Approximate Solutions of a Linear Differential Equation of Third Order

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Abstract. We will investigate the approximate solutions of the differential equation

$$y'''(x) + (\alpha + \beta + \gamma)y''(x) + (\alpha\beta + \beta\gamma + \gamma\alpha)y'(x) + \alpha\beta\gamma y(x) = 0$$

under some conditions imposed on α, β, γ, and on the domain of y, and we will compare the approximate solutions with the exact ones.

1 Introduction

The stability problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem (see [32] and [8]). Thereafter, Rassias [29] attempted to solve the stability problem of the Cauchy additive functional equation in a more general setting.

The stability concept introduced by Rassias's theorem significantly influenced a number of mathematicians to investigate the stability problems for various functional equations (see [3, 5, 6, 7, 9, 10, 17, 25, 30] and the references therein).

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Assume that Y is a normed space and I is an open subset of $\mathbb{R}$. If for any function $f : I \to Y$ satisfying the differential inequality

$$\|a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x)\| \leq \varepsilon$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0$$

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$), where $a_i : I \to \mathbb{K}$ and $h : I \to Y$ are (given) continuous functions and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, over which Y is a normed space. We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, we refer to [5, 6, 8, 9, 10, 17, 29, 30].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [26, 27]). Here, we will introduce a result of Alsina and Ger (see [2]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a constant C such that $|f(x) - Ce^x| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura and Miyajima: they proved in [31] that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [22]).

In [23], Miura, Miyajima and Takahasi also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(x) + g(x)y(x) = 0$, where $g(x)$ is a continuous function, while the author [11] proved the Hyers-Ulam stability of differential equations of the form $c(x)y'(x) = y(x)$. For more recent results about this subject, we can refer to [1, 4, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 24, 28, 33].

The aim of this paper is to prove a kind of Hyers-Ulam stability of a linear differential equation of third order,

$$y'''(x) + (\alpha + \beta + \gamma)y''(x) + (\alpha\beta + \beta\gamma + \gamma\alpha)y'(x) + \alpha\beta\gamma y(x) = 0, \tag{1.1}$$

where α, β, and γ are nonzero real numbers. More precisely, we will investigate the (approximate) solutions of the differential inequality

$$|y'''(x) + (\alpha + \beta + \gamma)y''(x) + (\alpha\beta + \beta\gamma + \gamma\alpha)y'(x) + \alpha\beta\gamma y(x)| \leq \varepsilon \tag{1.2}$$

and compare them with the (exact) solutions of the differential equation (1.1).

2 Preliminaries

The author recently obtained a result concerning the Hyers-Ulam stability of linear differential equations of the form

$$y'(x) + g(x)y(x) + h(x) = 0$$

which includes the following theorem as a special case (see [13, Remark 3]).

Theorem 2.1 *Let $I = (a, b)$ be an open interval with $-\infty \leq a < b \leq \infty$. Assume that $g, h : I \to \mathbb{R}$ are continuous functions and $\varphi : I \to [0, \infty)$ is a function such that*

(i) *$g(x)$ and $\exp\{\int_a^x g(u)du\}h(x)$ are integrable on (a, d) for each $d \in I$;*

(ii) *$\varphi(x)\exp\{\int_a^x g(u)du\}$ is integrable on I.*

If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality

$$|y'(x) + g(x)y(x) + h(x)| \leq \varphi(x)$$

for all $x \in I$, then there exists a unique real number c such that

$$\left| y(x) - \exp\left\{ -\int_a^x g(u)du \right\} \left(c - \int_a^x \exp\left\{ \int_a^v g(u)du \right\} h(v)dv \right) \right|$$
$$\leq \exp\left\{ -\int_a^x g(u)du \right\} \int_x^b \varphi(v)\exp\left\{ \int_a^v g(u)du \right\} dv$$

for every $x \in I$.

The following corollaries are essential for the proof of our main theorems. We can prove them easily by using Theorem 2.1.

