is a product of a finite number of irreducible factors.

**Proof.** From the last lemma, $R_w$ satisfies the ACCP. Take any non-zero, non-unit $F$ in $R_w$. If $F$ is irreducible, we have nothing to prove. Assume that $F$ is reducible. Then there exists non-zero, non-units $F_1, G_1$ in $R_w$ such that $F = F_1G_1$. This yields an inclusion of principal ideals $(F) \subset (F_1)$. If both $F_1$ and $G_1$ are irreducible, we are done; otherwise at least one of them, say $F_1$ is reducible. Thus there are non-zero, non-units $F_2, G_2$ in $R_w$ such that $F_1 = F_2G_2$ yielding another inclusion of principal ideals $(F_1) \subset (F_2)$. Should $F$ not be written as a product of finitely many irreducible factors, we would get an infinite ascending chain of principal ideals generated by proper factors of $F$. But the ACCP insists that the chain must be finite, a desired contradiction.

Let $j \in \mathbb{N}$ and $F(x_1, x_2, \cdots) \in R_w$ or $R_m$ with $m \geq j$. By $(F)_j$, the projection of $F$ on $R_j$, we mean the series $F(x_1, \cdots, x_j, 0, 0, \cdots)$ obtained from $F$ by putting equal to 0 all terms of $F$ actually involving any $x_i$ with $i > j$. The map $F \mapsto (F)_j$ is a ring homomorphism of $R_w$ or $R_m$ onto $R_j$ and $(FG)_j = (F)_j(G)_j$.

**Lemma 2.3.** Let $F$ be a non-zero, non-unit element of $R_w$. Then there is a least positive integer $J = J(F)$, hereby called the index of $F$, for which $(F)_j$ is a non-zero, non-unit element of $R_j$ for all $j \geq J$.

**Proof.** Since $F$ is a non-zero non-unit element of $R_w$, then $F$ contains a non-zero coefficient of some monomial term $x_1^{n_1} x_2^{n_2} \cdots$ with $n_i$ not all zero. If $x_k$ is the last variable in this term with positive $n_k$, then $(F)_k \neq 0$. Among all such $k$, the least one is our desired $J$.

**Lemma 2.4.** Let $F$ be a non-zero, non-unit element of $R_w$. Let $J$ be the index of $F$. If $(F)_j$ is irreducible in $R_j$ for some $j \geq J$, then $(F)_m$ is irreducible in $R_m$ for all $m \geq j$ and $F$ is also irreducible in $R_w$. 
Proof. Take any \( m \geq j \geq J \) and suppose we have a factorization \((F)_{m} = G^{(m)}H^{(m)}\), for some \( G^{(m)}, H^{(m)} \in R_{m} \). Observe that \((F)_{j} = G^{(m)}H^{(m)}\) for some \( G^{(m)} \) and \( H^{(m)} \). By the irreducibility of \((F)_{j}\), either \( G^{(m)} \) or \( H^{(m)} \) is a unit of \( R_{j} \) and so \( G^{(m)} \) or \( H^{(m)} \) is a unit of \( R_{m} \). Hence, \((F)_{m}\) is irreducible. The proof of the last assertion is similar.

The proof of the next lemma requires a number of new concepts and definitions which we now elaborate. Let \( F \) be a non-zero non-unit in \( R_{w} \) and let \( j \) be a positive integer. A proper divisor of \( (F)_{j} = \alpha \beta \), where \( \alpha \) is a non-unit in \( R_{j} \). By a true factor of \( (F)_{j} \), we mean a non-unit proper divisor of \( (F)_{j} \) in \( R_{j} \), and call such factorization of \( (F)_{j} \), a true factorization.

\( F \) is said to be finitely irreducible if there is a least integer \( I \geq j \) for which \((F)_{I} \) is irreducible in \( R_{I} \) for all \( I \geq j \). We call a chain \( \{G^{(1)}, G^{(2)}, \ldots\} \), with each \( G^{(i)} \in R_{i} \), telescopic, respectively pseudo-telescopic, if \( G^{(i)} = G^{(i+1)} \) in \( R_{i} \) for all \( i \). Following Lu [12], we introduce a topology which eases the discussion considerably. We say that a non-zero monomial \( c x_{1}^{k_{1}} x_{2}^{k_{2}} \cdots x_{i}^{k_{i}} \in R_{j} \) is of weight \( r \) when \( r = 1k_{1} + 2k_{2} + \cdots + jk_{j} \). Clearly any \( F \in R_{j} \) can be written as \( F = F_{0} + F_{1} + \cdots \) where each \( F_{i} \) is a sum of all monomials of weight \( i \). Define an order function \( \text{ord} \) on \( R_{j} \) as follows:

\[
\text{ord}(0) = \infty, \quad \text{ord}(F) = \min\{n; F^{n} \neq 0\} \quad \text{if} \quad F \neq 0.
\]

