

Jordan Curves in the Digital Plane

JOSEF ŠLAPAL

Institute of Mathematics, IT4Innovations, Brno University of Technology,
616 69 Brno, Czech Republic
slapal@fme.vutbr.cz

Abstract

We discuss certain interrelated pretopologies on the digital plane \mathbb{Z}^2 including the Khalimsky topology and several other topologies on \mathbb{Z}^2 . We present a digital analogue of the Jordan curve theorem for each of the pretopologies to demonstrate that they can provide background structures on \mathbb{Z}^2 convenient for the study of geometric and topological properties of two-dimensional digital images.

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1 Introduction

Geometric and topological properties of (two-dimensional) digital images are crucial in creating new, efficient algorithms to solve various problems of computer image processing. To study such properties, we need the digital plane \mathbb{Z}^2 to be provided with a convenient structure. Here, convenience means that such a structure satisfies some analogues of basic geometric and topological properties of the Euclidean topology on \mathbb{R}^2 . Most importantly, it is usually required that an analogue of the Jordan curve theorem be valid. (Recall that the classical Jordan curve theorem states that any simple closed curve in the Euclidean plane separates this plane into exactly two components.) In the classical approach to this problem (see e.g. [11] and [12]), graph theoretic tools were used for structuring \mathbb{Z}^2 , namely the well-known binary relations of 4-adjacency and 8-adjacency. Unfortunately, neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem - cf. [8]. To overcome this, a combination of the two binary relations has to be used. Despite this inconvenience, the graph-theoretic approach is used to solve many problems of digital image processing and to create useful graphic software. In [5], a new, purely topological approach to the problem was proposed which utilizes a convenient

topology on \mathbb{Z}^2 , called Khalimsky topology (cf. [4]), for structuring the digital plane. At present, this topology is one of the most important concepts of the theory called digital topology. It has been studied and used by many authors, e.g., [2] and [6]-[9]. The possibility of structuring \mathbb{Z}^2 by using closure operators more general than the Kuratowski ones is discussed in [13]-[14] and [17]-[18].

In [15], a new, digital topology on \mathbb{Z}^2 has been introduced and studied. This topology was shown to have some advantages over the Khalimsky one. More precisely, it was proved in [15] that all cycles in a certain natural graph with the vertex set \mathbb{Z}^2 are Jordan curves with respect to the new topology. Some further Jordan curves with respect to it were identified in [19]. The topology was further investigated in [16] where four of its quotient pretopologies including the Khalimsky topology were studied. Jordan curve theorems with respect to the four quotient pretopologies were then proved in [17]. In the present paper, we continue to discuss pretopologies on \mathbb{Z}^2 possessing with a rich variety of convenient Jordan curves. Some of the pretopologies discussed here are shown to be quotient pretopologies of a pretopology introduced in [18] and this fact is then used to prove Jordan curve theorems for them. Since Jordan curves represent boundaries of regions in digital images, the structures proposed by the pretopologies may especially be used for solving problems of computer image processing which are related to these boundaries.

2 Preliminaries

By a *pretopology* p on a set X we mean a map $p: \exp X \rightarrow \exp X$ (where $\exp X$ denotes the power set of X) fulfilling the following three axioms:

- (i) $p\emptyset = \emptyset$,
- (ii) $A \subseteq pA$ for all $A \subseteq X$,
- (iii) $p(A \cup B) = pA \cup pB$ whenever $A, B \subseteq X$.

The pair (X, p) is then called a *pretopological space*. A subset $A \subseteq X$ is called *closed* if $pA = A$, and it is called *open* if $X - A$ is closed. Observe that the closure of a subset of X need not be closed.

Pretopological spaces were studied in great detail by E. Čech in [1] (and, therefore, they are sometimes called Čech closure operators). A pretopology p on a set X that is idempotent, i.e., satisfies $ppA = pA$ whenever $A \subseteq X$, is nothing but the well-known Kuratowski closure operator or, briefly, a topology on X (and the pair (X, p) is then a topological space).

If a pretopology p on a set X satisfies the axiom

- (iv) $pA = \bigcup_{x \in A} p\{x\}$ whenever $A \subseteq X$

(which is stronger than (iii)), then p and (X, p) will be called *Alexandroff* (in [1], they are called *quasi-discrete*). So, if p is an Alexandroff pretopology on a set X , then it is given by determining the closures of all points of X and there

is an Alexandroff pretopology \bar{p} on X given by $x \in \bar{p}\{y\} \Leftrightarrow y \in p\{x\}$ whenever $x, y \in X$. The pretopology \bar{p} is said to be *dual* to p . Clearly, $\bar{\bar{p}} = p$ and a subset $A \subseteq X$ is closed (open) in (X, p) if and only if it is open (closed) in (X, \bar{p}) .

We will work with some basic topological concepts (see e.g. [3]) naturally extended from topological spaces to pretopological ones. A pretopological space (X, p) is said to be a *subspace* of a pretopological space (Y, q) if $X \subseteq Y$ and $pA = qA \cap X$ for each subset $A \subseteq X$. In this case we may simply say that X is a subspace of (Y, q) without explicitly mentioning the pretopology on X . A pretopological space (X, p) is said to be *connected* if \emptyset and X are the only subsets of X which are both closed and open. A subset $X \subseteq Y$ is considered to be connected in a pretopological space (Y, q) if the subspace X of (Y, q) is connected. A maximal connected subset of a pretopological space is called a *component* of this space. Basic properties of connected sets and components in topological spaces (see e.g. [4]) are preserved also in pretopological ones. A pretopology p on a set X is said to be a T_0 -*pretopology* if, for arbitrary points $x, y \in X$, from $x \in p\{y\}$ and $y \in p\{x\}$ it follows that $x = y$, and it is called a $T_{1/2}$ -*pretopology* if each singleton subset of X is closed or open (so that $T_{1/2}$ implies T_0).