Corollary 2.2 *Let $I = (a, b)$ be an open interval with $-\infty < a < b \leq \infty$. Assume that $\alpha \neq 0$, $\beta \neq 0$, γ are real numbers and $e^{\alpha(x-a)}$ is integrable on I. If a twice continuously differentiable function $f : I \to \mathbb{R}$ satisfies the differential inequality*

$$|f''(x) + (\alpha+\beta)f'(x) + \alpha\beta f(x) + \gamma| \leq \varepsilon \tag{2.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, then there exists a unique real number c such that

$$\left| f'(x) + \beta f(x) - ce^{-\alpha(x-a)} + \frac{\gamma}{\alpha} \right| \leq \frac{\varepsilon}{|\alpha|} \left| e^{\alpha(b-x)} - 1 \right|$$

for all $x \in I$, where $e^{\alpha(b-x)}$ stands for $\lim\limits_{w\to b} e^{\alpha(w-x)}$ and it exists even for $b = \infty$.

Proof. If we set $z(x) = f'(x) + \beta f(x)$ for all $x \in I$, then it follows from (2.1) that

$$|z'(x) + \alpha z(x) + \gamma| \leq \varepsilon$$

for any $x \in I$. According to Theorem 2.1, there exists a unique real number c such that

$$\left| z(x) - ce^{-\alpha(x-a)} + \frac{\gamma}{\alpha} \right| \leq \frac{\varepsilon}{|\alpha|} \left| e^{\alpha(b-x)} - 1 \right|$$

for $x \in I$. □

The inequality (2.1) is symmetric with respect to α and β. If α and β interchange their roles, then we obtain the following corollary.

Corollary 2.3 *Let $I = (a, b)$ be an open interval with $-\infty < a < b \leq \infty$. Assume that $\alpha \neq 0$, $\beta \neq 0$, γ are real numbers and $e^{\beta(x-a)}$ is integrable on I. If a twice continuously differentiable function $f : I \to \mathbb{R}$ satisfies the differential inequality* (2.1) *for all $x \in I$ and some $\varepsilon \geq 0$, then there exists a unique real number c such that*

$$\left| f'(x) + \alpha f(x) - ce^{-\beta(x-a)} + \frac{\gamma}{\beta} \right| \leq \frac{\varepsilon}{|\beta|} \left| e^{\beta(b-x)} - 1 \right|$$

for all $x \in I$, where $e^{\beta(b-x)}$ stands for $\lim\limits_{w\to b} e^{\beta(w-x)}$ and it exists even for $b = \infty$.

If $I = (a, \infty)$ with $a > -\infty$, $\alpha < 0$, and $\beta < 0$, then both $e^{\alpha(x-a)}$ and $e^{\beta(x-a)}$ are integrable on I. Thus, the following corollary is an immediate consequence of Corollaries 2.2 and 2.3.

Corollary 2.4 *Let $I = (a, \infty)$ be an open interval with $a > -\infty$. Assume that $\alpha < 0$, $\beta < 0$, γ are real numbers. If a twice continuously differentiable function $f : I \to \mathbb{R}$ satisfies the inequality* (2.1) *for all $x \in I$ and for some $\varepsilon \geq 0$, then there exist real numbers c_α and c_β such that*

$$\left| f'(x) + \beta f(x) - c_\alpha e^{-\alpha(x-a)} + \frac{\gamma}{\alpha} \right| \leq \frac{\varepsilon}{|\alpha|}$$

and

$$\left| f'(x) + \alpha f(x) - c_\beta e^{-\beta(x-a)} + \frac{\gamma}{\beta} \right| \leq \frac{\varepsilon}{|\beta|}$$

for all $x \in I$. The real numbers c_α and c_β are uniquely determined.

3 Main Theorems

In this section, we investigate the approximate solutions of the differential equation (1.1) in the class of three times continuously differentiable functions $y : (a, b) \to \mathbb{R}$ for the case of either $a \in \mathbb{R}$ and $b = \infty$ or $a = -\infty$ and $b \in \mathbb{R}$.

