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Range Symmetric Matrices in Minkowski Space

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Abstract. The concept of range symmetric matrix is introduced in Minkowski space $\mathfrak{m}$. Equivalent conditions for a matrix to be range symmetric are determined. The existence of the Minkowski inverse of a range symmetric matrix in $\mathfrak{m}$ is discussed.

1. Introduction

We shall index the components of a complex vector in C^n from 0 to $n-1$, that is, $u = (u_0, u_1, u_2, \cdots, u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu = (u_0, -u_1, -u_2, \cdots, -u_{n-1})$. Clearly the Minkowski metric matrix

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}$$

and $G^2 = I_n$. In [4], Minkowski inner product on C^n is defined by $(u, v) = < u, Gv >$, where $< \cdot, \cdot >$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space. With respect to the Minkowski inner product, the adjoint of a matrix $A \in C^{n \times n}$ is given by $A^+ = GA^*G$, where A^* is the usual Hermitian adjoint. Naturally, we call a matrix $\mathfrak{m}$-symmetric in Minkoswki space if $A^+ = A$ and $\mathfrak{m}$-orthogonal if $A^+ A = I$. As in unitary space $\mathfrak{m}$-orthogonal matrices form a group. For $A \in C^{m \times n}$, let us define the following:

Definition 1. *A^g is said to be a generalized inverse (g inverse) of A, if*

$$AA^g A = A \tag{1.1}$$

Definition 2. A^r *is said to be a reflexive g inverse of A if*

$$AA^r A = A \text{ and } A^r AA^r = A^r \qquad (1.2)$$

Definition 3. A^n *is a right (left) normalized g inverse of A if*

$$AA^n A = A; \ A^n AA^n = A^n \text{ and } AA^n \text{ is } \mathfrak{m}\text{-symmetric } (A^n A \text{ is } \mathfrak{m}\text{-symmetric}). \qquad (1.3)$$

Definition 4. A^M *is the Minkowski inverse of A, if*

$$AA^M A = A; \ A^M AA^M = A^M; \ AA^M \text{ and } A^M A \text{ are } \mathfrak{m}\text{-symmetric}. \qquad (1.4)$$

It is well known [5] that, for $A \in C^{m\times n}$, solutions exist for equations (1.1) and (1.2). In unitary space, for $A \in C^{m\times n}$, since $rk\ (A) = rk\ (AA^*) = rk\ (A^* A)$ ($rk\ (A)$ denotes the rank of *A*), solutions exist for equation (1.3) and unique solution exists for equation (1.4) which is called the Moore penrose inverse of *A*[3]. However this fails in Minkowski space $\mathfrak{m}$, since $rk(A) \neq rk(A^+A) \neq rk(AA^+)$. For $A \in C^{m\times n}$, let A^*, A^+, $R(A)$ and $N(A)$ denote the hermitian adjoint, Minkowski adjoint, range space and null space of *A* respectively. In [3] we have determined necessary and sufficient conditions for the existence of Minkowski inverse and normalized generalized inverse of a complex matrix. A matrix $A \in C^{n\times n}$ is said to be range symmetric in unitary space (or) equivalently *A* is said to be *EP* if $N(A) = N(A^*)$ [p.163, 1]. In this paper we define the concept of a range symmetric matrix in $\mathfrak{m}$. The existence of the Minkowski inverse of a range symmetric matrix in $\mathfrak{m}$ is also discussed. In the sequel, we shall make use of the following result obtained in [3].

Lemma 1.1. *For* $A \in C^{n\times n}$, *the Minkowski inverse* A^M *exists* $\Leftrightarrow rk\ A = rk\ AA^+ = rk\ A^+A$. *If* A^M *exists, then it is unique.*

2. Range symmetric matrices in $\mathfrak{m}$

In this section, we shall define a range symmetric matrix in $\mathfrak{m}$, analogous to that of an *EP* matrix (or) equivalently a range symmetric matrix in the unitary space. We present equivalent characterizations of a range symmetric matrix in $\mathfrak{m}$.

Definition 2.1. $A \in C^{n\times n}$ *is range symmetric in* $\mathfrak{m}$ *if and only if* $N(A) = N(A^+)$.

