ON THE EXTERIOR DEGREE OF THE WREATH PRODUCT OF FINITE ABELIAN GROUPS

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Abstract. The exterior degree $d^{\wedge}(G)$ of a finite group G has been recently introduced by Rezaei and Niroomand in order to study the probability that two given elements x and y of G commute in the nonabelian exterior square $G \wedge G$. This notion is related with the probability $d(G)$ that two elements of G commute in the usual sense. Motivated by a paper of Erovenko and Sury of 2008, we compute the exterior degree of a group which is the wreath product of two finite abelian p–groups (p prime). We find some numerical inequalities and study mostly abelian p-groups.

1. Introduction

The present paper deals only with finite groups. A consistent body of scientific results is devoted to study the combinatorial conditions which influence the structure of finite groups in [1, 4, 5, 6, 17]. Denoting with $k(G)$ the *number of the G–conjugacy classes* $[x]_G = \{x^g \mid g \in G\}$ of a group G and with $C_G(x)$ the centralizer of x in G, it is shown in [1, 4, 5, 6, 17] that the *commutativity degree*

$$d(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2} = \frac{1}{|G|^2}\sum_{x\in G}|C_G(x)| = \frac{k(G)}{|G|}$$

allows us to classify large classes of groups only looking at their numerical value of $d(G)$. The intriguing idea, which is behind most of the proofs of [1, 3, 4], is that $d(G)$ measures the distance of G from being abelian and so we may apply different techniques of combinatorial nature. We inform the reader that there are some recent contributions in [12, 19] which study the recognition of the structure of a group from inequalities of numerical nature. This approach might be useful to compare with our techniques of investigation.

Going back to illustrate our scopes, we mention that several authors call $d(G)$ the *probability of commuting pairs* of G. In fact, $\{(x, y) \in G\times G \mid [x, y] = 1\}$ can be regarded as a measurable subset of G^2 (with respect to the discrete measure over G^2) and $d(G)$ is defined exactly as a probability measure. Of course, $d(G) = 1$ if and only if G is abelian. As one may expect, $d(G)$ is an invariant, but it is not only invariant under isomorphisms of groups, but also under various generalizations, for instance the *isoclinisms* (see [5, 17]).

On the other hand, there is a recent interest in algebraic topology and in group theory in the study of the *nonabelian exterior square* $G \wedge G$ of G: we recall that $G \wedge G$ is the group generated by the symbols $g \wedge h$ and by the relations $gg' \wedge h =$

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$((g')^g \wedge h^g)\ (g \wedge h)$, $g \wedge hh' = (g \wedge h)\ (g^h \wedge (h')^h)$ and $g \wedge g = 1$ for all $g, g', h, h' \in G$, where G acts on itself by conjugation via $(g')^g = g^{-1}g'g$.

A recent number of papers is in fact devoted to investigate a more specific invariant, which allows us to measure how far is G from being an abelian group of a prescribed type, for instance, elementary abelian of given rank. Niroomand and Rezaei [14] introduced the *exterior degree* of G

$$d^\wedge(G) = \frac{|\{(x,y) \in G \times G \mid x \wedge y = 1_{G\wedge G}\}|}{|G|^2} = \frac{1}{|G|}\sum_{i=1}^{k(G)} \frac{|C^\wedge_G(x_i)|}{|C_G(x_i)|},$$

where the last equality is precisely [14, Lemma 2.2]. The set

$$C^\wedge_G(x) = \{a \in G \mid a \wedge x = 1_{G\wedge G}\}$$

is called *exterior centralizer* of x in G and turns out to be a subgroup of G (see [13]) contained in $C_G(x)$. The *exterior center* of G is the set

$$Z^\wedge(G) = \{g \in G \mid 1_{G\wedge G} = g \wedge y \in G \wedge G, \forall y \in G\} = \bigcap_{x\in G} C^\wedge_G(x)$$

which is a subgroup of the center $Z(G)$ of G (see [13, 14, 15]). Originally, $C^\wedge_G(x)$ and $Z^\wedge(G)$ have been introduced for the study of properties of $G \wedge G$ and this justifies the use of these subgroups in our perspective of research.

$H_2(G,\mathbb{Z}) = M(G)$ denotes the second homology group of G with integral coefficients (also called *Schur multiplier* of G, see [11]) and plays a fundamental role in the study of the exterior degree, as noted in [14, 15, 16]. There is a classical result in [11], known as *Poincaré Duality*, which shows $H_2(G,\mathbb{Z}) \simeq H^2(G,\mathbb{C}^*)$. This means that the second homology group with coefficients in $\mathbb{Z}$ is isomorphic with the second cohomology group with coefficients in $\mathbb{C}^*$ and, in principle, we may use indipendently $H_2(G,\mathbb{Z})$ or $H^2(G,\mathbb{Z})$ for denoting the Schur multiplier. We prefer to use $H_2(G,\mathbb{Z}) = M(G)$, following [13, 14, 15, 16].