As we know,

$$y(x) = \begin{cases} c_1 e^{-\alpha(x-a)} + c_2 e^{-\beta(x-a)} + c_3 e^{-\gamma(x-a)} & (\text{for distinct } \alpha,\ \beta,\ \gamma), \\ c_1 e^{-\alpha(x-a)} + c_2 x e^{-\alpha(x-a)} + c_3 e^{-\gamma(x-a)} & (\text{for } \alpha = \beta \neq \gamma), \\ c_1 e^{-\alpha(x-a)} + c_2 x e^{-\alpha(x-a)} + c_3 x^2 e^{-\alpha(x-a)} & (\text{for } \alpha = \beta = \gamma) \end{cases}$$

is the general solution of the differential equation (1.1) for any real coefficients c_1, c_2, and c_3.

We apply the methods introduced in [2, 11, 13, 21, 33] to the proof of the following main theorem.

Theorem 3.1 *Let $I = (a, \infty)$ be an open interval with a real number a. Assume that α, β, γ are real numbers. Suppose $y : I \to \mathbb{R}$ is a three times continuously differentiable function and the limits $y(a) = \lim_{x \to a^+} y(x)$ and $y'(a) = \lim_{x \to a^+} y'(x)$ exist. Moreover, assume that y satisfies the inequality* (1.2) *for all $x \in I$ and for some $\varepsilon \geq 0$.*

(*i*) *If $\alpha < 0$, $\beta < 0$, $\alpha \neq \beta$, and $\gamma \notin \{0, \alpha, \beta\}$, then there exist solutions y_1, $y_2 : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$|y(x) - y_1(x)| \leq \frac{\varepsilon}{\alpha\beta} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \beta} e^{-\beta(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \beta} \right) e^{-\gamma(x-a)} \right|$$

and

$$|y(x) - y_2(x)| \leq \frac{\varepsilon}{\alpha\beta}\left|\frac{1}{\gamma} - \frac{1}{\gamma - \alpha}e^{-\alpha(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha}\right)e^{-\gamma(x-a)}\right|$$

for all $x \in I$.

(*ii*) *If* $\alpha = \beta < 0$, *and* $\gamma \notin \{0, \alpha\}$, *then there exists a solution* $\hat{y} : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| \leq \frac{\varepsilon}{\alpha^2}\left|\frac{1}{\gamma} - \frac{1}{\gamma - \alpha}e^{-\alpha(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha}\right)e^{-\gamma(x-a)}\right|$$

for all $x \in I$.

(*iii*) *If* $\alpha = \beta = \gamma < 0$, *then there exists a solution* $\hat{y} : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| \leq \frac{\varepsilon}{\alpha^2}\left|\frac{1}{\alpha} - \left(\frac{1}{\alpha} - a\right)e^{-\alpha(x-a)} - xe^{-\alpha(x-a)}\right|$$

for all $x \in I$.

Proof. We will prove (*i*) only. The proofs for (*ii*) and (*iii*) run in the same way as the proof of (*i*).

Assume that $\alpha < 0$, $\beta < 0$, and $\gamma \neq 0$ are distinct real numbers. Let us define a twice continuously differentiable function $f : I \to \mathbb{R}$ by $f(x) = y'(x) + \gamma y(x)$ for all $x \in I$ and let $f(a) = y'(a) + \gamma y(a)$. It then follows from (1.2) that

$$|f''(x) + (\alpha + \beta)f'(x) + \alpha\beta f(x)| \leq \varepsilon$$

for any $x \in I$. According to Corollary 2.4, there exist real numbers c_α and c_β such that

$$\left|f'(x) + \beta f(x) - c_\alpha e^{-\alpha(x-a)}\right| \leq \frac{\varepsilon}{|\alpha|} \tag{3.1}$$

and

$$\left|f'(x) + \alpha f(x) - c_\beta e^{-\beta(x-a)}\right| \leq \frac{\varepsilon}{|\beta|} \tag{3.2}$$

for all $x \in I$, where the real numbers c_α and c_β are uniquely determined.