If p and q are pretopologies on X such that $pA \subseteq qA$ for every $A \subseteq X$, then we write $p \leq q$ and say that p is *finer* than q (or that q is *coarser* than p).

A map $f : (X, p) \rightarrow (Y, q)$ between pretopological spaces (X, p) and (Y, q) is said to be *continuous* if $f(pA) \subseteq q(f(A))$ whenever $A \subseteq X$ - cf. [3] (or [1], respectively).

Given a topological space (X, p) and a surjection $e : X \rightarrow Y$, a pretopology q on Y is called the *quotient pretopology* of p generated by e if q is the finest pretopology on Y for which $e : (X, p) \rightarrow (Y, q)$ is continuous.

By a *graph* on a set V , we always mean an undirected simple graph without loops whose vertex set is V . Recall that a *path* in a graph is a finite (nonempty) sequence x_0, x_1, \dots, x_n of pairwise different vertices such that x_{i-1} and x_i are adjacent (i.e., joined by an edge) whenever $i \in \{1, 2, \dots, n\}$. By a *cycle* in a graph we understand any finite set of at least three vertices which can be ordered into a path whose first and last members are adjacent.

The *connectedness graph* of a pretopology p on X is the graph on X in which a pair of vertices x, y is adjacent if and only if $x \neq y$ and $\{x, y\}$ is a connected subset of (X, p) . Let p be an Alexandroff pretopology on a set X . Then a subset $A \subseteq X$ is connected in (X, p) if and only if each pair of points of A may be joined by a path in the connectedness graph of (X, p) contained in A . Clearly, p is given by its connectedness graph provided that every edge of the graph is adjacent to a point which is known to be closed or to a point which is known to be open (in which case p is T_0). Indeed, the closure of a closed point consists of just this point, the closure of an open point consists of this point and all points adjacent to it and the closure of a mixed point (i.e., a point that is neither closed nor open) consists of this point and all closed points adjacent to it. In the sequel, only connected Alexandroff pretopologies on \mathbb{Z}^2 will be dealt with. In connectedness graphs of these pretopologies, the closed points will be ringed and the mixed ones boxed (so that the points neither ringed nor boxed

will be open - note that no of the points of \mathbb{Z}^2 may be both closed and open). Obviously, there are 2^{\aleph_0} Alexandroff T_0 -pretopologies on \mathbb{Z}^2 having the same given connectedness graph.

For every point $(x, y) \in \mathbb{Z}^2$, we denote by $A_4(x, y)$ and $A_8(x, y)$ the set of all points that are 4-adjacent to (x, y) and that of all points that are 8-adjacent to (x, y) , respectively. Thus, $A_4(x, y) = \{(x+i, y+j); i, j \in \{-1, 0, 1\}, ij = 0, i+j \neq 0\}$ and $A_8(x, y) = A_4(x, y) \cup \{(x+i, y+j); i, j \in \{-1, 1\}\}$. For natural reasons related to possible applications of our results in digital image processing, only such pretopologies on \mathbb{Z}^2 will be dealt with whose connectedness graphs are subgraphs of the 8-adjacency graph. (It is well known [3] that there are exactly two topologies on \mathbb{Z}^2 whose connectedness graphs lie between the 4-adjacency and 8-adjacency graphs. These topologies are the Khalimsky and Marcus-Wyse ones - see below.)

By a (*digital*) *simple closed curve* in a pretopological space (\mathbb{Z}^2, p) we mean, in accordance with [16], a nonempty, finite and connected subset $C \subseteq \mathbb{Z}^2$ such that, for each point $x \in C$, there are exactly two points of C adjacent to x in the connectedness graph of p . A simple closed curve C in (\mathbb{Z}^2, p) is said to be a (*digital*) *Jordan curve* if it separates (\mathbb{Z}^2, p) into precisely two components (i.e., if the subspace $\mathbb{Z}^2 - C$ of (\mathbb{Z}^2, p) consists of precisely two components).

3 Jordan curves with respect to Alexandroff pretopologies on \mathbb{Z}^2

Definition 3.1 The *square-diagonal graph* is the graph on \mathbb{Z}^2 in which two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{Z}^2$ are adjacent if and only if one of the following four conditions is fulfilled:

1. $|y_1 - y_2| = 1$ and $x_1 = x_2 = 4k$ for some $k \in \mathbb{Z}$,
2. $|x_1 - x_2| = 1$ and $y_1 = y_2 = 4l$ for some $l \in \mathbb{Z}$;
3. $x_1 - x_2 = y_1 - y_2 = \pm 1$ and $x_1 - 4k = y_1$ for some $k \in \mathbb{Z}$,
4. $x_1 - x_2 = y_2 - y_1 = \pm 1$ and $x_1 = 4l - y_1$ for some $l \in \mathbb{Z}$.

A section of the square-diagonal graph is shown in Figure 1.

When studying digital images, it may be helpful to equip \mathbb{Z}^2 with a closure operator with respect to which some or even all cycles in the square-diagonal graph are Jordan curves.

