On a Class of Functions whose Derivatives Map the Unit Disc into a Half Plane

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Abstract. Let $G(\alpha, \delta)$ denote the class of functions $f$, $f(0) = f(0) - 1 = 0$ for which $\text{Re} \ e^{ia} f'(z) > \delta$ in $D = \{z : |z| < 1\}$ where $|\alpha| \leq \pi$ and $\cos \alpha - \delta > 0$. We discuss some basic properties of the class including representation theorem, extremals and argument of $G(\alpha, \delta)$.

1. Introduction

We denote $G(\alpha, \delta)$ the class of normalized analytic functions $f$ in the unit disc $D$ where

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

satisfying $\text{Re} \ e^{ia} f'(z) > \delta$ where $|\alpha| \leq \pi$ and $\cos \alpha - \delta > 0$.

Many of the classes $G(\alpha, \delta)$ have been studied by several researchers such as MacGregors [3] for $G(0, 0)$, Goel and Mehrok [1] for $G(\alpha, \delta)(\delta \geq 0)$ and Silverman and Silvia [4] for $G(\alpha, 0)$. Writing

$$p(z) = \frac{e^{ia} f'(z) - i \sin \alpha - \delta}{\cos \alpha - \delta} \quad (z \in D),$$

clearly $f \in G(\alpha, \delta)$ if and only if $p \in P$, the class of functions with positive real parts. Solving (1) for $f'(z)$ yields

$$f'(z) = e^{-ia} (Ap(z) + i \sin \alpha + \delta) \quad (z \in D)$$

where $A = \cos \alpha - \delta$. 
2. Representation theorem

We obtain the representation theorem for $G(\alpha, \delta)$, sharing the same approach through Herglotz Representation Theorem for functions in $P$.

**Theorem 2.1.** Let $f \in g(\alpha, \delta)$. Then for some probability measure $\mu$ on the unit circle $X$,

$$f(z) = \int_X \left[ -\psi'_\delta (e^{-ia} - 2\delta)z - 2e^{-ia} A\log(1-zx) \right] d\mu(x). \quad (3)$$

Conversely, if $f$ is given by the above equation, then $f \in G(\alpha, \delta)$.

**Proof.** For some probability measure $\mu$ on the circle $X$,

$$p \in P \iff p(z) = \int \frac{1 + xz}{1 - xz} d\mu(x).$$

Using (2), we have

$$f'(z) = e^{-ia} \left[ A \int \frac{1 + xz}{1 - xz} + i \sin \alpha + \delta \right] d\mu(x)$$

and so

$$f(z) = e^{-ia} \left[ \int_0^z \left( \int_X \left( \frac{1 + x\psi}{1 - x\psi} \right) + (i \sin \alpha + \delta) d\mu(x) \right) d\psi \right]$$

$$= \int_0^z \left[ \int_X \frac{1 + (e^{-2ia} - 2\delta e^{-ia})x\psi}{1 - x\psi} d\mu(x) \right] d\psi \quad (4)$$

and the desired representation theorem is obtained by reversing the order of integration and integrating with respect to $\psi$. 
We note that the extreme points of $G(\alpha, \delta)$ are the unit point masses

$$f_\alpha(z) = -e^{-iz} (e^{-iz} - 2\delta)z - 2e^{-iz} A\bar{z} \log(1 - xz)$$

with $|x| = 1$ and the derivatives of the extreme points for $G(\alpha, \delta)$ are the point masses

$$f'_\alpha(z) = \frac{1 + (e^{-2iz} - 2\delta xz e^{-iz})}{1 - xz}, |x| = 1.$$

3. Extremal properties

Following Silverman and Silvia [4], we now obtain a coefficient bound for functions in $g(\alpha, \delta)$ and distortion theorems for the derivatives of these functions.