For \( F \in R_{w} \), \( k \) a non-negative integer, define \( \{B_{k}(F) := \{G \in R_{w} ; \text{ord}(G - F) \geq k\}\} \).

Observe that \( R_{w} = B_{0}(0) \) and that if \( B_{s}(F) \cap B_{t}(G) \neq \emptyset \) with \( s \leq t \), then \( B_{s}(G) \subseteq B_{s}(F) \). By Theorem 3.2, p.67 of Dugundji [8], it follows that \( \{B_{k}(F); F \in R_{w}, k \) a non-negative integer\} is a basis for a topology, called the weight topology, of \( R_{w} \). Indeed, by the same arguments as used in the proof of Theorem 1 in Lu[12], \( R_{w} \) is the completion of \( \bigcup_{j=1}^{\infty} R_{j} \) with respect to this topology. By the same proof as in Lemma 2 of Lu [12], any infinite telescopic chain is a Cauchy sequence and by completeness has a limit in \( R_{w} \). The case of pseudo-telescopic chain, which will be encountered, is more complicated and requires some compactness arguments. For convenience, we adopt the following definition. The ring \( R_{w} \) is said to be \( n \)-sequentially compact (with respect to the weight topology) if every sequence of units in \( R_{w} \) has a convergent subsequence. Clearly, the limit of such a subsequence must then be a unit in \( R_{w} \).
Lemma 2.5. If \( R_w \) is u-sequentially compact, then any pseudo-telescopic chain has a convergent subchain.

Proof. Take a pseudo-telescopic chain \( \{G^{(i)}, G^{(2)}, \ldots\} \). Thus

\[
G^{(i)} = (u^{(i)}G^{(i+1)})_i, \text{ where } u^{(i)} \text{ is a unit in } R_i.
\]

Put \( F^{(1)} = G^{(1)} \), \( F^{(2)} = u^{(1)}G^{(2)} \), \ldots, \( F^{(j)} = u^{(1)}u^{(2)} \cdots u^{(j-1)}G^{(j)} \). Clearly, \( G^{(j)} = v^{(j)}F^{(j)} \), where \( v^{(j)} = (u^{(1)} \cdots u^{(j-1)})^{-1} \) is a unit in \( R_j \) and more importantly, \( \{F^{(1)}, F^{(2)}, \ldots\} \) is a telescopic chain, which must then converge to a limit \( F \), say, in \( R_w \). Since \( R_w \) is u-sequentially compact, there is a subsequence \( v^{(j_k)} \) of \( v^{(j)} \) which converges to a unit \( v \), say, in \( R_w \). Hence, the subchain \( \{G^{(j_k)} = v^{(j_k)}F^{(j_k)}\} \) converges to the limit \( vF \).

Lemma 2.6. Let \( R \) be a UFR. If \( R_j \) is a UFR for every positive integer \( j \) and \( R_w \) is u-sequentially compact, then all irreducible elements of \( R_w \) are finitely irreducible.

Proof. We claim that for non-zero non-unit \( F \in R_w, \) if \( (F)_j \) is reducible in \( R_j \) for all \( j \geq \) the index \( J \) of \( F \), then \( F \) is reducible in \( R_w \). To prove the claim, take any non-zero non-unit \( F \) in \( R_w \). Assume that for each \( j \geq J, \)

\[
(F)_j = G^{(j)}H^{(j)} \text{ where } G^{(j)} \text{ and } H^{(j)} \text{ are true factors of } (F)_j \text{ in } R_j.
\]

