FIXED POINT RESULTS FOR COMPLETE DISLOCATED Gd-METRIC SPACE VIA C-CLASS FUNCTIONS

ABDULLAH SHOAIB, ARSLAN HOJAT ANSARI, QASIM MAHMOOD AND AQEEL SHAHZAD

Abstract. In this paper, we discuss unique fixed point results for mappings satisfying contractive condition via C-class functions for a complete dislocated Gd-metric space. Example is also given which shows the novelty of our work. Our results improve/generalize several well known recent and classical results.

1. Introduction and Basic Concepts

In the field of analysis the notion of metric spaces plays an important role in pure and applied science such as biology, physics and computer science. The notion of a G-metric space was introduced by Mustaфа et al. [29].

A point $x \in X$ is said to be a fixed point of mapping $T : X \to X$, if $x = Tx$. Many results appeared related to fixed point for mappings satisfying certain contractive conditions in complete G-metric spaces and dislocated metric spaces(see [1]-[43]). Recently, dislocated quasi G-metric space was introduced by Shoaib et al. [37, 39], which is a generalization of both G-metric spaces and dislocated metric spaces. A class of new C-class functions was recently introduced by Ansari et al. [6].

In this paper, we have obtained fixed point results for contractive self mappings in a complete dislocated Gd-metric space via C-class functions which extend and improve the recent fixed point results proved by Karapınar et al. [23]. An example is also given to support our results.

Definition 1.1 Let $X$ be a nonempty set, and let $G_d : X \times X \times X \to [0, \infty)$, be a function satisfying the following properties:

- $(G_1)$ If $G_d(a, b, c) = 0$, then $a = b = c$;
- $(G_2)$ $G_d(a, a, b) \leq G_d(a, b, c)$, for all $a, b, c \in X$ with $b \neq c$;
- $(G_3)$ $G_d(a, b, c) = G_d(a, c, b) = G_d(b, a, c) = G_d(b, a, c) = G_d(c, a, b) = G_d(c, b, a)$ for all $a, b, c \in X$;
- $(G_4)$ $G_d(a, b, c) \leq G_d(a, d, d) + G_d(d, b, c)$, for all $a, b, c, d \in X$, (rectangle inequality).

Then the function $G_d$ is called a dislocated $G_d$-metric on $X$ and the pair $(X, G_d)$ is called dislocated $G_d$-metric space.
Example 1.2 Let $X = [0, \infty)$ be a nonempty set and $G_d : X \times X \times X \to [0, \infty)$ be a function defined by

$$G_d(a, b, c) = \max\{a, b, c\}, \text{ for all } a, b, c \in X.$$

Then clearly $G_d : X \times X \times X \to [0, \infty)$ is dislocated $G_d$-metric space.

Definition 1.3 Let $(X, G_d)$ be a dislocated $G_d$-metric space, and let $\{x_n\}$ be a sequence of points in $X$, a point $x$ in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G_d(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is $G_d$-convergent to $x$. Thus, if $x_n \to x$ in a dislocated $G_d$-metric space $(X, G_d)$, then for any $\epsilon > 0$, there exist $n, m \in N$ such that $G_d(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 1.4 Let $(X, G_d)$ be a dislocated $G_d$-metric space. A sequence $\{x_n\}$ is called $G_d$-Cauchy sequence if, for $\epsilon > 0$ there exists a positive integer $n^* \in N$ such that $G_d(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n^*$; or $G_d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.5 A dislocated $G_d$-metric space $(X, G_d)$ is said to be $G_d$-complete if every $G_d$-Cauchy sequence in $(X, G_d)$ is $G_d$-convergent in $X$.

Proposition 1.6 Let $(X, G_d)$ be a dislocated $G_d$-metric space, then the following are equivalent:

(i) $\{x_n\}$ is $G_d$ convergent to $x$.
(ii) $G_d(x_n, x_n, x) \to 0$ as $n \to \infty$.
(iii) $G_d(x_n, x, x) \to 0$ as $n \to \infty$.
(iv) $G_d(x_n, x_m, x) \to 0$ as $m, n \to \infty$.

Lemma 1.7 Let $(X, G_d)$ be a dislocated $G_d$-metric space and $\{x_n\}$ be a sequence in $X$ such that $\{G_d(x_n, x_n, x_{n+1})\}$ is decreasing and

$$\lim_{n \to \infty} G_d(x_n, x_n, x_{n+1}) = 0.$$

If $\{x_{2n}\}$ is not a $G_d$-Cauchy sequence, then there exist an $\epsilon > 0$ and $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences $\{G_d(x_{m_k}, x_{n_k}, x_{n_k})\}$, $\{G_d(x_{m_k}, x_{n_k+1}, x_{n_k+1})\}$, $\{G_d(x_{m_k-1}, x_{n_k}, x_{n_k})\}$, $\{G_d(x_{m_k-1}, x_{n_k+1}, x_{n_k+1})\}$ and $\{G_d(x_{m_k}, x_{n_k+1}, x_{n_k+1})\}$ tend to $\epsilon > 0$, when $k \to \infty$.