It follows from (3.1) that

$$-\frac{\varepsilon}{|\alpha|}e^{\beta(x-a)} \leq f'(x)e^{\beta(x-a)} + \beta e^{\beta(x-a)}f(x) - c_\alpha e^{(\beta-\alpha)(x-a)} \leq \frac{\varepsilon}{|\alpha|}e^{\beta(x-a)}$$

or

$$\frac{d}{dx}\left\{\frac{\varepsilon}{\alpha\beta}e^{\beta(x-a)}\right\} \le \frac{d}{dx}\left\{f(x)e^{\beta(x-a)} - \frac{c_\alpha}{\beta-\alpha}e^{(\beta-\alpha)(x-a)}\right\} \le -\frac{d}{dx}\left\{\frac{\varepsilon}{\alpha\beta}e^{\beta(x-a)}\right\}.$$

If we integrate the last inequalities from a to x, then we get

$$\frac{\varepsilon}{\alpha\beta}\left[e^{\beta(x-a)} - 1\right] \le f(x)e^{\beta(x-a)} - f(a) - \frac{c_\alpha}{\beta-\alpha}\left[e^{(\beta-\alpha)(x-a)} - 1\right] \le \frac{\varepsilon}{\alpha\beta}\left[1 - e^{\beta(x-a)}\right]$$

or

$$\begin{aligned} &\frac{\varepsilon}{\alpha\beta}\left[1 - e^{-\beta(x-a)}\right] \\ &\le \; y'(x) + \gamma y(x) - f(a)e^{-\beta(x-a)} - \frac{c_\alpha}{\beta-\alpha}\left[e^{-\alpha(x-a)} - e^{-\beta(x-a)}\right] \\ &\le \; \frac{\varepsilon}{\alpha\beta}\left[e^{-\beta(x-a)} - 1\right]. \end{aligned}$$

If we multiply by $e^{\gamma(x-a)}$ each term of the last inequalities, then we have

$$\begin{aligned} &\frac{\varepsilon}{\alpha\beta}\frac{d}{dx}\left\{\frac{1}{\gamma}e^{\gamma(x-a)} - \frac{1}{\gamma-\beta}e^{(\gamma-\beta)(x-a)}\right\} \\ &\le \; \frac{d}{dx}\Bigg[y(x)e^{\gamma(x-a)} - \frac{f(a)}{\gamma-\beta}e^{(\gamma-\beta)(x-a)} \\ &\qquad - \frac{c_\alpha}{\beta-\alpha}\left\{\frac{1}{\gamma-\alpha}e^{(\gamma-\alpha)(x-a)} - \frac{1}{\gamma-\beta}e^{(\gamma-\beta)(x-a)}\right\}\Bigg] \\ &\le \; \frac{\varepsilon}{\alpha\beta}\frac{d}{dx}\left\{\frac{1}{\gamma-\beta}e^{(\gamma-\beta)(x-a)} - \frac{1}{\gamma}e^{\gamma(x-a)}\right\}. \end{aligned}$$

If we integrate the last inequalities from a to x and then multiply by $e^{-\gamma(x-a)}$ the resulting inequalities, then we obtain

$$\begin{aligned} &\frac{\varepsilon}{\alpha\beta}\left\{\frac{1}{\gamma} - \frac{1}{\gamma-\beta}e^{-\beta(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma-\beta}\right)e^{-\gamma(x-a)}\right\} \\ &\le \; y(x) - \frac{c_\alpha}{(\beta-\alpha)(\gamma-\alpha)}e^{-\alpha(x-a)} - \frac{1}{\gamma-\beta}\left(f(a) - \frac{c_\alpha}{\beta-\alpha}\right)e^{-\beta(x-a)} \\ &\qquad - \left(y(a) - \frac{f(a)}{\gamma-\beta} - \frac{c_\alpha}{(\beta-\alpha)(\gamma-\alpha)} + \frac{c_\alpha}{(\beta-\alpha)(\gamma-\beta)}\right)e^{-\gamma(x-a)} \\ &\le \; \frac{\varepsilon}{\alpha\beta}\left\{-\frac{1}{\gamma} + \frac{1}{\gamma-\beta}e^{-\beta(x-a)} + \left(\frac{1}{\gamma} - \frac{1}{\gamma-\beta}\right)e^{-\gamma(x-a)}\right\}, \end{aligned}$$

that is, there exist real numbers c_1, c_2, c_3 such that

$$\begin{aligned} &\left|y(x) - c_1e^{-\alpha(x-a)} - c_2e^{-\beta(x-a)} - c_3e^{-\gamma(x-a)}\right| \\ &\quad \le \; \frac{\varepsilon}{\alpha\beta}\left|\frac{1}{\gamma} - \frac{1}{\gamma-\beta}e^{-\beta(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma-\beta}\right)e^{-\gamma(x-a)}\right| \end{aligned}$$

for all $x \in I$.