Theorem 2.2. *For* $A \in C^{n\times n}$, *the following are equivalent:*

(1) *A is range symmetric in* $\mathfrak{m}$

(2) *GA is EP*

(3) *AG is EP*

(4) $N(A^*) = N(AG)$

(5) $R(A) = R(A^+)$

(6) $A^+ = HA = AK$ *for some nonsingular matrices H and K*

(7) $R(A^*) = R(GA)$

(8) $C_n = R(A^*) \oplus N(A)$

(9) $C_n = R(A) \oplus N(A^*)$

Proof. The proof for the equivalence of (1), (2) and (3) runs as follows:

A is range symmetric in $\mathfrak{m}$ $\Leftrightarrow N(A) = N(A^+)$

$\Leftrightarrow N(GA) = N(A^*G)$

$\Leftrightarrow N(GA) = N((GA)^*)$

$\Leftrightarrow GA$ is EP

$\Leftrightarrow G(GA)G^*$ is EP

$\Leftrightarrow AG$ is EP

Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.

(1) $\Leftrightarrow$ (4) :

A is range symmetric in $\mathfrak{m}$ $\Leftrightarrow N(A) = N(A^+)$

$\Leftrightarrow N(A) = N(GA^*G)$

$\Leftrightarrow N(A) = N(A^*G)$

$\Leftrightarrow A^*G = A^*GA^g A$ (by[1, p.21])

$\Leftrightarrow A^* = A^*GA^g AG$

$\Leftrightarrow A^* = A^*(AG)^g(AG)$

$\Leftrightarrow N(A^*) = N(AG)$

Thus the equivalence of (1) and (4) is proved.

(3) $\Leftrightarrow$ (5):

AG is EP $\Leftrightarrow R(AG) = R(AG)^*)$

$\Leftrightarrow R(A) = R(GA^*)$

$\Leftrightarrow R(A) = R(GA^*G)$

$\Leftrightarrow R(A) = R(A^+)$

Thus the equivalence of (3) and (5) is proved.

(2) $\Leftrightarrow$ (6):

GA is $EP \Leftrightarrow (GA)^* = (GA)K$ for a nonsingular $n \times n$ matrix K (by[1, p.166])

$\Leftrightarrow A^*G = GAK$

$\Leftrightarrow GA^*G = AK$

$\Leftrightarrow A^+ = AK$

$\Leftrightarrow A = (AK)^+ = K^+A^+$ (by using $(A^+)^+ = A$)

$\Leftrightarrow A^+ = (K^+)^{-1}A$

$\Leftrightarrow A^+ = HA$ where $H = (K^+)^{-1}$.

Thus the equivalence of (2) and (6) is proved.

(5) $\Leftrightarrow$ (7)

$R(A) = R(A^+)$ $\Leftrightarrow R(A) = R(GA^*G)$

$\Leftrightarrow R(A) = R(GA^*)$

$\Leftrightarrow GA^* = AA^gGA^*$

$\Leftrightarrow A^* = GAA^gGA^*$

$\Leftrightarrow A^* = (GA)(GA)^g A^*$

$\Leftrightarrow R(A^*) = R(GA)$

Thus the equivalence of (5) and (7) is proved.

(2) $\Leftrightarrow$ (8):

GA is $EP \Leftrightarrow C_n = R(GA) \oplus N(GA)$

$= R((GA)^*) \oplus N(A)$

$= R(A^*G) \oplus N(A)$

$= R(A^*) \oplus N(A)$

Thus the equivalence of (2) and (8) is proved.

(3) $\Leftrightarrow$ (9):

$$AG \text{ is } EP \Leftrightarrow C_n = R(AG) \oplus N(AG)$$
$$= R(AG) \oplus N((AG)^*)$$
$$= R(A) \oplus N(A)^*$$

Thus the equivalence of (3) and (9) is proved. Hence the Theorem.

It is well known that in unitary space a complex normal matrix is *EP*. However an 𝔪-normal matrix (*A* is 𝔪-normal if $AA^+ = A^+A$) need not be range symmetric in 𝔪.

Example 2.3.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is } EP; \quad A^+ = GA^*G = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } N(A) \neq N(A^+).$$

Hence *A* is not range symmetric in 𝔪. However *A* is 𝔪-normal.

Theorem 2.4. *If* $A \in C^{n\times n}$ *is 𝔪-normal and* $rk\ A = rk\ AA^+$, *then A is range symmetric in* 𝔪.

Proof. Since *A* is 𝔪-normal, $AA^+ = A^+A$. $rk\ A = rk\ AA^+ = rk\ A^+A = rk\ A^+$. Hence $N(A) = N(A^+A) = N(AA^+) = N(A^+)$. Thus *A* is range symmetric in 𝔪.

