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On Sums of *k-EP* Matrices

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Abstract. Necessary and Sufficient conditions for the sums of *k-EP* matrices to be *k-EP* are discussed. As an application it is shown that sum and parallel sum of parallel summable *k-EP* matrices are *k-EP*.

1. Introduction

Throughout we shall deal with $C_{n\times n}$, the space of $n \times n$ complex matrices. Let C_n be the space of complex *n*-tuples. For $A\, \varepsilon\, C_{n\times n}$, let A^T, A^* denote the transpose, conjugate transpose of *A*, let A^- be a generalized inverse $\left(AA^- A = A\right)$ and $A^{\dagger}$ be the Moore-Penrose inverse of *A*[5]. A matrix *A* is called EP_r if $\rho(A) = r$ and $N(A) = N(A^*)$ or $R(A) = R(A^*)$ where $\rho(A)$ denotes the rank of *A*; $N(A)$ and $R(A)$ denote the null space and range space of *A* respectively. Throughout let '*k*' be a fixed product of disjoint transpositions in $S_n = \{1, 2, \cdots, n\}$ and *K* be the associated permutation matrix. A matrix $A = (a_{ij})\, \varepsilon\, C_{n\times n}$ is *k*-hermitian if $a_{ij} = \overline{a}_{k(j),k(i)}$ for $i, j = 1, \cdots, n$. A theory for *k*-hermitian matrices is developed in [1]. For $x = (x_1, x_2, \cdots, x_n)^T\, \varepsilon\, C_n$, let us define the function $\mathsf{k}(x) = (x_{k(1)}, x_{k(2)}, \cdots, x_{k(n)})^T\, \varepsilon\, C_n$. A matrix $A\, \varepsilon\, C_{n\times n}$, is said to be *k-EP* if it satisfies the condition $Ax = 0 \Leftrightarrow A^* \mathsf{k}(x) = 0$ or equivalently $N(A) = N(A^* K)$. In addition to that, *A* is $k\text{-}EP \Leftrightarrow KA$ is *EP* or *AK* is *EP* and *A* is $k\text{-}EP \Leftrightarrow A^*$ is *k-EP*. Moreover, *A* is said to be $k\text{-}EP_r$ if *A* is *k-EP* and of rank *r*. For further properties of *k-EP* matrix one may refer [4]. In this paper we give necessary and sufficient conditions for sums of *k-EP* matrix to be *k-EP*. As an application it is shown that sum and parallel summable *k-EP* matrices are *k-EP*.

2. Sums of *k-EP* matrices

Lemma 2.1. Let $A_1, A_2, \cdots, A_m\, \varepsilon\, C_{n\times n}$ and let $A = \sum_{i=1}^{m} A_i$. Consider the following conditions:

(a) $N(A) \subseteq N(A_i)$ for $i = 1, \cdots, m$;

(b) $N(A) = \bigcap_{i=1}^{m} N(A_i)$;

(c) $\rho(A) = \rho\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$;

(d) $\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^{*} A_j = 0$;

(e) $\rho(A) = \sum_{i=1}^{m} \rho(A_i)$.

Then the following statements hold:

(i) Conditions (a), (b) and (c) are equivalent.
(ii) Condition (d) implies (a), but condition (a) does not implies (d).
(iii) Condition (e) implies (a), but condition (a) does not implies (e).

Proof.

(i) ***(a)* ⇔ *(b)* ⇔ *(c)*:** $N(A) \subseteq N(A_i)$ for each $i \Rightarrow N(A) \subseteq \cap N(A_i)$.

Since $N(A) = N(\sum A_i) \supseteq N(A_1) \cap N(A_2) \cdots \cap N(A_m)$, it follows that $N(A) \supseteq \cap N(A_I)$.

Always $\bigcap_{i=1}^{m} N(A_i) \subseteq N(A)$. Hence $N(A) = \bigcap_{i=1}^{m} N(A_i)$. Thus (b) holds.

Now,

$$N(A) = \bigcap_{i=1}^{m} N(A_i) = N\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right)$$

Therefore,

$$\rho(A) = \rho\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right) \text{ and (c) holds.}$$

Conversely, Since $\rho\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right) = \rho(A)$ and

$$N\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right) = \bigcap_{i=1}^{m} N(A_i) \subseteq N(A) \Rightarrow N(A) = \bigcap_{i=1}^{m} N(A_i)$$

and (b) holds.