Recall [5] that the Khalimsky topology on \mathbb{Z}^2 is the Alexandroff topology t given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$t\{z\} = \begin{cases} \{z\} \cup A_8(z) & \text{if } x, y \text{ are even,} \\ \{(x+i, y); i \in \{-1, 0, 1\}\} & \text{if } x \text{ is even and } y \text{ is odd,} \\ \{(x, y+j); j \in \{-1, 0, 1\}\} & \text{if } x \text{ is odd and } y \text{ is even,} \\ \{z\} & \text{otherwise.} \end{cases}$$

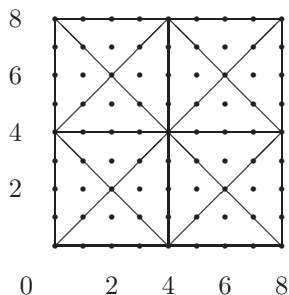


Figure 1: A section of the square-diagonal graph.

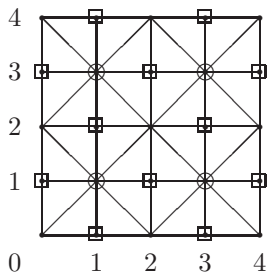


Figure 2: A section of the connectedness graph of the Khalimsky topology.

The Khalimsky topology is connected and T_0 ; a section of its connectedness graph is shown in Figure 2.

Another well-known topology on \mathbb{Z}^2 is the Marcus-Wyse one (cf. [10]), i.e., the Alexandroff topology s on \mathbb{Z}^2 given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$s\{z\} = \begin{cases} \{z\} \cup A_4(z) & \text{if } x + y \text{ is odd,} \\ \{z\} & \text{otherwise.} \end{cases}$$

The Marcus-Wyse topology is connected and $T_{1/2}$. A section of its connectedness graph is shown in Figure 3.

The topologies \bar{t} and \bar{s} dual to t and s are also called the Khalimsky and Marcus-Wyse topologies, respectively. It is readily confirmed that a cycle in the square-diagonal graph is a Jordan curve in the Marcus-Wyse topological space if and only if it does not employ diagonal edges. And a cycle in the square-diagonal graph is a Jordan curve in the Khalimsky topological space if and only if it does not turn, at any of its points, at the acute angle $\pi/4$ - cf. [5]. It could therefore be useful to replace the Khalimsky and Marcus-Wyse topologies with some more convenient connected topologies or pretopologies on \mathbb{Z}^2 that allow Jordan curves to turn at the acute angle $\pi/4$ at some points.

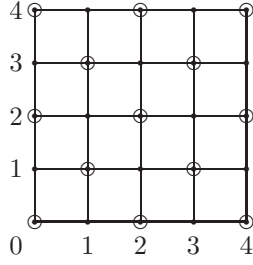


Figure 3: A section of the connectedness graph of the Marcus-Wyse topology.

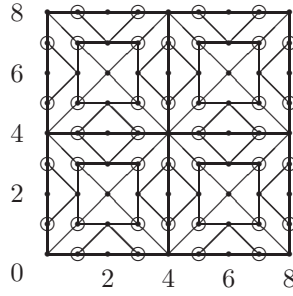


Figure 4: A section of the connectedness graph of w .

We denote by w the Alexandroff pretopology on \mathbb{Z}^2 given as follows:
For any point $z = (x, y) \in \mathbb{Z}^2$,

$$w\{z\} = \begin{cases} \{z\} \cup A_8(z) & \text{if } x = 4k, y = 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - A_4(z)) & \text{if } x = 2 + 4k, y = 2 + 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x+i, y+1); i \in \{-1, 0, 1\}\} \cup \{(x, y-1)\})) & \text{if} \\ & x = 2 + 4k, y = 1 + 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x+i, y-1); i \in \{-1, 0, 1\}\} \cup \{(x, y+1)\})) & \text{if} \\ & x = 2 + 4k, y = 3 + 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x+1, y+j); j \in \{-1, 0, 1\}\} \cup \{(x-1, y)\})) & \text{if} \\ & x = 1 + 4k, y = 2 + 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x-1, y+j); j \in \{-1, 0, 1\}\} \cup \{(x+1, y)\})) & \text{if} \\ & x = 3 + 4k, y = 2 + 4l, k, l \in \mathbb{Z}, \\ \{(x+i, y); i \in \{-1, 0, 1\}\} & \text{if } x = 2 + 4k, y = 4l, k, l \in \mathbb{Z}, \\ \{(x, y+j); j \in \{-1, 0, 1\}\} & \text{if } x = 4k, y = 2 + 4l, k, l \in \mathbb{Z}, \\ \{z\} & \text{otherwise.} \end{cases}$$

Clearly, w is a connected $T_{1/2}$ -topology. A section of the connectedness graph of w is shown in Figure 6.

The topology w was introduced in [15] and then studied also in [16] and [19]. The following statement is proved in [15] (Theorem 11):

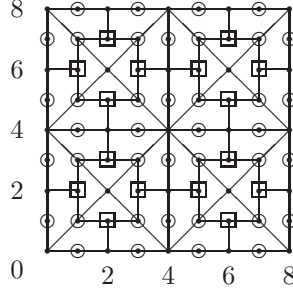


Figure 5: A section of the connectedness graph of r .