Similarly, if α and β interchange their roles, then it follows from (3.2) and the last inequality that there exist real numbers c_4, c_5, c_6 satisfying

$$\begin{aligned}&\left| y(x) - c_4 e^{-\alpha(x-a)} - c_5 e^{-\beta(x-a)} - c_6 e^{-\gamma(x-a)} \right| \\ &\quad \leq \frac{\varepsilon}{\alpha\beta} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \alpha} e^{-\alpha(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha} \right) e^{-\gamma(x-a)} \right|\end{aligned}$$

for any $x \in I$. □

We will now prove a counterpart of Theorem 3.1 for the case of $I = (-\infty, b)$, $\alpha > 0$, $\beta > 0$, and $\gamma \neq 0$.

Theorem 3.2 *Let $I = (-\infty, b)$ be an open interval with a real number b. Assume that α, β, γ are real numbers. Suppose $y : I \to \mathbb{R}$ is a three times continuously differentiable function and the limits $y(b) = \lim_{x \to b^-} y(x)$ and $y'(b) = \lim_{x \to b^-} y'(x)$ exist. Moreover, assume that y satisfies the differential inequality* (1.2) *for all $x \in I$ and for some $\varepsilon \geq 0$.*

(*i*) *If $\alpha > 0$, $\beta > 0$, $\alpha \neq \beta$, and $\gamma \notin \{0, \alpha, \beta\}$, then there exist solutions y_1, $y_2 : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$\begin{aligned}&|y(x) - y_1(x)| \leq \frac{\varepsilon}{\alpha\beta} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \beta} e^{\beta(b-x)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \beta} \right) e^{\gamma(b-x)} \right|, \\ &|y(x) - y_2(x)| \leq \frac{\varepsilon}{\alpha\beta} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \alpha} e^{\alpha(b-x)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha} \right) e^{\gamma(b-x)} \right|\end{aligned} \tag{3.3}$$

for all $x \in I$.

(*ii*) *If $\alpha = \beta > 0$, and $\gamma \notin \{0, \alpha\}$, then there exists a solution $\hat{y} : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| \leq \frac{\varepsilon}{\alpha^2} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \alpha} e^{\alpha(b-x)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha} \right) e^{\gamma(b-x)} \right|$$

for all $x \in I$.

(*iii*) *If $\alpha = \beta = \gamma > 0$, then there exists a solution $\hat{y} : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| \leq \frac{\varepsilon}{\alpha^2} \left| \frac{1}{\alpha} - \left(\frac{1}{\alpha} - b \right) e^{\alpha(b-x)} - x e^{\alpha(b-x)} \right|$$

for all $x \in I$.

Proof. We will prove (i) only. The parts (ii) and (iii) can be proved similarly. Hence, we omit their proofs.

Assume that $\alpha > 0$, $\beta > 0$, and $\gamma \neq 0$ are distinct real numbers. Let us define a three times continuously differentiable function $\tilde{y} : \tilde{I} \to \mathbb{R}$ by $\tilde{y}(x) = y(-x)$, where we set $\tilde{I} = (-b, \infty) =: (\tilde{a}, \infty)$. By the chain rule, if we set $t = -x$, then we have

$$y'(x) = -\tilde{y}'(t), \qquad y''(x) = \tilde{y}''(t), \qquad y'''(x) = -\tilde{y}'''(t).$$