Hence, $N(A) \subseteq N(A_i)$ for each i and (a) holds.

(ii) **(d) ⇒ (a):**

Since $\sum_{i \neq j} A_i^* A_j = 0$,

$$\begin{aligned} A^* A &= (\Sigma A_i)^* \quad (\Sigma A_i) \\ &= (\Sigma A_i^*) \quad (\Sigma A_i) \\ &= \Sigma A_i^* A_i \end{aligned}$$

$$\begin{aligned} N(A) = N(A^* A) &= N(\Sigma A_i^* A_i) \\ &= N\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}^* \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\right) \\ &= N(A_1) \cap N(A_2) \cdots \cap N(A_m) \\ &= \bigcap_{i=1}^{m} N(A_i). \end{aligned}$$

Hence $N(A) \subseteq N(A_i)$ for each i and (a) holds.

$(a) \not\Rightarrow (d)$: Let us consider the following example.

Let $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

and $A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. $A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Clearly, $N(A_1 + A_2) \subseteq N(A_1)$. Also $N(A_1 + A_2) \subseteq N(A_2)$. But $A_1^* A_2 + A_2^* A_1 \neq 0$.

(iii) **(e) $\Rightarrow$ (a) :**

If rank is additive, that is $\rho(A) = \sum \rho(A_i)$, then by [3], $R(A_i) \cap R(A_j) = \{0\}$, $i \neq j \Rightarrow N(A) \subseteq N(A_i)$ for each i and (a) holds.

$(a) \not\Rightarrow (e)$: Consider the example,

Let $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$$A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}.$$

Here, $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$.
But $\rho(A_1 + A_2) \neq \rho(A_1) + \rho(A_2)$.

Theorem 2.2. *Let $A_1, A_2, \cdots, A_m \varepsilon C_{n\times n}$ be k-EP matrices. If any one of the conditions (a) to (e) of Lemma 2.1 holds, then*

$$A = \sum_{i=1}^{m} A_i \quad \text{is } k\text{-}EP.$$

Proof. Since each A_i is *k-EP*, $N(A_i) = N(A_i^* K)$ for each i.

Now , $N(A) \subseteq N(A_i)$ for each i

$$\Rightarrow N(A) \subseteq \bigcap_{i=1}^{m} N(A_i) = \bigcap_{i=1}^{m} N(A_i^* K)$$

$$\subseteq N(A^* K)$$

and $\rho(A) = \rho(A^* K)$. Hence $N(A) = N(A^* K)$. Thus A is *k-EP*. Hence the Theorem.

Remark 2.3. In particular, if A is non-singular the conditions automatically hold and A is *k-EP*. Theorem 2.2 fails if we relax the conditions on the A_i's.

Example 2.4. Consider $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let $k = (1, 2)$, then the associated permutation matrix

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad KA_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ is } EP.$$

Therefore, A_1 is *k-EP*.

$$KA_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ is not } EP. \text{ Therefore } A_2 \text{ is not } k\text{-}EP.$$

$$A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } K(A_1 + A_2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

which is not *EP*. Therefore $(A_1 + A_2)$ is not *k-EP*. However,

$$N(A_1 + A_2) \subseteq N(A_1{}^* K) \subseteq N(A_1) \text{ and } N(A_1 + A_2) \subseteq N(A_2{}^* K) \subseteq N(A_2).$$

Moreover, $$\rho\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right) = \rho(A_1 + A_2).$$

Remark 2.5. Theorem 2.2 fails if we relax the condition that A_i's are *k-EP*. For, let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and let the associated permutation matrix be $K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$$KA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is not } EP.$$

Therefore A_1 is not *k-EP*.

$$KA_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ is not } EP.$$

Therefore A_2 is not *k-EP*.

$$A_1 + A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}. \quad K(A_1 + A_2) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

is not *EP*. Therefore, $(A_1 + A_2)$ is not *k-EP*. But $A_1^* A_2 + A_2^* A_1 = 0$.

Remark 2.6. The conditions given in Theorem 2.2 are only sufficient for the sum of *k-EP* matrices to be *k-EP*, but not necessary is illustrated in the following example.