The relation between *EP* matrices and range symmetric matrices in 𝔪 are discussed in the following:

Theorem 2.5. *For* $A \in C^{n\times n}$ *any two of the following conditions imply the other one:*

(1) *A is EP*
(2) *A is range symmetric in* 𝔪
(3) $R(A) = R(GA)$

Proof. First we prove that whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds, then by definition $R(A) = R(A^*)$. Now by Theorem 2.2, *A* is range symmetric in 𝔪 $\Leftrightarrow R(A^*) = R(GA)$. Thus *A* is range symmetric in 𝔪 $\Leftrightarrow R(A) = R(GA)$. This completes the proof of [(1) and (2)] $\Rightarrow$ (3) and [(1) and (3)] $\Rightarrow$ (2). Now let us prove [(2) and (3)] $\Rightarrow$ (1). Since *A* is range symmetric in 𝔪, by Theorem 2.2, *GA* is *EP*. Hence $R(GA) = R((GA)^*) = R(A^*G) = R(A^*)$. By (3), $R(A) = R(GA)$ from which it follows that $R(A) = R(A^*)$. Hence *A* is *EP*. Thus (1) holds.

We note that, for $A \in C^{n \times n}$, there exist unique matrices P and Q which are $\mathfrak{m}$-symmetric such that $A = P + iQ$ where $P = \frac{1}{2}(A + A^{+})$ and $Q = \frac{1}{2i}(A - A^{+})$. In the following Theorem, an equivalent condition for a matrix A to be range symmetric in $\mathfrak{m}$ is obtained in terms of P, the symmetric part of A in $\mathfrak{m}$.

Theorem 2.6. *For $A \in C^{n \times n}$, A is range symmetric in $\mathfrak{m}$ $\Leftrightarrow N(A) \subseteq N(P)$ where P is the symmetric part of A in $\mathfrak{m}$.*

Proof. If A is range symmetric in $\mathfrak{m}$ then $N(A) = N(A^{+})$. For $x \in N(A)$, $Ax = 0$ and $A^{+}x = 0$. Hence $Px = 0$. Thus $N(A) \subseteq N(P)$. Conversely, if $N(A) \subseteq N(P)$, then $Ax = 0 \Rightarrow Px = 0$ and hence $Qx = 0$. Therefore $N(A) \subseteq N(Q)$. Thus $N(A) \subseteq N(P) \cap N(Q)$. Since both P and Q are $\mathfrak{m}$-symmetric, they are range symmetric in $\mathfrak{m}$.

$$N(P) = N(P^{+}) = N(GP^{*}G) = N(P^{*}G)$$

and $$N(Q) = N(Q^{+}) = N(GQ^{*}G) = N(Q^{*}G).$$

Now, $$N(A) \subseteq N(P) \cap N(Q) = N(P^{*}G) \cap N(Q^{*}G) \subseteq N[(P^{*} - iQ^{*})G] = N(A^{*}G)$$

and $rkA = rkA^{*} = rkA^{*}G$. Therefore $N(A) = N(A^{*}G) = N(GA^{*}G) = N(A^{+})$. Thus A is range symmetric in $\mathfrak{m}$.

We shall discuss the existence of the Minkowski inverse of a range symmetric matrix in $\mathfrak{m}$. First, we shall prove certain lemmas, to simplify the proof of the main result.

Lemma 2.7. *For $A \in C^{m \times n}$, if A^{M} exists, then $R(A^{M}) = R(A^{+})$.*

Proof. If A^{M} exists, then $A^{M}A = (A^{M}A)^{+} = A^{+}(A^{M}A)^{+}(A^{M})^{+}$ and $A^{M} = A^{M}AA^{M} = A^{+}(A^{M})^{+}A^{M} \Rightarrow R(A^{M}) \subseteq R(A^{+})$ further $rk(A^{M}) = rk\,A = rk\,A^{+}$. Thus $R(A^{M}) = R(A^{+})$.

Lemma 2.8. *For $A \in C^{m \times n}$, if A^{M} exists, then in Minkowski space, AA^{M} is the projection on R(A) and $A^{M}A$ is the projection on $R(A^{M})$.*

Proof. $x \in R(A) \Leftrightarrow x = Ay = AA^{M}Ay = AA^{M}x.$ By Definition 4, AA^{M} being $\mathfrak{m}$-symmetric idempotent is the projection on $R(A)$. Similarly, $x \in R(A^{M}) \Leftrightarrow x = A^{M}AA^{M}y = A^{M}Ax.$ and $A^{M}A$ is $\mathfrak{m}$-symmetric idempotent. Hence, $A^{M}A$ is the projection on $R(A^{M})$.