Any true factorization \( (F)_m = G^{(m)}H^{(m)} \), with \( m > J \), induces a true factorization of \( (F)_{m-1} = (F)_m \) \( (H^{(m)})_{m-1} = G^{(m-1)}H^{(m-1)} \). Continuing down to \( (F)_j = G^{(j)}H^{(j)} \), we get a telescopic chain of true factors \( \{G^{(j)}, G^{(j+1)}, \ldots, G^{(m)}\} \). From the original assumption that for each \( j \geq J, (F)_j \) is expressible as a product of two true factors in \( R_j \), we have the existence of a sequence

\[
k_0 = [G_{00}], \quad k_i = [G_{10} = (G_{11})_j, G_{11}], \quad k_2 = [G_{20} = (G_{21})_j, G_{21} = (G_{22})_{j+1}, G_{22}], \ldots,
\]

each \( k_i \) being a telescopic chain of true factors \( G_{ij}, \) \( j = 0, 1, \ldots, i \), of \( (F)_{j+i} \). By the unique factorization of \( R_j \), the number of true factors of \( (F)_j \) is finite for each \( j \). Thus there is a true factor \( T_0 \) of \( (F)_j \) such that there is an infinite set of the chains \( k_i \) having their first entry being an associate of \( T_0 \). Choose one of these and call it \( k'_0 \). Of this
infinite set, there is an infinite subset of $k_i$ whose second entry is an associate of some one true factor $T_i$ of $(F)_{j+1}$. Choose one and call it $k'_i$. Continuing in this way we get a sequence of telescopic chains

$$k'_0 = [g'_{00}, \ldots], \quad k'_1 = [g'_{10}, g'_{11}, \ldots], \quad k'_2 = [g'_{20}, g'_{21}, g'_{22}, \ldots], \ldots$$

each of which extends, at least to the main diagonal, such that the entries on the diagonal and below have the property that, for each $j \geq 0$, $g'_{jj} \sim T_j$ for all $i \geq j$. Now construct the telescopic infinite chain $k^* \dagger$ by working only with the main diagonal and the diagonal next below it, as follows: define $g^*_j = g'_{00}$. Since $g'_{10} \sim T_0 \sim g'_{00}$ in $R_j$, there is a unit $u^{(J)} \in R_j$ such that $g^*_j = g_{10}'u^{(J)} = (g'_{11}u^{(J)})_j$. Define $g^*_{j+1} = g'_{11}'u^{(J)}$ in $R_{j+1}$. Then $g^*_j = (g^*_{j+1})_j$. Note that $g^*_{j+1}$ is a true factor of $(F)_{j+1}$ and $g^*_j \sim T_i$ in $R_{j+1}$. Since $g^*_{21} \sim T_1 \sim g^*_{11}$ in $R_{j+1}$, there is a unit $u^{(J+1)} \in R_{j+1}$ such that $g^*_{j+1} = (g^*_{21}'u^{(J+1)} = g_{22}'u^{(J+1)})_{j+1}$. Define $g^*_{j+2} = g'_{22}'u^{(J+1)}$ in $R_{j+2}$. Then $g^*_{j+1} = (g^*_{j+2})_{j+1}$. Note that $g^*_{j+2}$ is a true factor of $(F)_{j+2}$ and $g^*_{j+2} \sim T_2$ in $R_{j+2}$. Continuing in the same manner, we get an infinite telescopic chain of true factors

$$k^* = [g^*_j, g^*_{j+1}, g^*_{j+2}, \ldots],$$

where $g^*_j = g'_{00} = g'_{10}u^{(J)} = (g'_{11}u^{(J)})_j$, $g^*_{j+1} = g'_{11}u^{(J+1)} = g'_{21}u^{(J+1)} = (g'_{22}u^{(J+1)})_{j+1}$, $g^*_{j+2} = g'_{22}u^{(J+1)} = g'_{22}u^{(J+2)} = (g'_{23}u^{(J+2)})_{j+2}, \ldots$, which must converge to a limit $g$, say, in $R_w$. Now for all $j \geq 0$, we have $(F)_{j+1} = g^*_j h_{j+1}$, where $h_{j+1}$ are non-zero, non-unit in $R_{j+1}$, and so $g^*_j h_{j+1} = (F)_{j+1} = (F)_{j+1}(j+1)$, which by the weak cancellation law gives $h_{j+1} \sim (h_{j+1})_{j+1}$. Thus we get a pseudo-telescopic chain $[h_j, h_{j+1}, \ldots]$, which by $u$-sequential compactness of $R_w$ has a subchain converging to a limit $h$, say, in $R_w$. Now $(F)_{j+1} = g^*_j h_{j+1} = (g)_{j+1}h_{j+1}$ and so by passing to a subsequence, we arrive at $F = gh$. Clearly, both $g$ and $h$ are non-units in $R_w$ implying that $F$ is reducible in $R_w$ and the claim is established. It follows from the claim that if $F$ is irreducible in $R_w$, then there is a least integer $I \geq J$ for which $(F)_I$ is irreducible in $R_I$ and by Lemma 2.4 for which $(F)_j$ is irreducible in $R_j$ for all $j \geq I$. Consequently, $F$ is finitely irreducible.
Lemma 2.7. Let $R$ be a UFR, $F$ and $G \in R_w$, $D^{(j)}$ a greatest common divisor in $R_j$ of $(F)_j$ and $(G)_j$, $j \in \mathbb{N}$. If $R_j$ is a UFR for every $j \in \mathbb{N}$, then $(D^{(j+1)})_j \sim D^{(j)}$ for all $j \geq L(F,G)$, where $L(F,G)$ is a certain non-negative integer.