Example 2.7. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. For $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, A_1 and A_2 are $k\text{-}EP_2$. The conditions in Theorem 2.2 does not hold. However $(A_1 + A_2)$ is *k-EP*.

Remark 2.8. If A_1 and A_2 are *k-EP* matrices, then by Theorem 2.4(p.221,[4]), $A_1^* = H_1 K A_1 K$ and $A_2^* = H_2 K A_2 K$ where H_1 and H_2 are non-singular $n \times n$ matrices.

If $H_1 = H_2$, then $A_1^* + A_2^* = H_1 K (A_1 + A_2) K$

$\Rightarrow (A_1 + A_2)^* = H_1 K (A_1 + A_2) K \Rightarrow (A_1 + A_2)$ is *k-EP*.

If $(H_1 - H_2)$ is non-singular, then the above conditions are also necessary for the sum of *k-EP* matrices to be *k-EP* is given in the following Theorem.

Theorem 2.9. *Let K be the permutation matrix associated with the fixed transposition 'k'. Let* $A_1^* = H_1 K A_1 K$ *and* $A_2^* = H_2 K A_2 K$ *such that* $(H_1 - H_2)$ *is non-singular. Then* $(A_1 + A_2)$ *is k-EP if and only if* $N(A_1 + A_2) \subseteq N(A_i)$ *for some (and hence both)* $i \varepsilon \{1, 2\}$.

Proof. Since $A_1^* = H_1KA_1K$ and $A_2^* = H_2KA_2K$, by Remark 2.8, A_1 and A_2 are *k-EP* matrices. Since, $N(A_1 + A_2) \subseteq N(A_2)$ by Theorem 2.2, $(A_1 + A_2)$ is *k-E*P. Conversely, let us assume that $(A_1 + A_2)$ is *k-EP*. By Remark 2.8, there exists a non-singular matrix G such that

$$
\begin{aligned}
& (A_1 + A_2)^* &=& \; GK(A_1 + A_2)K \\
\Rightarrow \quad & A_1^* + A_2^* &=& \; GK(A_1 + A_2)K \\
\Rightarrow \quad & H_1KA_1K + H_2KA_2K &=& \; GK(A_1 + A_2)K \\
\Rightarrow \quad & (H_1KA_1 + H_2KA_2)K &=& \; GK(A_1 + A_2)K \\
\Rightarrow \quad & H_1KA_1 + H_2KA_2 &=& \; GKA_1 + GKA_2 \\
\Rightarrow \quad & (H_1K - GK)A_1 &=& \; (GK - H_2K)A_2 \\
\Rightarrow \quad & (H_1 - G)KA_1 &=& \; (G - H_2)KA_2 \\
\Rightarrow \quad & LKA_1 &=& \; MKA_2 \quad \text{where} \\
& L &=& \; H_1 - G \quad \text{and} \\
& M &=& \; G - H_2
\end{aligned}
$$

$$
\begin{aligned}
\text{Now} \quad (L + M)(KA_1) &= LKA_1 + MKA_1 \\
&= MKA_2 + MKA_1 \\
&= MK(A_1 + A_2) \\
\text{and} \quad (L + M)(KA_2) &= LK(A_1 + A_2)
\end{aligned}
$$

By hypothesis, $L + M = H_1 - G + G - H_2 = H_1 - H_2$ is non-singular. Therefore,

$$
\begin{aligned}
N(A_1 + A_2) &\subseteq N(MK(A_1 + A_2)) \\
&= N\big((L + M)KA_1\big) \\
&= N(KA_1) \\
&= N(A_1).
\end{aligned}
$$

Therefore, $N(A_1 + A_2) \subseteq N(A_1)$. Similarly $N(A_1 + A_2) \subseteq N(A_2)$. Hence the Theorem.

Remark 2.10. The condition $(H_1 - H_2)$ to be non-singular is essential in Theorem 2.9 is illustrated in the following example.

Example 2.11. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ are both *k-EP* matrices for $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Further $A_1^* = A_1 = KA_1K$ and $A_2^* = A_2 = KA_2K \Rightarrow H_1 = H_2 = I$.

$(A_1 + A_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is also *k-EP*. But $N(A_1 + A_2) \not\subseteq N(A_1)$ (or) $N(A_1 + A_2) \not\subseteq N(A_2)$. Thus Theorem 2.9 fails.