Very briefly, we mention that the interest for $C^\wedge_G(x)$ and $Z^\wedge(G)$ is due to the fact that they allow us to decide whether G is a *capable group* or not, that is, whether G is isomorphic to $E/Z(E)$ for some group E or not. Beyl and others [2] illustrate that capable groups are well–known and subject to interesting classifications.

We noted that it is not available a precise computation of the exterior degree of wreath products of abelian groups as in [7], even if some general bounds are known by [14, 15, 16]. The present paper has been written to cover this aspect of the literature. Since the dihedral group D_8 of order 8 is isomorphic to the wreath product $C_2 \wr C_2$ of two copies of the cyclic group C_2 of order 2, we have precise values for $d^\wedge(D_8)$ already in [14, 15] and several other extraspecial p–groups (p any prime) can be constructed directly as wreath products of cyclic p–groups (see [10]). In fact we confirm not only the main results of [16], but provide new formulas for the exterior degree of wreath products of cyclic p–groups.

2. Preliminaries

Let L and H be groups and Ω a set with H acting on it. Let K be the direct product $K = \prod_{\omega\in\Omega} L_\omega$ of copies of $L_\omega = L$ indexed by the set Ω. The elements of K can be seen as arbitrary sequences (l_ω) of elements of L indexed by Ω with componentwise multiplication. Then the action of H on Ω extends in a natural way to an action of H on the group K by $h(l_\omega) = (l_{h^{-1}\omega})$. In this way, we have defined the group $L \wr_\Omega H$, wreath product of L by H with respect to Ω. The subgroup K of $L \wr_\Omega H$ is called *base*. Since H acts in a natural way on itself by left multiplication (notion of *left Cayley action*), we can choose $\Omega = H$. In this case, we write briefly $L \wr H$, omitting Ω, and the wreath product turns out to be the semidirect product $H \ltimes K$, that is, $L \wr H = H \ltimes K$. We will consider only this type of wreath product, also called *standard wreath product*. More specifically, we will focus on two abelian groups A and B and on $A \wr B$, considering the left Cayley action as just said. We will have

$$A \wr B = B \ltimes \underbrace{A \times A \times \ldots \times A}_{|B|-\text{times}} = B \ltimes A^{|B|},$$

that is, the semidirect product of B by $|B|$–copies of A (see [11, Chapter 6] or [10]). Several examples, which motivated our investigations, are listed below.

Example 2.1. The symmetric group

$$S_3 = \langle x, y \mid x^2 = y^3 = 1, x^{-1}yx = y^{-1} \rangle = \langle x \rangle \ltimes \langle y \rangle \simeq C_2 \ltimes A_3 \simeq C_2 \ltimes C_3$$

on 3 letters is isomorphic to the dihedral group D_6 of order 6, where $A_3 \simeq C_3$ denotes the alternating group on 3 elements. It is easy to check that $Z(S_3) = Z^\wedge(S_3) = 1$, $C_{S_3}(A_3) = A_3$ and $C_{S_3}(\langle x \rangle) = \langle x \rangle$. More generally, the dihedral group of order $2q$ is

$$D_{2q} = \langle x, y \mid x^2 = y^q = 1, x^{-1}yx = y^{-1} \rangle \simeq C_2 \ltimes C_q$$

(see [10]) and, in case $q \geq 3$ is an odd prime, it is possible to extend our considerations, up to isomorphisms, to all dihedral groups D_{2q}. We find again $C_{D_{2q}}(C_q) = C_q$, $C_{D_{2q}}(\langle x \rangle) = \langle x \rangle$ and $Z(D_{2q}) = Z^\wedge(D_{2q}) = 1$.

One of the key results in [14, 15] is the following bound, which restricts the values of the exterior degree by two functions depending on the size of the Schur multiplier.

Theorem 2.2 (See [14], Theorem 2.3)**.** *Let G be a group. Then*

$$\frac{d(G)}{|M(G)|} + \frac{|Z^\wedge(G)|}{|G|}\left(1 - \frac{1}{|M(G)|}\right) \leq d^\wedge(G) \leq d(G) - \left(\frac{p-1}{p}\right)\left(\frac{|Z(G)| - |Z^\wedge(G)|}{|G|}\right)$$

where p is the smallest prime number dividing the order of G.