Proof. If $F$ or $G$ is zero, then the assertion is trivial. Assume that both $F$ and $G$ are non-zero. Let $n$ be the smallest integer such that $(F)_n$ and $(G)_n$ are both non-zero and let $i$ be an integer $\geq n$. Since $R_i$ is a UFR, we can represent $D^{(i)}$ as a finite product of irreducible elements of $R_i$. Denote by $N_i$ the number of irreducible factors counted with multiplicity of $D^{(i)}$. Since $D^{(i)}$ is a greatest common divisor of $(F)_i$ and $(G)_i$ in $R_i$, $i \geq n$, $(D^{(i+1)})_i \neq 0$, and the number of irreducible factors of $(D^{(i+1)})_i$ is not less than the number of irreducible factors of $(D^{(i)})$, then $(D^{(i+1)})_i | (D^{(i)})$, which thus gives $N_i \geq N_{i+1}$. Note that the projection of each irreducible factor of $(D^{(i+1)})$ on $R_i$ may not be irreducible in $R_i$. Thus we have the following descending chain of non-negative integers $N_n \geq N_{n+1} \geq \cdots$ and so there are integers $j$ and $k$ such that $k = N_m + j$, for every integer $r \geq 0$. Thus for every $m \geq n + j$, the projection of each irreducible factor of $D^{(m+1)}$ on $R_m$ is also irreducible, yielding $(D^{(m+1)})_m \sim D^{(m)}$. Taking $L(F,G) = n + j$, we are done.

4. Main results

Theorem 2. Let $R$ be a UFR. If $R_j$ is a UFR for each positive integer $j$ and $R_w$ is $u$-sequentially compact, then $R_w$ is a UFR.