3. Parallel summable *k-EP* matrices

In this section we shall show that sum and parallel sum of parallel summable *k-EP* matrices are *k-EP*. First we shall give the definition and some properties of parallel summable matrices as in (p.188, [5]).

Definition 3.1. *A_1 and A_2 are said to be parallel summable (p.s.) if $N(A_1 + A_2) \subseteq N(A_2)$ and $N(A_1 + A_2)^* \subseteq N(A_2{}^*)$ (or) equivalently $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2)^* \subseteq N(A_1{}^*)$.*

Definition 3.2. *If A_1 and A_2 are parallel summable then parallel sum of A_1 and A_2 denoted by $A_1 \pm A_2$ is defined as $A_1 \pm A_2 = A_1(A_1 + A_2)^- A_2$. The product $A_1(A_1 + A_2)^- A_2$ is invariant for all choices of generalized inverse $(A_1 + A_2)^-$ of $(A_1 + A_2)$ under the conditions that A_1 and A_2 are parallel summable (p.188, [5]).*

Properties 3.3. *Let A_1 and A_2 be a pair of parallel summable (p.s.) matrices. Then the following hold:*

P.1 $A_1 \pm A_2 = A_2 \pm A_1$

P.2 $A_1{}^*$ *and* $A_2{}^*$ *are p.s. and* $(A_1 \pm A_2)^* = A_1{}^* \pm A_2{}^*$

P.3 *If U is non-singular then* UA_1 *and* UA_2 *are p.s. and* $(UA_1 \pm UA_2) = U(A_1 \pm A_2)$

P.4 $R(A_1 \pm A_2) = R(A_1) \cap R(A_2)$

$N(A_1 \pm A_2) = N(A_1) + N(A_2)$

P.5 $(A_1 \pm A_2) \pm A_3 = A_1 \pm (A_2 \pm A_3)$

if all the parallel sum operations involved are defined.

Lemma 3.4. *Let A_1 and A_2 be k-EP matrices. Then A_1 and A_2 are p.s if and only if $N(A_1 + A_2) \subseteq N(A_i)$ for some (and hence both) $i \,\varepsilon \{1, 2\}$.*

Proof. A_1 and A_2 are p.s. $\Rightarrow N(A_1 + A_2) \subseteq N(A_1)$ follows from the Definition 3.1. Conversely, if $N(A_1 + A_2) \subseteq N(A_1)$, then $N(KA_1 + KA_2) \subseteq N(KA_1)$. Also $N(KA_1 + KA_2) \subseteq N(A_2)$. Since A_1 and A_2 are *k-EP* matrices, KA_1 and KA_2 are

EP matrices. $N(KA_1+KA_2)\subseteq N(KA_1)$ and $N(KA_1+KA_2)\subseteq N(KA_2)$, therefore (KA_1+KA_2) is *EP*.

Hence

$N(KA_1+KA_2)^* = N(KA_1+KA_2) = N(KA_1)\cap N(KA_2) = N(KA_1)^*\cap N(KA_2)^*$.

Therefore, $N(KA_1+KA_2)^*\subseteq N(KA_1)^*$, $N(KA_1+KA_2)^*\subseteq N(KA_2)^*$.

Also, $N(KA_1+KA_2)\subseteq N(KA)$ by hypothesis. Hence, by Definition 3.1, KA_1 and KA_2 are p.s. $N(KA_1+KA_2)\subseteq N(KA_1)\Rightarrow N(K(A_1+A_2))\subseteq N(KA_1)\Rightarrow N(A_1+A_2)\subseteq N(A_1)$. Similarly, $N(A_1+A_2)^*\subseteq N(A_1^*)$. Therefore, A_1 and A_2 are p.s. Hence the Theorem.

Remark 3.5. Lemma 3.4 fails if we relax the condition that A_1 and A_2 are *k-EP*.

Let $A_1=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ and $A_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

Let the associated permutation matrix be

$$K=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

A_1 is *k-EP*. A_2 is not *k-EP*. $N(A_1+A_2)\subseteq N(A_1)$ and $N(A_1+A_2)\subseteq N(A_2)$, but $N(A_1+A_2)^*\not\subseteq N(A_1^*)$; $N(A_1+A_2)^*\not\subseteq N(A_2^*)$. Hence A_1 and A_2 are not parallel summable.