Since capable groups are characterized to have trivial exterior center (see [2, 11]), the following consequences are clear.

Corollary 2.3 (See [14], Corollary 2.5)**.** *Let G be a group. Then $d^\wedge(G) \leq d(G)$. Moreover, if G is capable, then $\frac{1}{|G|} \leq d^\wedge(G) \leq d(G)$.*

There are a series of informations which can be found in [11] about $M(A \wr B)$ that we list in the next lines. Given an arbitrary abelian group A,

$$A \sharp A = \frac{A \otimes A}{U(A)}, \text{ where } U(A) = \langle a \otimes b + b \otimes a \mid a, b \in A \rangle$$

and

$$\mathrm{Inv}(A) = \{a \in A \mid a^2 = 1\}.$$

The structure of $A \sharp A$ is described by the following result.

Theorem 2.4 (See [11], Lemma 6.3.4). *Let $A = C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_t}$ be a decomposition of an abelian group A for $n_1, n_2, \ldots, n_t \geq 1$ and s the number of even n_i for $1 \leq i \leq t$. Then*

$$A \sharp A = \bigoplus_{1 \leq i \leq j}^{t} C_{(n_i, n_j)} \oplus C_2^s.$$

Two classic results of Blackburn show that we may compute $M(A \wr B)$ once we know $A \sharp A$ and $\mathrm{Inv}(A)$. The first is very general.

Theorem 2.5 (See [11], Theorem 6.3.3). *Let A and B be two abelian groups. Then*

$$M(A \wr B) = M(A) \oplus M(B) \oplus (B \otimes B)^{\frac{1}{2}(|A| - |\mathrm{Inv}(A)| - 1)} \oplus (B \sharp B)^{|\mathrm{Inv}(A)|}.$$

The second is an application and deals with $M(P_n)$, where P_n is a Sylow p–subgroup of the symmetric group S_{p^n}. It is well known by a result of Kaloujnine (see [11, Section 6]) that P_n has order p^k with $k = 1 + p + p^2 + \ldots + p^{n-1}$ and that $P_1 \simeq C_p$, $P_2 \simeq C_p \wr C_p$, $P_3 = C_p \wr (C_p \wr C_p)$ and so on until $P_n = P_1 \wr P_{n-1}$. Moreover P_{n-1}/P'_{n-1} is an elementary abelian p–group of order p^{n-1} for all n. The following result is very important after we note that any p–group can be embedded in a p–group whose Schur multiplier is elementary abelian [11, Corollary 6.3.6]. Therefore most of the groups which have been studied in [1, 4, 5, 6, 13, 14, 15, 17] turns out to have the Schur multipliers equal to $M(P_n)$.

Theorem 2.6 (See [11], Theorem 6.3.5). *If P_n is a Sylow p–subgroup of the symmetric group S_{p^n}, then $M(P_n) = C_p^s$, where $s = \frac{1}{12}(p-1)(n-1)n(2n-1)$ if $p \neq 2$ and $s = \frac{1}{6}n(n^2-1)$ if $p = 2$.*

We may be more specific on $|\mathrm{Inv}(A)|$ when A is a cyclic group in Theorem 2.5. Before to proceed, the following observation is fundamental and motivates us to concentrate on p–groups.

Remark 2.7. An abelian group can be always written as direct sum of its Sylow p–subgroups by a well known result of decomposition (see [10]). On the other hand, we know that the exterior degree is a multiplicative function, that is, the exterior degree of a direct product (of finitely many groups) equals the product of the values of the exterior degree of each factor (see [14]). Therefore it is reasonable to reduce the study of the exterior degree of abelian groups only to the case of abelian p–groups. Therefore we will concentrare mostly on p–groups from now on.

We know in fact that each finite cyclic group C_n can be written as a direct sum

$$C_n \simeq C_{p_1^{m_1}} \oplus C_{p_2^{m_2}} \oplus \ldots \oplus C_{p_r^{m_r}}$$

of cyclic groups $C_{p_i^{m_i}}$, where $p_i \geq 2$ are primes such that $n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$.

There is a good description of $|\mathrm{Inv}(C_n)|$ in [8, 9] by the function

$$\xi : n \in \mathbb{N} \mapsto \xi(n) = \begin{cases} 1, \text{ if } 8|n, \\ -1, \text{ if } 2|n \text{ and } 4 \not| n, \\ 0, \text{ otherwise.} \end{cases} \in \{-1, 0, 1\}$$

Theorem 2.8 (See [8], Lemma 2, Theorem 2). *Let $n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$ be a prime decomposition of n with $p_i < p_{i+1}$ and $m_i > 0$ for all $1 \leq 1 \leq r-1$. Then*

$$|\mathrm{Inv}(C_n)| = 2^{r + \xi(n)}.$$

In particular, if $r = 1$, then $n = p^m$ and

$$|\mathrm{Inv}(C_{p^m})| = 2^{1+\xi(p^m)}.$$

The wreath product of cyclic p–groups is described below.