Theorem 3.2 *Every cycle in the square-diagonal graph is a Jordan curve in (\mathbb{Z}^2, w) .*

We denote by r the Alexandroff pretopology on \mathbb{Z}^2 given as follows:

For any point $z = (x, y) \in \mathbb{Z}^2$,

$$r\{z\} = \begin{cases} \{z\} \cup A_8(z) & \text{if } x = 4k, y = 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - A_4(z)) & \text{if } x = 2 + 4k, y = 2 + 4l, k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x-1, y), (x+1, y)\} & \text{if } x = 2 + 4k, y = 1 + 2l, k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x, y-1), (x, y+1)\} & \text{if } x = 1 + 2k, \text{ and } y = 2 + 4l, \\ \{z\} \cup A_4(z) & \text{if either } x = 4k \text{ and } y = 2 + 4l \text{ or } x = 2 + 4k \text{ and} \\ & y = 4l, k, l \in \mathbb{Z}, \\ \{z\} & \text{otherwise.} \end{cases}$$

Clearly, r is connected and T_0 . A section of the connectedness graph of r is shown in Figure 4.

Theorem 3.3 *Every cycle in the square-diagonal graph is a Jordan curve in (\mathbb{Z}^2, r) .*

Proof. A proof similar to that of Theorem 11 in [15] may be applied to prove the statement. We present only a sketch the proof. Clearly, any cycle in the square-diagonal graph is a simple closed curve in (\mathbb{Z}^2, r) . Let $z = (x, y) \in \mathbb{Z}^2$ be a point such that $x = 4k + p$ and $y = 4l + q$ for some $k, l, p, q \in \mathbb{Z}$ with $pq = \pm 2$. Then we define the *fundamental triangle* $T(z)$ to be the nine-point subset of \mathbb{Z}^2 given as follows:

$$T(z) = \begin{cases} \{(s, t) \in \mathbb{Z}^2; y-1 \leq t \leq y+1 - |s-x|\} & \text{if } x = 4k+2 \text{ and} \\ & y = 4l+1 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s, t) \in \mathbb{Z}^2; y-1 + |s-x| \leq t \leq y+1\} & \text{if } x = 4k+2 \text{ and} \\ & y = 4l-1 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s, t) \in \mathbb{Z}^2; x-1 \leq s \leq x+1 - |t-y|\} & \text{if } x = 4k+1 \text{ and} \\ & y = 4l+2 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s, t) \in \mathbb{Z}^2; x-1 + |t-y| \leq s \leq x+1\} & \text{if } x = 4k-1 \text{ and} \\ & y = 4l+2 \text{ for some } k, l \in \mathbb{Z}. \end{cases}$$

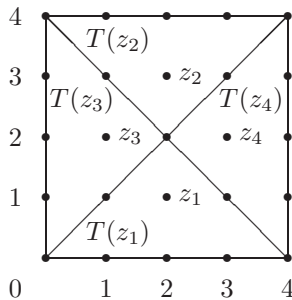


Figure 6: The four types of fundamental triangles.

Graphically, the fundamental triangle $T(z)$ consists of the point z and the eight points lying on the triangle surrounding z - the four types of fundamental triangles are represented in Figure 5.

Given a fundamental triangle, we speak about its sides - it is clear from the above picture what sets are understood to be the sides (note that each side consists of five or three points and that two different fundamental triangles may have at most one common side).

Now, one can easily see that:

1. Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected) in (\mathbb{Z}^2, r) .
2. If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected in (\mathbb{Z}^2, r) .
3. If S_1, S_2 are fundamental triangles having a common side D , then the set $(S_1 \cup S_2) - M$ is connected in (\mathbb{Z}^2, r) whenever M is the union of some sides of S_1 or S_2 different from D .
4. Every connected subset of (\mathbb{Z}^2, r) having at most two points is a subset of a fundamental triangle.

Further, using arguments analogous to those used in the proof of Theorem 11 in [15], may show that the following is also true:

5. For every cycle C in the square-diagonal graph, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, \mathcal{S}_F finite and \mathcal{S}_I infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:
 - (a) Each member of \mathcal{S} , excluding the first one, has a common side with at least one of its predecessors.
 - (b) C is the union of those sides of fundamental triangles from \mathcal{S} that are not shared by two different fundamental triangles from \mathcal{S} .

Given a cycle C in the square-diagonal graph, let S_F and S_I denote the union of all members of \mathcal{S}_F and \mathcal{S}_I , respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$.

Let \mathcal{S}_F^* and \mathcal{S}_I^* be the sequences obtained from \mathcal{S}_F and \mathcal{S}_I by subtracting C from each member of \mathcal{S}_F and \mathcal{S}_I , respectively. Let S_F^* and S_I^* denote the union of all members of \mathcal{S}_F^* and \mathcal{S}_I^* , respectively. Then S_F^* and S_I^* are connected by (1), (2) and (3) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So, S_F^* and S_I^* are the two components of $\mathbb{Z}^2 - C$ by (4) ($S_F - C$ is the so-called *inside* component and $S_I - C$ is the so-called *outside* component). This proves the statement. ■

Let r' , w' and w'' be the Alexandroff pretopologies on \mathbb{Z}^2 with sections of their connectedness graphs shown in Figure 7. Clearly, r' , w' and w'' are T_0 -pretopologies, w' is even a topology and we have $r \leq r'$ and $w \leq w' \leq w''$. Since the connectedness graph of each of the pretopologies r' , w' and w'' is obtained from the connectedness graph of r or w by inserting some new edges between pairs of different vertices belonging to the same fundamental triangle (see the proof of the Theorem 3.2), Theorems 3.2 and 3.3 imply:

Corollary 3.4 *Every cycle in the square-diagonal graph is a Jordan curve in each of the pretopological spaces (\mathbb{Z}^2, r') , (\mathbb{Z}^2, w') and (\mathbb{Z}^2, w'') .*

Of course, Theorems 3.2 and 3.3 and Corollary 3.4 remain valid when replacing Alexandroff pretopologies r , w and r' , w' , w'' by their duals, respectively (i.e., when interchanging open and closed points in these topologies).