Theorem 3.6. *Let* A_1 *and* A_2 *be p.s. k-EP matrices. Then* $(A_1 \mp A_2)$ *and* (A_1+A_2) *are k-EP.*

Proof. Since A_1 and A_2 are p.s. *k-EP* matrices, by Lemma 3.4,

$$N(A_1+A_2)\subseteq N(A_1) \quad\text{and}\quad N(A_1+A_2)\subseteq N(A_2).$$

$$N(K(A_1+A_2))\subseteq N(KA_1) \quad\text{and}\quad N(K(A_1+A_2))\subseteq N(KA_2).$$

$$N(KA_1+KA_2)\subseteq N(KA_1) \quad\text{and}\quad N(KA_1+KA_2)\subseteq N(KA_2).$$

Therefore, $KA_1+KA_2=K(A_1+A_2)$ is *EP*. Then (A_1+A_2) is *k-EP*. Since A_1 and A_2 are p.s. *k-EP* matrices, KA_1 and KA_2 are p.s. *EP* matrices.

Therefore,

$$R(KA_1)^* = R(KA_1) \text{ and } R(KA_2)^* = R(KA_2)$$

$$\begin{aligned} R(KA_1 \mp KA_2)^* &= R\left((KA_1)^* \mp (KA_2)^*\right) && \text{[By P.2]} \\ &= R\left((KA_1)^*\right) \cap R\left((KA_2)^*\right) && \text{[By P.4]} \\ &= R(KA_1) \cap R(KA_2) && \text{[Since } KA_1 \text{ and } KA_2 \text{ are } EP] \\ &= R(KA_1 \mp KA_2). \end{aligned}$$

Thus, $KA_1 \mp KA_2$ is $EP \Rightarrow K(A_1 \mp A_2)$ is $EP \Rightarrow (A_1 \mp A_2)$ is *k-EP*. Thus $(A_1 \mp A_2)$ is *k-EP* whenever A_1 and A_2 are *k-EP*. Hence the Theorem.

Corollary 3.7. *Let* A_1 *and* A_2 *be k-EP matrices such that* $N(A_1 + A_2) \subseteq N(A_2)$. *If* A_3 *is k-EP commuting with both* A_1 *and* A_2, *then* $A_3(A_1 + A_2)$ *and* $A_3(A_1 \mp A_2) = (A_3A_1 \mp A_3A_2)$ *are k-EP.*

Proof. A_1 and A_2 are *k-EP* with $N(A_1 + A_2) \subseteq N(A_2)$. By Theorem 2.2, $(A_1 + A_2)$ is *k-EP*. Now KA_1, KA_2 and $K(A_1 + A_2)$ are *EP*. Since A_3 commutes with A_1, A_2 and $(A_1 + A_2)$, KA_3 commutes with KA_1, KA_2 and $K(A_1 + A_2)$ and by Theorem (1.3) of [2], $K(A_3A_1)$, $K(A_3A_2)$ and $K(A_3(A_1 + A_2))$ are *EP*. Therefore, $(A_3A_1, A_3A_2, A_3(A_1 + A_2)$ are *k-EP*. Now by Theorem 3.6 $(A_3A_1 \mp A_3A_2)$ is *k-EP*. By P.3 (Properties 3.3),

$$K(A_3(A_1 \mp A_2) = K(A_3A_1 \mp A_3A_2).$$

Since $A_3A_1 \mp A_3A_2$ is *k-EP*, $K(A_3A_1 \mp A_3A_2)$ is $EP \Rightarrow K(A_3(A_1 \mp A_2)$ is EP $A_3(A_1 \mp A_2)$ is *k-EP*. Hence the corollary.

References

1. R.D. Hill and S.R. Waters, On k-Real and k-Hermitian matrices, *Linear Alg. Appl.* **169** (1992), 17-29.
2. I.J. Katz and M.H. Pearl, On EP_r and normal EP_r matrices, *J. Res. Nat. Bur. Stds.* **70B** (1966), 47-77.
3. G. Marsaglia and G.P.H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear Alg.* **2** (1974), 269-292.
4. A.R. Meenakshi and S. Krishnamoorthy, On *k-EP* matrices, *Linear Alg. Appl.* **269** (1998), 219-232.
5. C.R. Rao and S.K. Mitra, *Generalized Inverse of Matrices and its Applications*, Wiley and Sons, New York, 1971.