Lemma 2.9. *Let $A = C_{p^m}$ and $B = C_{p^n}$ where p is an odd prime and $m, n \geq 1$ integers. Then*

$$p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor} \leq |M(A \wr B)| \leq p^{\lfloor \frac{1}{2}n(p^m+1)\rfloor}.$$

Moreover, the lower bound is achieved when $U(A) = B \otimes B$ and the upper bound when $U(B) = 0$.

Proof. The Künneth Formula [11, Theorem 2.2.10] shows that

$$M(C_{p^m} \oplus C_{p^n}) = M(C_{p^m}) \oplus M(C_{p^n}) \oplus (C_{p^m} \otimes C_{p^n}) = C_{p^m} \otimes C_{p^n} = C_{p^{(m,n)}}$$

We apply Theorem 2.5 and find

$$M(A \wr B) = M(C_{p^m} \wr C_{p^n})$$

$$= M(C_{p^m}) \oplus M(C_{p^n}) \oplus (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m-|\mathrm{Inv}(C_{p^m})|-1)} \oplus (C_{p^n} \sharp C_{p^n})^{|\mathrm{Inv}(C_{p^m})|}$$

$$= (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m-|\mathrm{Inv}(C_{p^m})|-1)} \oplus (C_{p^n} \sharp C_{p^n})^{|\mathrm{Inv}(C_{p^m})|}$$

but p is odd, then $\xi(p) = \xi(p^m) = 0$ and $|\mathrm{Inv}(C_{p^m})| = 2$ by Theorem 2.8, and

$$= (C_{p^n} \otimes C_{p^n})^{\frac{1}{2}(p^m-3)} \oplus (C_{p^n} \sharp C_{p^n})^2 = C_{p^n}^{\frac{1}{2}(p^m-3)} \oplus (C_{p^n} \sharp C_{p^n})^2.$$

If $U(B) = B \otimes B$, then $B \sharp B = 0$ and

$$M(A \wr B) = C_{p^n}^{\frac{1}{2}(p^m-3)}.$$

If $U(B) = 0$, then $B \sharp B = B \otimes B$ and

$$M(A \wr B) = C_{p^n}^{\frac{1}{2}(p^m-3)} \oplus C_{p^n}^2 = C_{p^n}^{\frac{1}{2}(p^m+1)}.$$

If $U(B)$ is a nontrivial proper subgroup of $B \otimes B$, then $0 \leq |B \sharp B| \leq |B \otimes B|$ and

$$|C_{p^n}^{\frac{1}{2}(p^m-3)}| \leq |M(A \wr B)| \leq |C_{p^n}^{\frac{1}{2}(p^m+1)}|,$$

as claimed. □

Lemma 2.10. *Let $A = C_{2^m}$ and $B = C_{2^n}$ and $m, n \geq 1$ integers.*

(i) *If $m = 1$, then $|M(A \wr B)| \leq 2^{\lfloor \frac{1}{2}n\rfloor}$.*
(ii) *If $m = 2$, then $2^{\lfloor \frac{1}{2}n\rfloor} \leq |M(A \wr B)| \leq 2^{\lfloor \frac{5}{2}n\rfloor}$.*
(iii) *If $m \geq 3$, then $2^{\lfloor \frac{1}{2}n(2^m-5)\rfloor} \leq |M(A \wr B)| \leq 2^{\lfloor \frac{1}{2}n(2^m+5)\rfloor}$.*

Moreover, the lower bounds are achieved when $U(B) = B \otimes B$ and the upper bounds when $U(B) = 0$.