4 Jordan curves with respect to quotient pretopologies of r

The following three Lemmas immediately follow from [16], Theorems 10, 12 and 14.

Lemma 4.1 *The Khalimsky topology t is the quotient pretopology of r generated by the surjection $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given as follows:*

$$f(x, y) = \begin{cases} (2k, 2l) & \text{if } (x, y) = (4k, 4l), \quad k, l \in \mathbb{Z}, \\ (2k, 2l + 1) & \text{if } (x, y) \in A_4(4k, 4l + 2), \quad k, l \in \mathbb{Z}, \\ (2k + 1, 2l) & \text{if } (x, y) \in A_4(4k + 2, 4l), \quad k, l \in \mathbb{Z}, \\ (2k + 1, 2l + 1) & \text{if } (x, y) \in \{(4k + 2, 4l + 2)\} \cup (A_8(4k + 2, 4l + 2) - A_4(4k + 2, 4l + 2)), \quad k, l \in \mathbb{Z}. \end{cases}$$

The decomposition of the pretopological space (\mathbb{Z}^2, r) given by f is demonstrated in Figure 8 by the dashed lines. Every class of the decomposition is mapped by f to its center point expressed by the bold coordinates.

Let v be the Alexandroff topology on \mathbb{Z}^2 given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$v\{z\} = \begin{cases} \{(x + i, y); i \in \{-1, 0, 1\}\} & \text{if } x \text{ is odd and } y \text{ is even,} \\ \{(x, y + j); j \in \{-1, 0, 1\}\} & \text{if } x \text{ is even and } y \text{ is odd,} \\ \{z\} \cup (A_8(z) - A_4(z)) & \text{if } x, y \text{ are odd,} \\ \{z\} & \text{if } x, y \text{ are even.} \end{cases}$$

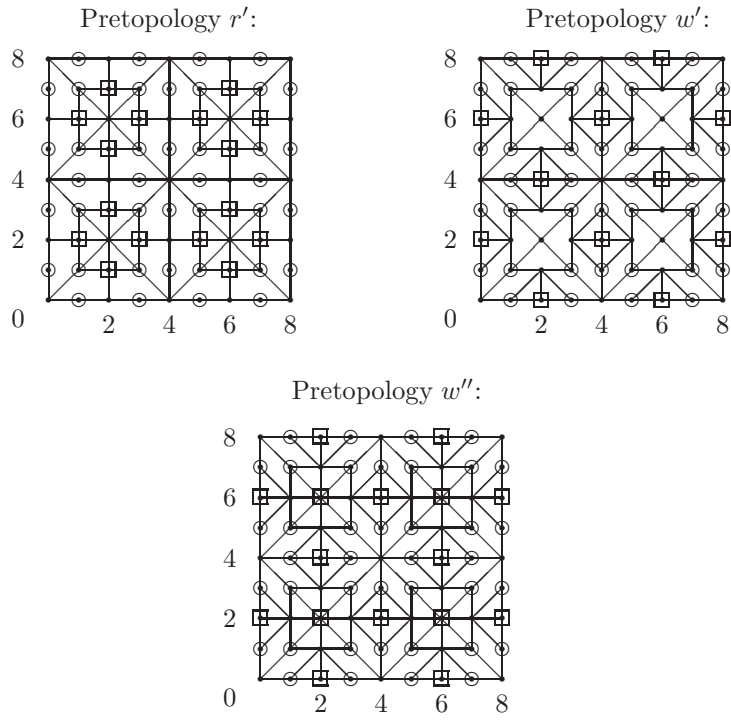


Figure 7: sections of the connectedness graphs of r' , w' and w'' .

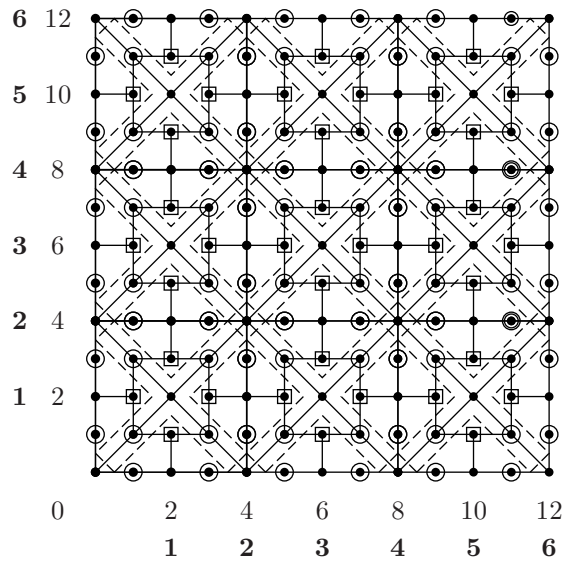


Figure 8: Decomposition of (\mathbb{Z}^2, r) given by the surjection f .

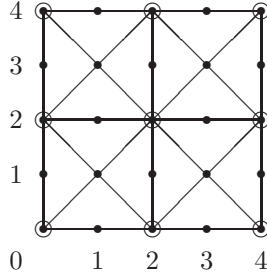


Figure 9: A section of the connectedness graph of v .

Evidently, v is connected and T_0 . A section of its connectedness graph is shown in Figure 9.