Thus, we get

$$\begin{aligned} & y'''(x) + (\alpha+\beta+\gamma)y''(x) + (\alpha\beta+\beta\gamma+\gamma\alpha)y'(x) + \alpha\beta\gamma y(x) \\ = \; & -\tilde{y}'''(t) + (\alpha+\beta+\gamma)\tilde{y}''(t) - (\alpha\beta+\beta\gamma+\gamma\alpha)\tilde{y}'(t) + \alpha\beta\gamma\tilde{y}(t) \\ = \; & -[\tilde{y}'''(t) + (\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\tilde{y}''(t) + (\tilde{\alpha}\tilde{\beta}+\tilde{\beta}\tilde{\gamma}+\tilde{\gamma}\tilde{\alpha})\tilde{y}'(t) + \tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{y}(t)], \end{aligned} \tag{3.4}$$

for all $t \in \tilde{I}$, where $\tilde{\alpha} = -\alpha < 0$, $\tilde{\beta} = -\beta < 0$, and $\tilde{\gamma} = -\gamma \neq 0$ are distinct real numbers, and it follows from (1.2) that

$$|\tilde{y}'''(t) + (\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\tilde{y}''(t) + (\tilde{\alpha}\tilde{\beta}+\tilde{\beta}\tilde{\gamma}+\tilde{\gamma}\tilde{\alpha})\tilde{y}'(t) + \tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{y}(t)| \leq \varepsilon$$

for all $t \in \tilde{I}$.

Moreover, $\tilde{y}(\tilde{a})$ and $\tilde{y}'(\tilde{a})$ exist as we see

$$\tilde{y}(\tilde{a}) = \lim_{t \to \tilde{a}^+} \tilde{y}(t) = \lim_{x \to b^-} y(x) = y(b)$$

and

$$\tilde{y}'(\tilde{a}) = \lim_{t \to \tilde{a}^+} \tilde{y}'(t) = \lim_{x \to b^-} (-y'(x)) = -\lim_{x \to b^-} y'(x) = -y'(b).$$

According to Theorem 3.1 (i), there exist solutions $\tilde{y}_1$, $\tilde{y}_2 : \tilde{I} \to \mathbb{R}$ of the differential equation,

$$\tilde{y}'''(t) + (\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\tilde{y}''(t) + (\tilde{\alpha}\tilde{\beta}+\tilde{\beta}\tilde{\gamma}+\tilde{\gamma}\tilde{\alpha})\tilde{y}'(t) + \tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{y}(t) = 0, \tag{3.5}$$

which satisfy

$$|\tilde{y}(t) - \tilde{y}_1(t)| \leq \frac{\varepsilon}{\tilde{\alpha}\tilde{\beta}} \left| \frac{1}{\tilde{\gamma}} - \frac{1}{\tilde{\gamma}-\tilde{\beta}} e^{-\tilde{\beta}(t-\tilde{a})} - \left(\frac{1}{\tilde{\gamma}} - \frac{1}{\tilde{\gamma}-\tilde{\beta}} \right) e^{-\tilde{\gamma}(t-\tilde{a})} \right|$$

and

$$|\tilde{y}(t) - \tilde{y}_2(t)| \leq \frac{\varepsilon}{\tilde{\alpha}\tilde{\beta}} \left| \frac{1}{\tilde{\gamma}} - \frac{1}{\tilde{\gamma}-\tilde{\alpha}} e^{-\tilde{\alpha}(t-\tilde{a})} - \left(\frac{1}{\tilde{\gamma}} - \frac{1}{\tilde{\gamma}-\tilde{\alpha}} \right) e^{-\tilde{\gamma}(t-\tilde{a})} \right|$$

for all $t \in \tilde{I}$. In view of (3.4), the differential equations (1.1) and (3.5) are equivalent in the sense that $y(x)$ is a solution of the differential equation (1.1) if and only if $\tilde{y}(t)$ is a solution of the differential equation (3.5). Hence, there exist solutions y_1, $y_2 : I \to \mathbb{R}$ of the differential equation (1.1) satisfying the inequalities in (3.3). □

4 Applications

The inequality (1.2) is symmetric with respect to α, β, and γ. If α, β, and γ are assumed to be distinct negative real numbers, then the following corollary is an immediate consequence of Theorem 3.1 (i).