Proof. By Theorem 2.8, we should distinguish three cases in order to apply the same argument of Lemma 2.9. If $m = 1$, then $\xi(2) = -1$ and $|\mathrm{Inv}(C_2)| = 1$. In this case we get

$$2^{\frac{1}{2}n(2^1-2)} \leq |M(A \wr B)| \leq 2^{\frac{1}{2}n(2^1-1)}.$$

If $m = 2$, then $\xi(4) = 0$ and $|\mathrm{Inv}(C_4)| = 2$. In this case, we get

$$2^{\frac{1}{2}n(2^2-3)} \leq |M(A \wr B)| \leq p^{\frac{1}{2}n(2^2+1)}.$$

If $m \geq 3$, then $\xi(2^m) = 1$ and $|\mathrm{Inv}(C_{2^m})| = 4$. In this case, we get

$$2^{\frac{1}{2}n(2^m-5)} \leq |M(A \wr B)| \leq 2^{\frac{1}{2}n(2^m+5)}.$$

□

Remark 2.11. Lemma 2.9 shows that

$$|M(A \wr B)| \in \{p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor}, p^{\lfloor \frac{1}{2}n(p^m-2)\rfloor}, p^{\lfloor \frac{1}{2}n(p^m-1)\rfloor}, p^{\lfloor \frac{1}{2}np^m\rfloor}, p^{\lfloor \frac{1}{2}n(p^m+1)\rfloor}\},$$

that is, we have just five choices for $|M(A \wr B)|$ and of the above type, for all $m, n \geq 1$. A similar situation happens in Lemma 2.10 (iii), where we find only eleven possible values of $|M(A \wr B)|$ between $2^{\lfloor \frac{1}{2}n(2^m-5)\rfloor}$ and $2^{\lfloor \frac{1}{2}n(2^m+5)\rfloor}$.

The following example is done for convenience of the reader.

Example 2.12. The Schur multipliers of metacyclic p-groups have been computed by Austin, Beyl and Ng independently, see [11, Theorem 2.11.3, Proposition 2.11.4] or [2]. It is well known that $C_2 \wr C_2 \simeq D_8$, which is a metacyclic 2–group, has $M(D_8) \simeq C_2$. We find exactly this result if $m = n = 1$ in Lemma 2.10 (i). On the other hand, P_2 is a Sylow 2–subgroup of S_4 of order 8 and is well known that $P_2 \simeq C_2 \wr C_2 \simeq D_8$. From Theorem 2.6, $s = 1$ and again $M(P_2) \simeq C_2$ is confirmed.

Erovenko and Sury [7] showed that if B is an abelian group of order n and A is an arbitrary abelian group, then the commutativity degree of the wreath product $A \wr B$ tends to $\frac{1}{n^2}$ as the order of A tends to infinity. By the way, Sury has recently investigated some combinatorial properties of wreath products in [18].

Theorem 2.13 (See [7], Theorem 1.1)**.** *Let A and $B = \{b_1, b_2, ..., b_n\}$ be two abelian groups. Then*

$$d(A \wr B) = \frac{1}{n^2|A|^n} \sum_{s,t=1}^{n} |A|^{\alpha(s,t)},$$

where $\alpha(s,t) = |B : \langle b_s, b_t\rangle|$.

Immediately, we may draw the following conclusion.

Corollary 2.14. *Let A and $B = \{b_1, b_2, ..., b_n\}$ be two abelian groups. If $A \wr B$ is capable, then*

$$\frac{1}{n^2\ |A|^n} \leq d^\wedge(A \wr B) \leq \frac{1}{n^2|A|^n} \sum_{s,t=1}^{n} |A|^{\alpha(s,t)}$$

Proof. The upper bound $d^\wedge(A \wr B) \leq d(A \wr B)$ is always true by Theorems 2.2 and 2.13. The lower bound follows by Corollary 2.3 because $A \wr B$ is capable. □

3. Main theorems

The p–group $E_1 = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle$ is extraspecial of order p^3 and exponent p and has $|M(E_1)| = p^2$. It was investigated recently in [16] under our perspective. [16, Theorem 2.2 (i)] shows that

$$d^\wedge(E_1) = \sum_{g \in E_1} |C^\wedge_{E_1}(g)| = \frac{p^3 + p^2 - 1}{p^5}, \tag{3.1}$$

where the first equality is clear from the definitions but the second depends on the fact that $|C^\wedge_{E_1}(g)| = p$ for all $g \in E_1$. Moreover, Niroomand [16] proved a series of results for $d^\wedge(P)$ in which the presence of a bound of the form (3.1) for an arbitrary p–group P implies that $P/Z^\wedge(P)$ is elementary abelian (see [16, Theorems 2.4 and 2.6]). Similar conditions were studied already in [1, 4, 5, 17] for the commutativity degree and have motivated us to look for a specific type of inequalities in our investigations, which has the formal aspect of (3.1).