**Theorem 3.1.** If $f \in G(\alpha, \delta)$, then $|a_n| \leq 2A/n, n = 2, 3, \ldots$ and equality is attained for each $n$ when $f$ is an extreme point of $G(\alpha, \delta)$.

**Proof.** Using (4) and since $1/(1 - x\psi) = \sum_0^\infty (x\psi)^n$, we can write

$$f(z) = z + 2e^{-iz} A \int \sum_0^\infty x^{n-1} d\mu(x) \frac{z^n}{n}.$$

Now, let $f(z) = z + \sum_{n=2}^\infty a_n z^n$. Then $a_n = \frac{2e^{-iz} A}{n} \int x^{n-1} d\mu(x)$ and the result follows immediately.

Our further result will be based on the following theorem.

**Theorem 3.2.** Let $f \in G(\alpha, \delta)$. Then $f'$ maps $|z| \leq r$ into the disc $D_r$ with center $-e^{-iz} (e^{-iz} - 2\delta) + (2e^{-iz} A)/(1 - r^2)$ and radius $2Ar/(1 - r^2)$.

**Proof.** If $a$ and $b$ are complex numbers with $|b| < 1$, and if $0 < r < 1$, the range of the function $(1 + a \omega)/(1 + b \omega)$ ($|\omega| \leq 1$) is the disc with center and radius

$$\frac{1 - abr^2}{1 - |b|^2 r^2}, \quad \frac{|a - b|r}{1 - |b|^2 r^2}.$$
respectively. By taking \( a = (e^{-2ia} - 2\delta e^{-ia})xr \) and \( b = xr \) where \(|x| = 1\), we see that

\[
1 + \frac{(e^{-2ia} - 2\delta e^{-ia})xz}{1 - xz}
\]

maps \(|z| \leq r\) onto \( D_r \). By convexity, any linear combination of functions of this form also maps \( D \) onto \( D_r \). Since for some probability measure \( \mu \), we have

\[
 f'(z) = \int_X \frac{1 + (e^{-2ia} - 2\delta e^{-ia})xz}{1 - xz} \, d\mu(x),
\]

the stated result now follows.

**Theorem 3.3.** If \( f \in \mathcal{G}(\alpha, \delta) \), then

\[
\frac{1 + r^2(2A(A + \delta) - l) - 2rA}{1 - r^2} \leq \text{Re} f'(z) \leq \frac{1 + r^2(2A(A + \delta) - l) + 2rA}{1 - r^2}
\]

and

\[
\frac{-2Ar(1 + r\sqrt{1 - (A + \delta)^2})}{1 - r^2} \leq \text{Im} f'(z) \leq \frac{2Ar(1 + r\sqrt{1 - (A + \delta)^2})}{1 - r^2}.
\]

All bounds are sharp for any extreme point \( f \) of \( \mathcal{G}(\alpha, \delta) \).

**Proof.** By Theorem 3.2, we can write

\[
\left| f'(z) - \left(\frac{-e^{-ia}(e^{-ia} - 2\delta)}{1 - r^2} + \frac{2e^{-ia}A}{1 - r^2}\right) \right| \leq \frac{2Ar}{1 - r^2}
\]

so that

\[
\frac{-2Ar}{1 - r^2} \leq \text{Re} \left\{ f'(z) - e^{-ia}(e^{-ia} - 2\delta) - \frac{2e^{-ia}A}{1 - r^2} \right\} \leq \frac{2Ar}{1 - r^2}
\]

and also

\[
\frac{-2Ar}{1 - r^2} \leq \text{Im} \left\{ f'(z) - e^{-ia}(e^{-ia} - 2\delta) - \frac{2e^{-ia}A}{1 - r^2} \right\} \leq \frac{2Ar}{1 - r^2}.
\]