Corollary 4.1 *Let $I = (a, \infty)$ be an open interval with a real number a. Assume that $\alpha < 0$, $\beta < 0$, $\gamma < 0$ are distinct real numbers. Suppose $y : I \to \mathbb{R}$ is a three times continuously differentiable function and the limits $y(a) = \lim_{x \to a^+} y(x)$ and $y'(a) = \lim_{x \to a^+} y'(x)$ exist. If y satisfies the inequality* (1.2) *for all $x \in I$ and for some $\varepsilon \geq 0$, then there exists a solution $y_1 : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$|y(x) - y_1(x)| \ \leq \ \frac{\varepsilon}{\alpha\beta} \left| \frac{1}{\gamma} - \frac{1}{\gamma - \beta} e^{-\beta(x-a)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma - \beta} \right) e^{-\gamma(x-a)} \right|$$

for all $x \in I$. Analogous inequalities hold for every permutation of α, β, γ.

The following corollary follows from the 4th or the 5th inequality of Corollary 4.1 and Theorem 3.1 (iii).

Corollary 4.2 *Let $I = (a, \infty)$ be an open interval with $a > -\infty$. Assume that α, β, γ are negative real numbers. Suppose $y : I \to \mathbb{R}$ is a three times continuously differentiable function and the limits $y(a) = \lim_{x \to a^+} y(x)$ and $y'(a) = \lim_{x \to a^+} y'(x)$ exist. Moreover, assume that y satisfies the inequality* (1.2) *for all $x \in I$ and for some $\varepsilon \geq 0$.*

(i) *If $\gamma < \beta < \alpha < 0$, then there exists a solution $\hat{y} : I \to \mathbb{R}$ of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| = o\Big(e^{-\gamma(x-a)}\Big)$$

as $x \to \infty$, where o stands for the Landau little-o notation.

(*ii*) *If* $\alpha = \beta = \gamma < 0$*, then there exists a solution* $\hat{y} : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| = O\Big(xe^{-\alpha(x-a)}\Big)$$

as $x \to \infty$*, where* O *stands for the Landau big-*O *notation.*

If α, β, and γ are assumed to be distinct positive real numbers, then the following corollary is an immediate consequence of Theorem 3.2 (*i*).

Corollary 4.3 *Let* $I = (-\infty, b)$ *be an open interval with a real number* b*. Assume that* $\alpha > 0$*,* $\beta > 0$*,* $\gamma > 0$ *are distinct real numbers. Suppose* $y : I \to \mathbb{R}$ *is a three times continuously differentiable function and the limits* $y(b) = \lim_{x \to b^-} y(x)$ *and* $y'(b) = \lim_{x \to b^-} y'(x)$ *exist. If* y *satisfies the inequality* (1.2) *for all* $x \in I$ *and for some* $\varepsilon \geq 0$*, then there exists a solution* $y_1 : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - y_1(x)| \;\leq\; \frac{\varepsilon}{\alpha\beta}\left|\frac{1}{\gamma} - \frac{1}{\gamma-\beta}e^{\beta(b-x)} - \left(\frac{1}{\gamma} - \frac{1}{\gamma-\beta}\right)e^{\gamma(b-x)}\right|$$

for all $x \in I$*. Analogous inequalities hold for every permutation of* α*,* β*,* γ*.*

The following corollary follows from the 4th or the 5th inequality of Corollary 4.3 and Theorem 3.2 (*iii*).

Corollary 4.4 *Let* $I = (-\infty, b)$ *be an open interval with* $b < \infty$*. Assume that* α*,* β*,* γ *are positive real numbers. Suppose* $y : I \to \mathbb{R}$ *is a three times continuously differentiable function and the limits* $y(b) = \lim_{x \to b^-} y(x)$ *and* $y'(b) = \lim_{x \to b^-} y'(x)$ *exist. Moreover, assume that* y *satisfies the inequality* (1.2) *for all* $x \in I$ *and for some* $\varepsilon \geq 0$*.*

(*i*) *If* $\gamma > \beta > \alpha > 0$*, then there exists a solution* $\hat{y} : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| = o\Big(e^{\gamma(b-x)}\Big)$$

as $x \to -\infty$*.*

(*ii*) *If* $\alpha = \beta = \gamma > 0$*, then there exists a solution* $\hat{y} : I \to \mathbb{R}$ *of the differential equation* (1.1) *such that*

$$|y(x) - \hat{y}(x)| = O\Big(xe^{\alpha(b-x)}\Big)$$

as $x \to -\infty$.