We need to recall from [13] that the map

$$\varphi : g \in C_G(x) \mapsto x \wedge g \in M(G) \tag{3.2}$$

is a monomorphism of groups such that $\ker \varphi = C^{\wedge}_G(x)$ and $C_G(x)/C^{\wedge}_G(x)$ is isomorphic to a subgroup of $M(G)$ for all $x \in G$. Consequently,

$$|C_G(x) : C^{\wedge}_G(x)| \leq |M(G)| \tag{3.3}$$

and, in case φ is surjective, we find

$$|C_G(x) : C^{\wedge}_G(x)| = |M(G)|. \tag{3.4}$$

The following example is instructive.

Example 3.1. (i). The group E_1 satisfies (3.3) properly, because $|C_{E_1}(x) : C^{\wedge}_{E_1}(x)| = p$ for all $x \in E_1$ and $|M(E_1)| = p^2$.

(ii). The extraspecial p–group of order p^3 and exponent p^2 with $p \neq 2$ is $E_2 = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle$ and it satisfies (3.4), because $|C_{E_2}(x) : C^{\wedge}_{E_2}(x)| = |M(E_2)| = 1$ for all $x \in E_2$.

(iii). A cyclic group C_n has $M(C_n) = 1$ (see [11]) and satisfies (3.4), because $|C_{C_n}(x) : C^{\wedge}_{C_n}(x)| = |M(C_n)| = 1$ for all $x \in C_n$.

If $G = P$ is a p–group, then it is not hard to see that $M(P)$ is also a p–group (see [11]) and it is meaningful to introduce

$$u_x = \log_p \frac{|M(P)|}{|C_P(x) : C^{\wedge}_P(x)|} \tag{3.5}$$

in order to measure the gap among (3.3) and (3.4).

Of course, u_x depends on x and $|C_P(x) : C^{\wedge}_P(x)| \cdot p^{u_x} = |M(P)|$ is a bound depending on x. In particular, $u_x = 0$ if and only if $|C_P(x) : C^{\wedge}_P(x)| = |M(P)|$, which is exactly (3.4). Immediately, we observe that all groups with trivial Schur multiplier must satisfy (3.4) and then they have $u_x = 0$. Example 3.1 (ii) and (iii) belong to this case and so they are indicative of a more general fact.

Theorem 3.2. *Let $A = C_{p^m}$, $B = C_{p^n}$, p odd prime, $\alpha(s, t) = |B : \langle b_s, b_t\rangle|$ for $b_s, b_t \in B$ and $m, n, s, t \geq 1$. Then*

$$\frac{1}{p^{\lfloor \frac{1}{2}(2mp^n + n(p^m+5))\rfloor}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} \leq d^{\wedge}(A \wr B).$$

Moreover, there exist elements $x_1, x_2, \ldots, x_{k(A \wr B)} \in A \wr B$ such that $u = u_{x_1} + u_{x_2} + \ldots + u_{x_{k(A\wr B)}}$ and

$$d^{\wedge}(A \wr B) \leq \frac{1}{p^{m(p^n-1)+n}} + \frac{u}{p^{\lfloor \frac{1}{2}(2mp^n + n(p^m+1))\rfloor}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}.$$

Proof. First of all,

$$|A \wr B| = |B| \cdot |A|^{|B|} = p^n \cdot (p^m)^{p^n} = p^n \cdot p^{mp^n} = p^{n+mp^n}. \tag{3.6}$$

Notice that $Z(A \wr B) = \{(a, a, \ldots, a) \mid a \in A\}$ is the set of elements of $A^{|B|}$ in which the components are equal, that is, the diagonal subgroup of $A^{|B|}$ and so

$|Z(A \wr B)| = |A| \geq |Z^\wedge(A \wr B)|$. We will prove before the upper bound and then the lower bound.

Since for all $i = 1, 2, \ldots, k(A \wr B)$

$$\left|\frac{C^\wedge_{A\wr B}(x_i)}{C_{A\wr B}(x_i)}\right| = \frac{u_{x_i}}{|M(A \wr B)|},$$

we get

$$d^\wedge(A \wr B) = \frac{1}{|A \wr B|} \sum_{i=1}^{k(A\wr B)} \left|\frac{C^\wedge_{A\wr B}(x_i)}{C_{A\wr B}(x_i)}\right|$$

$$= \frac{1}{|A \wr B|}\left(|Z^\wedge(A \wr B)| + \frac{k(A \wr B) - |Z^\wedge(A \wr B)|}{|M(A \wr B)|}\right)$$

and, if $u = u_{x_1} + u_{x_2} + \ldots + u_{k(A\wr B)}$, then the above quantity becomes

$$= \frac{u\; k(A \wr B)}{|A \wr B|\; |M(A \wr B)|} + \frac{|Z^\wedge(A \wr B)|}{|A \wr B|}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$

$$= u\; \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{|Z^\wedge(A \wr B)|}{|A \wr B|}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$

$$\leq u\; \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{|A|}{|B| \cdot |A|^{|B|}}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$

$$= u\; \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{1}{|B| \cdot |A|^{|B|-1}}\left(1 - \frac{u}{|M(A \wr B)|}\right). \tag{3.7}$$