Proof. Since $R$ is a UFR, it satisfies the ACCP and so by Lemma 2.2 any non-zero non-unit element of $R_w$ can be written as a finite product of irreducible elements of $R_w$. We shall use the characterization in Theorem 1. First, we show that any two elements $F$ and $G$ of $R_w$ have a greatest common divisor. Since this is trivial when $F$ or $G$ is zero, we assume that both are non-zero. Let $D^{(j)}$ be a greatest common divisor of $(F)_j$ and $(G)_j$. We construct an infinite telescopic chain $[E^{(L)}, E^{(L+1)}, \cdots]$ with the initial term in $R_L$, where $L = L(F,G)$ is as in Lemma 2.7, as follows: put $E^{(L)} = D^{(L)}$. Assume $E^{(j)}$, $j \geq L$, has been defined and let $D^{(j+1)}$ be any greatest common divisor of $(F)_j$ and $(G)_j$. Then by Lemma 2.7 there is a unit $u^{(j)}$ in $R_j$ such that $E^{(j)} = (u^{(j)}D^{(j+1)})_j = (u^{(j)}(D^{(j+1)}))_j$. Taking $E^{(j+1)} = u^{(j)}D^{(j+1)}$ we get a
telescopic chain \( [E^{(L)}, E^{(L+1)}, \cdots] \). This chain has a limit \( E \in R_w \). Note that \( (E)_j = E^{(j)} \) or \( (E^{(L)}_j) \) according as \( j \geq L \) or \( 0 \leq j < L \). Let \( F^{(j)} \) and \( G^{(j)} \) be two elements of \( R_j \) such that \( (F)_j = F^{(j)}(E)_j \), \( (G)_j = G^{(j)}(E)_j \) for each \( j \geq L \). Then \( (F^{(j+1)}_j) \sim F^{(j)}(E)_j \) and \( (G^{(j+1)}_j) \sim G^{(j)}(E)_j \) by the weak cancellation law. Hence we have two pseudo-telescopic chains \( [F^{(L)}_j, F^{(L+1)}_j, \cdots] \) and \( [G^{(L)}_j, G^{(L+1)}_j, \cdots] \) with the initial terms in \( R_j \). By Lemma 2.5, a subchain of \( [F^{(L)}_j, F^{(L+1)}_j, \cdots] \), respectively a subchain of \( [G^{(L)}_j, G^{(L+1)}_j, \cdots] \) converge to limits \( f \), respectively \( g \) in \( R_w \). Passing to subchain in \( (F)_j = F^{(j)}(E)_j \), we deduce that \( F = fE \). In the same manner, we get \( G = gE \) for the weight topology, i.e. \( E \) is a common divisor of \( F \) and \( G \). To show that \( E \) is a greatest divisor of \( F \) and \( G \), let \( E^* \) be any common divisor of \( F \) and \( G \) in \( R_w \). Then \( E^*_j \) is also a common divisor of \( (F)_j \) and \( (G)_j \) in \( R_j \) for each \( j \geq L \). Since \( (E)_j \) is a greatest common divisor of \( (F)_j \) and \( (G)_j \) for such \( j \), then \( (E^*_j) \mid (E)_j \). Thus there is an element \( a^{(j)} \in R_j \) such that \( (E)_j = a^{(j)}(E^*_j) \), \( j \geq L \). Thus for \( j \geq L \), we have

\[
a^{(j)}(E^*_j) = (E)_j = (E)_{j+1} = (a^{(j+1)}(E^*_j))_{j+1} = (a^{(j+1)}_j)(E^*_j)
\]

and so by the weak cancellation law \( a^{(j)} \sim (a^{(j+1)}_j) \) which yields a pseudo-telescopic chain \( \{a^{(L)}, a^{(L+1)}, \cdots\} \). By Lemma 2.5, let its subchain converge to a limit \( a \) in \( R_w \). Passing to subchain in \( (E)_j = (a)_j(E^*_j) \), we get \( E = aE^* \). Hence, \( E \) is a greatest common divisor of \( F \) and \( G \). Lastly, we show that \( R_w \) satisfies the weak cancellation law. Let \( F, G \) and \( H \in R_w \) be such that \( FH \neq 0 \). Then there are integers \( n \) and \( k \) for which \( (FG)_j = (FH)_j \neq 0 \) for all \( j \geq k \) and \( (F)_i \neq 0 \) for all \( i \geq n \). Put \( m = \max(k, n) \). Then for all \( j \geq m \), we get \( (F)_j(G)_j = (FG)_j = (FH)_j = (F)_j(H)_j \neq 0 \). By the weak cancellation in \( R_j \), we deduce that \( (G)_j = u^{(j)}(H)_j \) for some unit \( u^{(j)} \in R_j \). Now the definition of \( u \)-sequential compactness implies that \( (u^{(j)}) \) has a subsequence converging to a unit \( u \) in \( R_w \). Passing to subchain in \( (G)_j = u^{(j)}(H)_j \) we get \( G = uH \). The result now follows from Theorem 1.

As an immediate consequence, we mention
Corollary. If $R$ is a UFR such that $R_j$ is a UFR for each positive integer $j$ and $R_u$ is $u$-sequentially compact, then $A_R$ is a UFR.

Final remarks. The main results in this paper can be extended by replacing the domain $N$ of arithmetic function with an arithmetical semigroup $G$. This is done in the same manner as that in the proof of Proposition 1.3 in Knopfmacher [11] as follows: if $G$ has infinitely many primes, then $G$ is algebraically isomorphic to $N$ and we are done. If $G$ has only finitely many primes, say $j$, then the ring $A_R(G) = \{ f : G \to R \}$ is isomorphic to $R[[x_1, x_2, \cdots, x_j]] = R_j$ and the condition of $R_j$ being a UFR thus implies that $A_R(G)$ is a UFR. Note that $u$-sequential compactness does not come into consideration in this case.

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References


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