Lemma 4.2 v is the quotient pretopology of r generated by the surjection $h : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given as follows:

$$h(x, y) = \begin{cases} (2k, 2l) & \text{if } (x, y) \in \{(4k, 4l)\} \cup A_8(4k, 4l), \quad k, l \in \mathbb{Z}, \\ (2k, 2l + 1) & \text{if } (x, y) \in \{(4k + i, 4l + 2)\}; \quad i \in \{-1, 0, 1\}, \quad k, l \in \mathbb{Z}, \\ (2k + 1, 2l) & \text{if } (x, y) \in \{(4k + 2, 4l + j)\}; \quad j \in \{-1, 0, 1\}, \quad k, l \in \mathbb{Z}, \\ (2k + 1, 2l + 1) & \text{if } (x, y) = (4k + 2, 4l + 2), \quad k, l \in \mathbb{Z}. \end{cases}$$

The decomposition of the pretopological space (\mathbb{Z}^2, r) given by h is demonstrated in Figure 10 by the dashed lines. Every class of the decomposition is mapped by h to its center point expressed in the bold coordinates.

Let u be the Alexandroff pretopology on \mathbb{Z}^2 defined as follows:
For any point $z = (x, y) \in \mathbb{Z}^2$,

$$u\{z\} = \begin{cases} \{z\} \cup A_4(z) & \text{if both } x \text{ and } y \text{ are odd or } (x, y) = (4k + 2l, 2l + 2), \\ & k, l \in \mathbb{Z}, \\ \{z\} \cup A_8(z) & \text{if } (x, y) = (4k + 2l, 2l), \quad k, l \in \mathbb{Z}, \\ \{z\} & \text{otherwise.} \end{cases}$$

Then u is connected and T_0 . A section of its connectedness graph is shown in Figure 11.

Lemma 4.3 u is the quotient pretopology of r generated by the surjection $d : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given as follows:

$$d(x, y) = \begin{cases} (2k + 2l + 1, 2l - 2k + 1) & \text{if } (x, y) \in \{(4k, 4l + 2)\} \cup A_4(4k, 4l + 2), \\ & k, l \in \mathbb{Z}, \\ (2k + 2l + 1, 2l - 2k - 1) & \text{if } (x, y) \in \{(4k + 2, 4l)\} \cup A_4(4k + 2, 4l), \\ & k, l \in \mathbb{Z}, \\ ((x + y)/2, (y - x)/2) & \text{if } x, y \text{ are odd or } (x, y) = (4k + 2l, 2l), \quad k, l \in \mathbb{Z}. \end{cases}$$

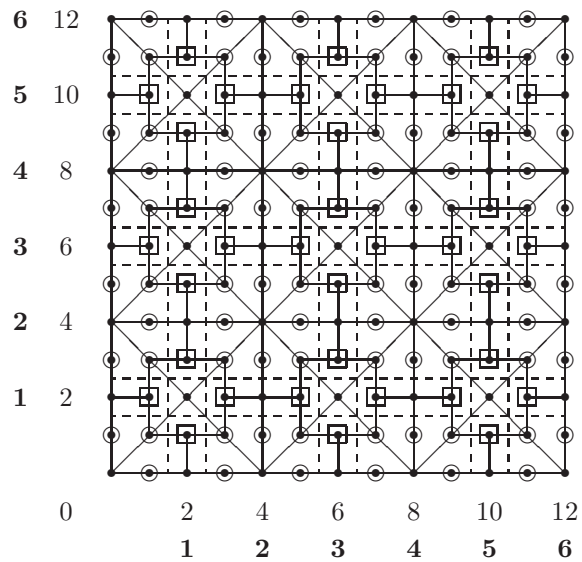


Figure 10: Decomposition of (\mathbb{Z}^2, r) given by the surjection h .

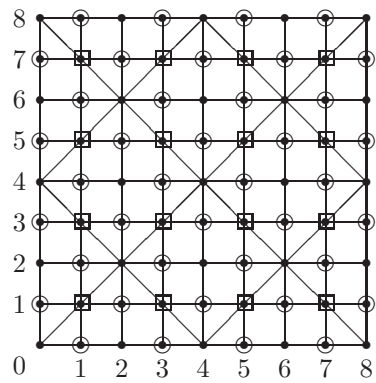


Figure 11: A section of the connectedness graph of u .

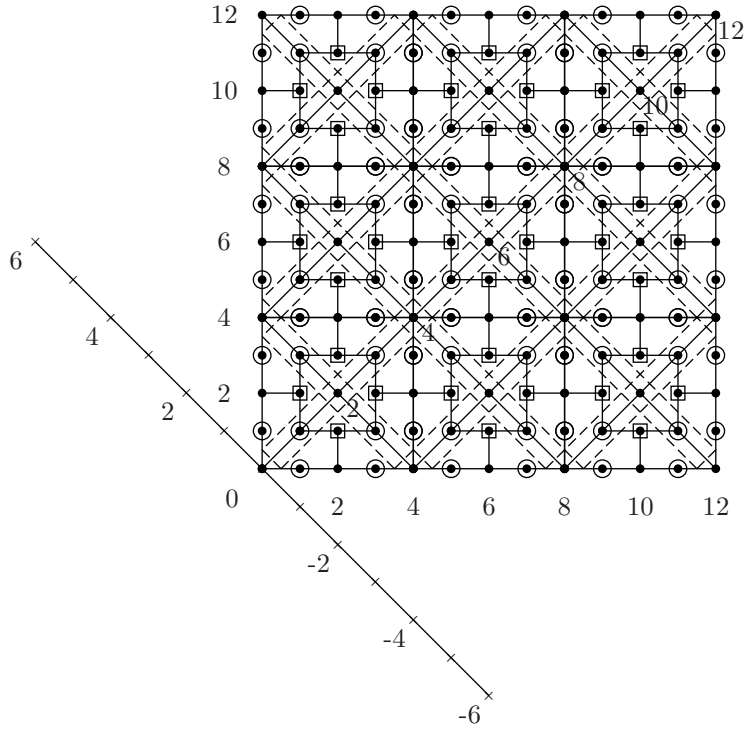


Figure 12: Decomposition of (\mathbb{Z}^2, r) given by the surjection d .