Open Problem 1. Are Theorems 3.1 and 3.2 also true for the case when some of α, β, and γ are complex numbers and the range of y is $\mathbb{C}$?

Open Problem 2. Are Theorems 3.1 and 3.2 also true for the case of $I = \mathbb{R}$?

5 Discussion

Let $I = (a, \infty)$ be an open interval with a real number a. Suppose $y : I \to \mathbb{R}$ is a three times continuously differentiable function and the limits $y(a) = \lim_{x \to a^+} y(x)$ and $y'(a) = \lim_{x \to a^+} y'(x)$ exist. Moreover, assume that y satisfies the inequality

$$\big| y'''(x) - 6y''(x) + 11y'(x) - 6y(x) \big| \leq \varepsilon \tag{5.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$.

According to Theorem 3.1 (i), there exist solutions y_1, $y_2 : I \to \mathbb{R}$ of the differential equation

$$y'''(x) - 6y''(x) + 11y'(x) - 6y(x) = 0 \tag{5.2}$$

such that

$$|y(x) - y_1(x)| \leq \varepsilon \left| \frac{1}{3} e^{3(x-a)} - \frac{1}{2} e^{2(x-a)} + \frac{1}{6} \right|$$

and

$$|y(x) - y_2(x)| \leq \varepsilon \left| \frac{1}{12} e^{3(x-a)} - \frac{1}{4} e^{x-a} + \frac{1}{6} \right|$$

for all $x \in I$. Strictly speaking, this is not a Hyers-Ulam stability of the differential equation (5.2).

Under stronger conditions, however, the differential equation (5.2) has the Hyers-Ulam stability. We assume that $\vec{y} : \mathbb{R} \to \mathbb{R}^3$ is a continuously differentiable vector function. We now consider the inequality

$$\big\| \vec{y}\,'(x) - \mathbf{A}\vec{y}(x) \big\|_\infty \leq \varepsilon \tag{5.3}$$

for all $x \in \mathbb{R}$ and for some $\varepsilon \geq 0$, where

$$\vec{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}.$$

According to [14, Theorem 2], there exists a differentiable vector function $\vec{w}$: $\mathbb{R} \to \mathbb{R}^3$ such that

$$\vec{w}\,'(x) = \mathbf{A}\vec{w}(x)$$

and

$$\big\|\vec{y}(x) - \vec{w}(x)\big\|_\infty \le \varepsilon \|\mathbf{N}\|_\infty \big\|\mathbf{N}^{-1}\big\|_\infty \|\mathbf{B}\vec{e}\|_\infty,$$

where

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \qquad \mathbf{N}^{-1} = \begin{pmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

and $\vec{e} = (\,1\;1\;1\,)^{\mathrm{tr}}$. That is, if we set $w_1(x) = w(x)$, then there exists a differentiable function $w : \mathbb{R} \to \mathbb{R}$ such that

$$w'''(x) - 6w''(x) + 11w'(x) - 6w(x) = 0$$

and

$$|y_1(x) - w(x)| \le 112\varepsilon, \quad |y_2(x) - w'(x)| \le 112\varepsilon, \quad |y_3(x) - w''(x)| \le 112\varepsilon$$

for every $x \in \mathbb{R}$. This provides the Hyers-Ulam stability of the differential equation (5.2). (We know that $\vec{y}\,'(x) = \mathbf{A}\vec{y}(x)$ is equivalent to the differential equation (5.2).)

We remark that the inequality (5.3) is equivalent to the inequalities

$$\begin{cases} |y_1'(x) - y_2(x)| \le \varepsilon, \\ |y_2'(x) - y_3(x)| \le \varepsilon, \\ |y_3'(x) - 6y_1(x) + 11y_2(x) - 6y_3(x)| \le \varepsilon \end{cases}$$

for all $x \in \mathbb{R}$, which in general seem to be stronger than the condition (5.1).

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