The results are obtained by simplifying the above inequalities.
We note that if \( f \in \mathcal{G}(\alpha, \delta) \), then since \( f_\alpha'(0) = 1 \), we have Re \( f'(z) > 0 \) for \( |z| < \rho \) and some \( \rho \) in \((0,1]\). However if

\[
f_\alpha(z) = \frac{1 + (e^{-2\alpha} - 2\delta e^{-\alpha})z}{1 - z}, \quad (z \in D),
\]

then the left side of inequality (5) is sharp so that

\[
(1-r^2) \text{Re} f_\alpha'(-r) = 1 + r^2(2A(A + \delta) - 1) - 2rA \to 2(\cos \alpha - \delta)(\cos \alpha - 1) \quad (r \to 1)
\]

and the last expression is negative if \( |\alpha| \neq 0 \). This shows that \( \rho \neq 1 \) in general, and it is natural to ask for the best possible value of \( \rho \). We answer this question in the following application of Theorem 3.2.

**Theorem 3.4.** Let \( f \in \mathcal{G}(\alpha, \delta) \) and put \( \rho = 1/(A + \sqrt{1-A(2\delta + A)}) \). Then

0 < \( \rho \leq 1 \) and \( \text{Re} f'(z) \geq 0 \) for \( r \mid z \mid < \rho \). If \( \rho \leq r \leq 1 \), then \( \text{Re} f_\alpha'(z) < 0 \) for some \( z \) on \( |z| < r \).

**Proof.** Let \( f \in \mathcal{G}(\alpha, \delta) \) and define \( \rho \) as above. Obviously \( \rho > 0 \) since \( A > 0 \), and

\[
1 - A(2\delta + A) = 1 + \delta^2 - \cos \alpha \geq 0.
\]

Then \( \rho \leq 1 \) is equivalent to \( A + \sqrt{1-A(2\delta + A)} \geq 1 \) and this is obviously true if \( A \geq 1 \). If \( A < 1 \), it is true if \( 1 - A(2\delta + A) \geq (1 - A)^2 \), and thus reduces to the trivially true inequality \( \cos \alpha \leq 1 \). So in both cases, \( \rho \leq 1 \).

Now, put \( \sigma(x) = (2A(A + \delta) - 1)x^2 - 2x + 1 \) for real values of \( x \). From (5), we have \( (1-r^2)\text{Re} f'(z) \geq \sigma(r) \quad (0 \leq \mid z \mid = r < 1) \) with equality for each \( r \) when \( f = f_\alpha \) and \( z \) is a suitable value on \( |z| = r \). To prove the theorem, it is sufficient to show that \( \sigma(x) \) is positive on \([0, \rho)\) and non-positive on \([\rho, 1]\).

If \( 2A(A + \delta) = 1 \), so that \( \sigma(x) \) is linear in \( x \), then \( \rho = 1/(2A) \) and it is clear that \( \sigma(x) \) is positive on \([0, \rho)\) and non-positive on \([\rho, 1]\). When \( 2A(A + \delta) \neq 1 \), \( \sigma(x) \) is quadratic and has zeros

\[
x = \frac{A \pm \sqrt{1-A(2\delta + A)}}{2A(A + \delta) - 1} = \frac{1}{A \mp \sqrt{1-A(2\delta + A)}}.
\]

One of the zeros is \( \rho \). Let the other zero be \( \mu \). If \( 2A(A + \delta) < 1 \), then \( \mu \rho < 0 \) and (7) shows that \( \mu < 0 \) and \( \rho > 0 \). Since \( \sigma \) is concave, \( \sigma(x) \) is positive on \([0, \rho)\) and
non-positive on $[\rho, 1]$. If $2A(A + \delta) > 1$, then $\mu, \rho > 0$ since $\mu \rho > 0$, $\mu + \rho > 0$. Also $\rho < \mu$ by (5). In this case $\sigma$ is convex so $\sigma(x)$ is positive on $[0, \rho)$ and non-positive on $[\rho, \mu]$. In particular, since $\sigma(1) = 2A(\cos \alpha - 1) \leq 0$, $\sigma(x)$ is non-positive on $[\rho, 1]$. This completes the proof.