The decomposition of the pretopological space (\mathbb{Z}^2, r) given by d is demonstrated in Figure 12 by the dashed lines. Every class of the decomposition is mapped by d to its center point expressed by the coordinates with respect to the diagonal axes (where the first coordinate relates to the axis with only the non-negative part displayed).

Remark 4.4 It follows from [16], Theorem 11, that also the Marcus-Wyse topology s is a quotient pretopology of r , namely that one generated by the surjection $g : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given as follows:

$$g(x, y) = \begin{cases} (k + l, l - k) & \text{if } (x, y) \in \{(4k, 4l)\} \cup A_8(4k, 4l), \quad k, l \in \mathbb{Z}, \\ (k + l + 1, l - k) & \text{if } (x, y) = (4k + 2, 4l + 2) \text{ for some } k, l \in \mathbb{Z} \\ & \text{with } k + l \text{ odd or } (x, y) \in \{(p, q) \in \mathbb{Z}^2; \text{ both } x = 4k + 2 \\ & \text{and } |y - 4l - 2| \leq 3 \text{ or both } |x - 4k - 2| \leq 3 \text{ and} \\ & y = 4l + 2\} \text{ for some } k, l \in \mathbb{Z} \text{ with } k + l \text{ even.} \end{cases}$$

In [16], the statements of Lemmas 4.1-4.3 and Remark 4.4 are proved for the topology w instead of the pretopology r . It is obvious that the statements remain valid when replacing w by r . Moreover, then the inverse image of every singleton under any of the three mappings f , h and d is a connected subset of

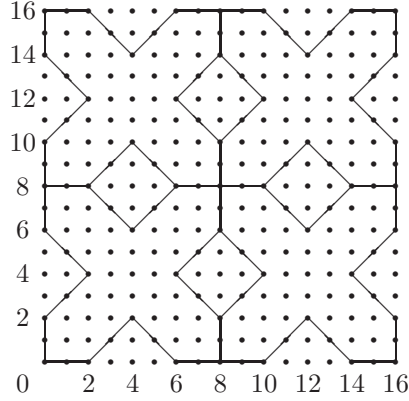


Figure 13: A subgraph of the connectedness graph of v .

(\mathbb{Z}^2, r) which is evidently not true for h in the case when (\mathbb{Z}^2, w) is considered instead of (\mathbb{Z}^2, r) . (As for the inverse images of singletons under the mapping g from Remark 4.4, they need not be connected subsets of either (\mathbb{Z}^2, r) or (\mathbb{Z}^2, w) .) Thus, if $p = r$, $q \in \{t, v, u\}$ and e is the respective surjection f , h or d , then the assumptions of the following statement proved in [17] are satisfied:

Proposition 4.5 *Let (\mathbb{Z}^2, p) be a pretopological space and let q be the quotient pretopology of p on \mathbb{Z}^2 generated by a surjection $e : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$. Let e have the property that $e^{-1}(\{y\})$ is connected for every point $y \in \mathbb{Z}^2$ and let $D \subseteq \mathbb{Z}^2$ be a simple closed curve in (\mathbb{Z}^2, q) . Then D is a Jordan curve in (\mathbb{Z}^2, p) if the following two conditions are fulfilled:*

- (1) *There is a Jordan curve C in (\mathbb{Z}^2, p) such that $e(C) = D$.*
- (2) *$C_i - e^{-1}(D)$ is nonempty and connected in (\mathbb{Z}^2, p) for $i = 1, 2$ where C_1 and C_2 are the two components of $\mathbb{Z}^2 - C$.*

Using Proposition 4.5, Theorem 3.2 and Lemmas 4.1-4.3, we may identify Jordan curves among the simple closed curves in the Khalimsky plane, in (\mathbb{Z}^2, v) and in (\mathbb{Z}^2, u) . This will be done in the following three Theorems which can be proved in much the same way and, therefore, we will present only the proof of the last of them.

Theorem 4.6 *Let D be a simple closed curve in the Khalimsky plane such that every point $z \in D$ with both coordinates odd satisfies $A_4(z) \cap D = \emptyset$. Then D is a Jordan curve in the Khalimsky plane.*

Theorem 4.7 *Let D be a simple closed curve in (\mathbb{Z}^2, v) having more than four points and such that every pair of different points $z_1, z_2 \in D$ with both coordinates even satisfies $A_4(z_1) \cap A_4(z_2) \subseteq D$. Then D is a Jordan curve in (\mathbb{Z}^2, v) .*

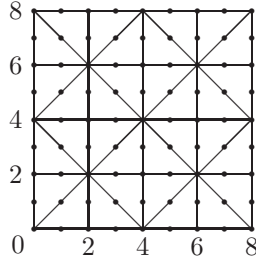


Figure 14: A subgraph of the connectedness graph of u .