Now Theorem 2.13 implies

$$d(A \wr B) = \frac{1}{p^{2n}p^{mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} = \frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} \tag{3.8}$$

and, if we replace (3.8) in (3.7) and use (3.6), then we get

$$= \frac{u}{|M(A \wr B)|}\left(\frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right) + \frac{1}{p^{n+mp^n-m}}\left(1 - \frac{u}{|M(A \wr B)|}\right)$$

$$\leq \frac{u}{|M(A \wr B)|}\left(\frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right) + \frac{1}{p^{n+mp^n-m}}.$$

But the lower bound in Lemma 2.9 implies $\frac{1}{|M(A\wr B)|} \leq \frac{1}{p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor}}$ and so we may upper bound with

$$\leq \frac{u}{p^{\lfloor \frac{1}{2}n(p^m-3)\rfloor}}\left(\frac{1}{p^{2n+mp^n}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right) + \frac{1}{p^{n+mp^n-m}}$$

$$= \frac{u}{p^{\lfloor \frac{1}{2}(n(p^m+1)+2mp^n)\rfloor}} \sum_{s,t=1}^{p^n} p^{m\alpha(s,t)} + \frac{1}{p^{n+m(p^n-1)}},$$

as claimed.

On the other hand,

$$d^\wedge(A \wr B) = \frac{d(A \wr B)}{|M(A \wr B)|} + \frac{|Z^\wedge(A \wr B)|}{|A \wr B|}\left(1 - \frac{1}{|M(A \wr B)|}\right) \geq \frac{d(A \wr B)}{|M(A \wr B)|}$$

and by Theorem 2.13 and the upper bound of Lemma 2.9 we get

$$= \frac{1}{|M(A \wr B)|}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right) \geq \frac{1}{p^{\lfloor \frac{1}{2}n(p^m+1)\rfloor}}\left(\frac{1}{p^{2n+mp^n}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}\right)$$

$$= \frac{1}{p^{\lfloor \frac{1}{2}(n(p^m+5)+2mp^n)\rfloor}}\sum_{s,t=1}^{p^n} p^{m\alpha(s,t)}$$

as claimed. □

The even case is described below.

Theorem 3.3. *Let $A = C_{2^m}$, $B = C_{2^n}$, $\alpha(s,t) = |B : \langle b_s, b_t\rangle|$ for $b_s, b_t \in B$, $m, n, s, t \geq 1$ and suitable $x_1, x_2, \ldots, x_{k(A\wr B)} \in A \wr B$ such that $u = u_{x_1} + u_{x_2} + \ldots + u_{x_{k(A\wr B)}}$.*

(i) *If $m = 1$, then*

$$\frac{1}{2^{\lfloor \frac{1}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \leq d^{\wedge}(A \wr B) \leq \frac{1}{2^{n+m2^n-m}} + \frac{u}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}$$

(ii) *If $m = 2$, then*

$$\frac{1}{2^{\lfloor \frac{5}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \leq d^{\wedge}(A\wr B) \leq \frac{1}{2^{n+m2^n-m}} + \frac{u}{2^{\lfloor \frac{1}{2}(m2^{n+1}+5n)\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}$$

(iii) *If $m \geq 3$, then*

$$\frac{1}{2^{\lfloor \frac{1}{2}(m2^{n+1}+n(2^m+9))\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \leq d^{\wedge}(A \wr B) \leq \frac{1}{2^{n+m2^n-m}}$$

$$+\frac{u}{2^{\lfloor \frac{1}{2}(m2^{n+1}+n(2^m-1))\rfloor}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}.$$

Proof. We follow the argument of the proof of Theorem 3.2. From Theorem 2.13,

$$d^{\wedge}(A \wr B) \leq \frac{u}{|M(A \wr B)|}\left(\frac{1}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\right) + \frac{1}{2^{n+m2^n-m}}$$

and we should distinguish three cases in view of Lemma 2.10. If $m = 1$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} + \frac{1}{2^{n+m2^n-m}}.$$

If $m = 2$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{\lfloor \frac{1}{2}n\rfloor}}\left(\frac{1}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\right) + \frac{1}{2^{n+m2^n-m}}.$$