We next obtain a distortion theorem for $G(\alpha, \delta)$.

**Theorem 3.5.** If $f \in G(\alpha, \delta)$, then

$$|f'(z)| \leq C(r) + \frac{2Ar}{1 - r^2}$$

where

$$C(r) = \sqrt{\frac{4Ar^2}{1 - r^2} \left( \frac{A}{1 - r^2} + \delta \right) + 1}$$

and the bound is sharp for any extreme point $f$ of $G(\alpha, \delta)$.

**Proof.** Let $\Gamma(r) = -e^{-ia}(e^{-ia} - 2\delta) + \frac{2e^{-ia}A}{1 - r^2}$. By using (6) we have

$$|f'(z)| \leq |\Gamma(r)| + \frac{2Ar}{1 - r^2}$$

$$= C(r) + \frac{2Ar}{1 - r^2}$$

as required.

4. **Argument of $f'(z)$**

We see that if $\delta \geq 0$, then $f'$ is non-zero throughout $D$, and has continuous argument. But if $\delta < 0$, and if $f_0$ is any extreme function of $G(\alpha, \delta)$, then at some point of $D$, $f_0'$ has a zero and hence no argument. So to obtain result for argument of $f'$, we restrict the values of $|z|$ considered in the case $\delta < 0$. We will also use the following property for argument: for a given $\alpha$ in $[-\pi, \pi]$ and as $x$ varies in some interval $[0, c]$, so that $e^{ia} + x \neq 0$, $\phi_\alpha(x)$ is the continuous argument of $e^{ia} + x$, for which $\phi_\alpha(0) = \alpha$. We have
\[ \phi_\alpha(x) = \begin{cases} 
\tan^{-1}\left(\frac{\sin \alpha}{\cos \alpha + x}\right), & \text{if } x + \cos \alpha > 0 \\
\pi + \tan^{-1}\left(\frac{\sin \alpha}{\cos \alpha + x}\right), & \text{if } x + \cos \alpha < 0 \\
\pi / 2, & \text{if } x + \cos \alpha = 0 
\end{cases} \]

when \( 0 < \alpha < \pi \), and similar formulae for the case \( -\pi < \alpha < 0 \), \( \alpha = 0, \pm \pi \).

**Theorem 4.1.** Let \( f \in \mathcal{G}(\alpha, \delta) \), and put \( x(r) = 2Ar^2/(1-r^2) \) (\( 0 \leq r < 1 \)). Let

\[ r_o = \begin{cases} 
1, & \delta \geq 0 \\
\frac{1}{\sqrt{1-4A\delta}}, & \delta < 0.
\end{cases} \]

Then, for \( 0 < |z| = r < r_o \), and for suitable determination of argument

\[ |\arg f'(z) + \alpha - \phi_\alpha(x(r))| \leq \sin^{-1}\frac{2Ar}{(1-r^2)C(r)} \quad (9) \]

where \( \phi_\alpha(x) \) is defined on \( [0, x(r_o)] \) as above and \( C(r) \) is given by (8).

**Proof.** We restrict the value of \(|z| = r\) by the condition

\[ \left| \frac{2A}{1-r^2} + 2\delta - e^{-i\alpha} \right| > \frac{2Ar}{1-r^2} \]

to ensure that \( f'(z) \neq 0 \). Squaring both sides and simplifying, we have

\[ \frac{4A\delta}{1-r^2} - 4A\delta + 1 > 0. \]

The inequality holds for all \( r \) in \([0, 1)\) if \( \delta \geq 0 \) and for \( 0 \leq r < \frac{1}{\sqrt{1-4\delta}} \) if \( \delta < 0 \). This establishes the restriction on \(|z|\). By using (6) and Theorem 3.5, we deduce that

\[ |\arg f'(z) - \arg \Gamma(r)| \leq \sin^{-1}\frac{2Ar}{(1-r^2)C(r)} \quad (10) \]
and also
\[
\arg \Gamma(r) = \arg \left[ -e^{-i\alpha} (e^{-i\alpha} - 2\delta) + \frac{2e^{-i\alpha} A}{1 - r^2} \right] \\
= -\alpha + \arg \left[ e^{i\alpha} + \frac{2Ar^2}{1 - r^2} \right].
\]

Put \( x(r) = \frac{2Ar^2}{1 - r^2} \), then \( \arg \Gamma(r) = -\alpha + \phi_{\alpha}(x(r)) \) and the desired result follows using (10).