Corollary 4.8 *Every cycle in the graph a section of which is shown in Figure 13 is a Jordan curve in (\mathbb{Z}^2, v) .*

Theorem 4.9 *Every simple closed curve D in (\mathbb{Z}^2, u) which is a cycle in the graph a section of which is shown in Figure 14 is a Jordan curve in (\mathbb{Z}^2, u) .*

Proof. Let D be a simple closed curve in (\mathbb{Z}^2, u) which is a cycle in the graph a section of which is shown in the above figure. By Lemma 4.3, u is the quotient pretopology of r generated by d . It immediately follows from the definition of d that there exists a unique cycle C in the square-diagonal graph such that $d(C) = D$. By Theorem 3.2, C is a Jordan curve in (\mathbb{Z}^2, r) . Let C_1, C_2 be the two components of $\mathbb{Z}^2 - C$ in (\mathbb{Z}^2, r) and put $C'_i = C_i - d^{-1}(D)$ for $i = 1, 2$. Let $(x, y) \in D$ be a point and write $d^{-1}(x, y)$ briefly instead of $d^{-1}(\{(x, y)\})$. Clearly, $d^{-1}(x, y) \not\subseteq C$ if and only if both x and y are odd ($d^{-1}(x, y)$ is a singleton if x or y is even). Thus, let $(x, y) \in D$ be a point with both x and y odd. Then one of the following two cases occurs:

(1) $(x, y) = (2k + 2l + 1, 2k - 2l + 1)$ for some $k, l \in \mathbb{Z}$. Then $d^{-1}(x, y) = A_4(4k, 4l + 2)$, hence $C \cap d^{-1}(x, y) = \{(4k, 4l + 2 + i), i \in \{-1, 0, 1\}\}$. Therefore, there is $i \in \{1, 2\}$ such that $(4k - 1, 4l + 2) \in C_i - C'_i$ and $(4k + 1, 4l + 2) \in C_{3-i} - C'_{3-i}$ while $(4k - 2, 4l + 2) \in C'_i$ and $(4k + 2, 4l + 2) \in C'_{3-i}$, $i \in \{1, 2\}$. The points $(4k - 1, 4l + 1)$ and $(4k - 1, 4l + 3)$ are the only points that belong to C_i and are adjacent (in the connectedness graph of r) to $(4k - 1, 4l + 2)$. But both of these points are adjacent also to $(4k - 2, 4l + 2) \in C'_i$. Thus, $C_i - \{(4k - 1, 4l + 2)\}$ is connected and, by similar arguments, also $C_{3-i} - \{(4k + 1, 4l + 2)\}$ is connected.

(2) $(x, y) = (2k + 2l + 1, 2l - 2k - 1)$ for some $k, l \in \mathbb{Z}$. Then $d^{-1}(x, y) = A_4(4k + 2, 4l)$, hence $C \cap d^{-1}(x, y) = \{(4k + 2 + i, 4l), i \in \{-1, 0, 1\}\}$. Now, analogously to (1), we can easily show that there is $i \in \{1, 2\}$ such that $C_i - \{(4k + 2, 4l - 1)\}$ and $C_{3-i} - \{(4k + 2, 4l + 1)\}$ are connected.

It follows that $C'_1 = d^{-1}(d(C'_1))$ and $C'_2 = d^{-1}(d(C'_2))$ are connected. Therefore, D is a Jordan curve in (\mathbb{Z}^2, u) by Proposition 4.5. ■

Remark 4.10 a) The three Jordan curve theorems presented in Theorems 4.6-4.7 and 4.9 build on the Jordan curve theorem for (\mathbb{Z}^2, r) given in Theorem 3.2. Analogously, building on other Jordan curve theorems for (\mathbb{Z}^2, r) , we may

obtain further Jordan curve theorems in the two topological spaces and the closure space that are quotients of (\mathbb{Z}^2, r) .

b) The procedure based on applying Proposition 4.5 and Theorem 3.2 and used to prove Theorem 4.9 does not lead to the classical result that, in the Khalimsky plane, any simple closed curve having at least four points is a Jordan curve (Khalimsky, Kopperman and Meyer [5]). For example, if D is a simple closed curve in (\mathbb{Z}^2, t) (having at least four points) which turns at a point with both coordinates odd at the angle $3\pi/4$, then none of the Jordan curves C in (\mathbb{Z}^2, w) identified in Theorem 3.2 fulfills $f(C) = D$. Theorem 4.6 is weaker than the classical result.

c) For the Marcus-Wyse topological space, the following Jordan curve theorem may easily be proved: If D be a cycle in the connectedness graph of the Marcus-Wyse topology such that there exists a point $(x, y) \in \mathbb{Z}^2$ with $(x + 2k, y + 2l) \notin D$ whenever $k, l \in \mathbb{Z}$, then D is a Jordan curve in the Marcus-Wyse plane. Another digital Jordan curve theorem for the Marcus-Wyse topology was proved in [7].

Conclusion. We discussed nine connectedness structures on \mathbb{Z}^2 allowing for a considerable variety of Jordan curves. All of these structures are pretopologies on \mathbb{Z}^2 , namely the Khalimsky and Marcus-Wyse topologies, the topologies v , w and w' and the pretopologies u , r , r' and w'' . For each of these nine pretopologies, we presented a Jordan curve theorem identifying Jordan curves admitted by the pretopology. While the Khalimsky and Marcus-Wyse topologies are well known and the Jordan curves they provide have been discussed in the literature by a number of authors (e.g. [5] and [7]), the other pretopologies are new and give specific varieties of Jordan curves. Five of the pretopologies, namely r , w , r' , w' and w'' , allow for especially convenient Jordan curves (all cycles in the square-diagonal graph) which may turn, in some points, at the acute angle $\pi/4$ - such Jordan curves are not allowed in the Khalimsky (or Marcus-Wyse) topology. These pretopologies may advantageously be used as background structures for solving problems of computer image processing, particularly those closely related to boundaries of regions in digital images (image data compression, pattern recognition, boundary detection and contour filling, etc.).

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