If $m \geq 3$, then

$$d^{\wedge}(A \wr B) \leq \frac{u}{2^{\lfloor \frac{1}{2}n(2^m-5)\rfloor}}\left(\frac{1}{2^{2n+m2^n}}\sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}\right) + \frac{1}{2^{n+m2^n-m}}.$$

On the other hand,

$$d^{\wedge}(A \wr B) \geq \frac{d(A \wr B)}{|M(A \wr B)|}$$

and the following cases should be considered by Lemma 2.10 and Theorem 2.13. If $m = 1$, then we may lower bound with

$$\geq \frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \geq \frac{1}{2^{\lfloor \frac{1}{2}n \rfloor}} \frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)}.$$

If $m = 2$, then we have analogously

$$\geq \frac{1}{2^{\lfloor \frac{5}{2}n \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right).$$

If $m \geq 3$, then we have analogously

$$\geq \frac{1}{2^{\lfloor \frac{1}{2}n(2^m+5) \rfloor}} \left(\frac{1}{2^{2n+m2^n}} \sum_{s,t=1}^{2^n} 2^{m\alpha(s,t)} \right).$$

□

We end with an application to the Sylow p–subgroups P_n of the symmetric group S_{p^n}, described in Theorem 2.6.

Theorem 3.4. *Let P_n be a capable Sylow p–subgroup of S_{p^n} and $u = u_{x_1} + \ldots + u_{x_{k(P_n)}}$ for suitable $x_1, \ldots, x_{k(P_n)} \in P_n$.*

(i) *If $p \neq 2$, then*

$$d^{\wedge}(P_n) = \frac{u\ d(P_n)}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}}\right).$$

(ii) *If $p = 2$, then*

$$d^{\wedge}(P_n) = \frac{u\ d(P_n)}{p^{\frac{1}{6}n(n^2-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{6}n(n^2-1)}}\right).$$

Proof. (i). We know from Theorem 2.6 that $P_n = P_1 \wr P_{n-1}$,

$$|P_n| = 1 + p + p^2 + \ldots + p^{n-1} = \frac{1-p^n}{1-p}$$

and $M(P_n) = C_p^s$, where $s = \frac{1}{12}(p-1)(n-1)n(2n-1)$ if $p \neq 2$. Moreover, P_n is capable, then $Z^{\wedge}(P_n) = 1$. We may repeat the proof of Theorem 3.2 and get

$$d^{\wedge}(P_n) = \frac{1}{|P_n|} \sum_{i=1}^{k(P_n)} \left| \frac{C^{\wedge}_{P_n}(x_i)}{C_{P_n}(x_i)} \right| = \frac{1}{|P_n|} \left(|Z^{\wedge}(P_n)| + \frac{k(P_n) - |Z^{\wedge}(P_n)|}{|M(P_n)|} \right)$$

$$= \frac{u\ k(P_n)}{|P_n|\ |M(P_n)|} + \frac{|Z^{\wedge}(P_n)|}{|P_n|} \left(1 - \frac{u}{|M(P_n)|}\right) = u\ \frac{d(P_n)}{|M(P_n)|} + \frac{|Z^{\wedge}(P_n)|}{|P_n|} \left(1 - \frac{u}{|M(P_n)|}\right)$$

$$= u\ \frac{d(P_n)}{|M(P_n)|} + \frac{1}{|P_n|} \left(1 - \frac{u}{|M(P_n)|}\right) = \frac{u}{|M(P_n)|} \left(d(P_n) - \frac{1}{|P_n|}\right) + \frac{1}{|P_n|}$$

$$= \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \left(d(P_n) - \frac{1}{p^{1+p+p^2+\ldots+p^{n-1}}}\right) + \frac{1}{p^{1+p+p^2+\ldots+p^{n-1}}}$$

$$= \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \left(d(P_n) - \frac{1}{p^{\frac{1-p^n}{1-p}}} \right) + \frac{1}{p^{\frac{1-p^n}{1-p}}}$$

$$= \frac{u\ d(P_n)}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} + \frac{1}{p^{\frac{1-p^n}{1-p}}} \left(1 - \frac{u}{p^{\frac{1}{12}(p-1)(n-1)n(2n-1)}} \right).$$

(ii). In case $p = 2$, it is enough to replace the term $\frac{1}{12}(p-1)(n-1)n(2n-1)$ with $\frac{1}{6}n(n^2-1)$ by Theorem 2.6. □

The importance of Theorem 3.4 is due to the fact that it provides a relation among $d^{\wedge}(P_n)$ and $d(P_n)$. Since there are several results on the commutativity degree in [1, 4, 5, 6], the term $d(P_n)$ is well known and then Theorem 3.4 is significative.

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