We obtain another result for argument of \( G(\alpha, \delta) \), features \( \arg(f'(z) + k) \) for some real \( k \) that satisfy \( f'(z) + k \neq 0 \) for \( z \in D \) and for all \( f \in G(\alpha, \delta) \). When \( |\alpha| = \pi / 2 \), such a choice is impossible, for if \( f_o \) is an extreme function in \( G(\alpha, \delta) \), then \( f_o'(z) + k \) maps \( D \) onto either \( \text{Im} w > \delta \) or \( \text{Im} w < -\delta \) and since \( \delta < 0 \) both these half planes contain \( 0 \). If \( |\alpha| \neq \pi / 2 \), any choice of \( k \) with \( k \cos \alpha + \delta > 0 \) ensures that \( f_o'(z) + k \neq 0 \) for \( z \in D \), \( f \in G(\alpha, \delta) \).

In the statement of the following theorem, for a given \( \alpha \in [-\pi, \pi] \), and as \( x \) varies in some interval \( [0, c] \), so that \( (k+1)e^{i\alpha} + x \neq 0 \), \( \psi_{\alpha}(\alpha) \) is the continuous argument of \( (k+1)e^{i\alpha} + x \) for which \( \psi_{\alpha}(0) \) is principal.

**Theorem 4.2.** Let \( f \in G(\alpha, \delta) \), where \( |\alpha| \neq \pi / 2 \). Put \( x(r) = 2A/(1 - r^2) \) \((0 \leq r < 1)\) and let \( k \) be a real number such that \( k \cos \alpha + \delta > 0 \). Then

\[
\left| \arg(f'(z) + k) + \alpha - \psi_{\alpha}(x(r)) \right| \leq \sin^{-1} \frac{2Ar}{(1 - r^2)C_1(r)}
\]

where \( \psi_{\alpha}(\alpha) \) is defined on \([0, \infty)\) as above, and

\[
C_1(r) = \sqrt{\frac{4Ar^2}{1 - r^2} \left( \frac{A}{1 - r^2} + k \cos \alpha + \delta \right) + (k+1)^2}. \tag{11}
\]

**Proof.** Let \( |\alpha| \neq \pi / 2 \), and let \( k \) satisfy \( k \cos \alpha + \delta > 0 \). We have, using (6),

\[
\left| f'(z) + k - (\Gamma(r) + k) \right| \leq \frac{2Ar}{1 - r^2}
\]

where

\[ \Gamma(r) = -e^{-i\alpha} (e^{-i\alpha} - 2\delta) + \frac{2Ae^{-i\alpha}}{1-r^2} = 1 + \frac{2Ar^2}{1-r^2} e^{-i\alpha}. \]

Hence

\[ |\text{arg}(f'(z) + k) - \text{arg}(\Gamma(r) + k)| \leq \sin^{-1} \frac{2Ar}{(1-r^2)C_1(r)} \]  \hspace{1cm} (12)

where \( C_1(r) = |\Gamma(r) + k | \) and is written as in (11). Now

\[ \text{arg}(\Gamma(r) + k) = -\alpha + \text{arg} \left[ 2\delta e^{-i\alpha} + \frac{2A}{1-r^2} + ke^{i\alpha} \right] = -\alpha + \psi_\alpha(x(r)) \]

and the proof is complete by using (12).

**References**