Theorem 2.9. *For* $A \in C^{n\times n}$ *, the following are equivalent:*

(1) *A is range symmetric in* $\mathfrak{m}$ *and* $rk(A) = rk(A^2)$

(2) A^M *exists and* A^M *is range symmetric in* $\mathfrak{m}$.

(3) *There exists a symmetric idempotent E such that* $AE = EA$ *and* $R(A) = R(E)$.

Proof. (1) $\Rightarrow$ (2). Since, $rk(A) = rk(A^2)$ and A is range symmetric in $\mathfrak{m}$, by using Theorem 2.2, we have, $rk(A^+A) = rk(HA^2) = rk(A^2) = rk(A^2) = rkA$ and $rk(AA^+) = rk(A^2K) = rk(A^2) = rk\ A$. Thus $rk\ A = rk\ AA^+ = rk\ A^+A$ and by Lemma 1.1, it follows that A^M exists. By Lemma 2.7 and Theorem 2.2, $R(A^M) = R(A^+) = R(A) = R((A^+)^+) = R((A^M)^+)$. Hence A^M is range symmetric. Thus (2) holds.

(2) $\Rightarrow$ (3). Since A^M exists, by Lemma 2.7, $R(A^M) = R(A^+)$ by Theorem 2.2, A^M is range symmetric in $\mathfrak{m}$ implies that $R(A^M) = R((A^M)^+)$. Hence, $R(A^+) = R((A^M)^+)$. By Lemma 2.8, it follows that $A^+(A^+)^M = (A^M)^+A^+$, hence $(A^MA)^+ = (AA^M)^+$. By Definition 4, $A^MA = AA^M = E$ is $\mathfrak{m}$-symmetric idempotent and $AE = EA = A$; hence $R(A) \subseteq R(E)$ and $rk\ (E) = rk\ (AA^M) = rk\ A$, which implies $R(A) = R(E)$. Thus (3) holds.

(3) $\Rightarrow$ (1). Since, E is $\mathfrak{m}$-symmetric idempotent $E^+ = E = E^2$, by Lemma 1.1, E^M exists and $E^M = E$. $R(A) = R(E)$ implies E is the projection on $R(A)$. For all reflexive g-inverses A^r of A, $AA^r = EE^M = E$ (p.52, [1]). Since E is $\mathfrak{m}$-symmetric idempotent, AA^r is $\mathfrak{m}$-symmetric and by Definition 3, A^n exists and $AA^n = EE^M = E$ which implies $EA = A$. By hypothesis $AE = EA = A$. Therefore $AA^n = A^nA = E$. Thus both AA^n and A^nA are $\mathfrak{m}$-symmetric, by Definition 4, A^M exists and $E = AA^M = A^MA$. By taking Minkowski adjoint on $AE = EA = A$ we get $EA^+ = A^+E = A^+$. $R(A^+) \subseteq R(E) = R(A)$ and $rk\ A^+ = rk\ A$. Therefore $R(A) = R(A^+)$. By Theorem 2.2, A is range symmetric in $\mathfrak{m}$. $rk(AA^+) \geq rk\ A(A^+(A^M)^+ = rk\ A(A^MA)^+\ rk\ AE = rk\ A \geq rk\ AA^+$. Thus $rk(A) = rk(AA^+) = rk(A^2K) = rk(A^2)$. Thus (1) holds. Hence the Theorem.

Remark 2.10. An analogue of Theorem 2.9 for matrices over an arbitrary field has been proved in our earlier work [2].

Corrollary 2.11. *Let* $A \in C^{n\times n}$ *be range symmetric in* $\mathfrak{m}$. *Then* A^M *exists* $\Leftrightarrow rk\ A = rk\ A^2$.

Proof. Since A is range symmetric in $\mathfrak{m}$ and $rk\ A = rk\ A^2$, the existence of A^M follows from Theorem 2.9. Conversely, if A is range symmetric in $\mathfrak{m}$ and A^M exists, then by Lemma 1.1, $rk\ A = rk\ AA^+ = rk\ A^+A$ and by Theorem 2.2, $A^+ = AH$. Hence $rk\ A = rk\ AAH = rk\ A^2$.

Remark 2.12. The condition on A to be range symmetric in $\mathfrak{m}$ is essential in the above Corollary 2.11 can be seen by the matrix A in Example 2.3. A is not range symmetric in $\mathfrak{m}$ and $rk\ A = rk\ A^2$. Since $AA^+ = A^+A$ is the null matrix, by Lemma 1.1, A^M does not exists. Hence the Corollary fails.

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