Definition 1.8 \[4\] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a $C$-class function if it is continuous and satisfies the following axioms:

(i) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;
(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Mention that some $C$-class function $F$ verifies $F(0, 0) = 0$. We denote by $\mathcal{C}$ the set of $C$-class functions.

Example 1.9 \[4\] Following examples show that the class $\mathcal{C}$ is nonempty:

(i) $F(s, t) = s - t$.
(ii) $F(s, t) = ms t$, for some $m \in (0, 1)$.
(iii) $F(s, t) = \frac{t}{1+st}$.

[4] Let $\Phi_{m}$ denote the class of all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(i) $\varphi$ is continuous ;
(ii) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$. 

2. Main Result

**Theorem 2.1:** Let \((X, G_d)\) be a complete dislocated \(G_d\)-metric space, let \(T : X \to X\) be a mapping satisfying
\[
G_d(Ta, Tb, Tc) \leq F(W(a, b, c), \varphi(W(a, b, c))) \tag{2.1}
\]
for all \(a, b, c \in X\), where \(\varphi \in \Phi_u\), and \(F\) is a \(C\) class function.

Here,
\[
W(a, b, c) = \frac{1}{2} \max\{G_d(b, T^2a, Tb), G_d(Ta, T^2a, Tb), G_d(a, Ta, b), G_d(a, Ta, c),
G_d(c, T^2a, Tb), G_d(b, Ta, Tb), G_d(Ta, T^2a, Tc), G_d(c, Ta, Tb),
G_d(a, b, c), G_d(a, Ta, Ta), G_d(b, Tb, Tb), G_d(c, Tc, Tc),
G_d(a, Tb, Tb), G_d(b, Tc, Tc), G_d(c, Ta, Ta)\}. \tag{2.2}
\]

Then, there exists a unique fixed point \(a \in X\) such that \(Ta = a\).

**Proof:** Consider a Picard sequence \(\{a_n\}\) with initial guess \(a_0 \in X\) such that
\[
a_{n+1} = T a_n, \text{ for all } n \in N.
\]
Suppose \(a_{n+1} \neq a_n\), for all \(n \in N \cup \{0\}\). Now, consider the relation
\[
G_d(a_{n}, a_{n+1}, a_{n+1}) = G_d(Ta_{n-1}, Ta_{n}, Ta_{n}) \leq F(W(a_{n-1}, a_{n}, a_{n}), \varphi(W(a_{n-1}, a_{n}, a_{n}))). \tag{2.3}
\]
From (2.2),
\[
W(a_{n-1}, a_{n}, a_{n}) = \frac{1}{2} \max\{G_d(a_{n-1}, a_{n}, a_{n}), G_d(a_{n}, a_{n+1}, a_{n+1}), G_d(a_{n}, a_{n}, a_{n+1}),
G_d(a_{n-1}, a_{n+1}, a_{n+1}), G_d(a_{n}, a_{n}, a_{n})\}.
\]
By Definition 1.1, we have
\[
G_d(a_{n}, a_{n}, a_{n}) \leq G_d(a_{n}, a_{n+1}, a_{n+1}).
\]
So,
\[
W(a_{n-1}, a_{n}, a_{n}) \leq \frac{1}{2} \max\{G_d(a_{n-1}, a_{n}, a_{n}), G_d(a_{n}, a_{n+1}, a_{n+1}),
G_d(a_{n}, a_{n}, a_{n+1}), G_d(a_{n-1}, a_{n+1}, a_{n+1})\}.
\]
In first case, if
\[
W(a_{n-1}, a_{n}, a_{n}) = \frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}),
\]
then, by (2.3)
\[
\frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}) \leq G_d(a_{n}, a_{n+1}, a_{n+1}) \leq F(\frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}), \varphi(\frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}))) \leq \frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}).
\]
Then
\[
F(\frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}), \varphi(\frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}))) = \frac{1}{2} G_d(a_{n}, a_{n+1}, a_{n+1}).
\]
By the property of $F$, we get
\[ \frac{1}{2} G_d(a_n, a_{n+1}, a_{n+1}) = 0 \quad \text{or} \quad \varphi\left(\frac{1}{2} G_d(a_n, a_{n+1}, a_{n+1})\right) = 0. \]

Then,
\[ G_d(a_n, a_{n+1}, a_{n+1}) = 0. \]

It is a contradiction because $a_{n+1} \neq a_n$. Now, in second case, if
\[ W(a_{n-1}, a_n) = \frac{1}{2} G_d(a_n, a_{n+1}), \]
then, we have
\[
\frac{1}{2} G_d(a_n, a_{n+1}) \leq G_d(a_n, a_{n+1}) \leq G_d(a_n, a_{n+1}, a_{n+1}) \\
\leq F\left(\frac{1}{2} G_d(a_n, a_{n+1}), \varphi\left(\frac{1}{2} G_d(a_n, a_{n+1})\right)\right) \\
\leq \frac{1}{2} G_d(a_n, a_{n+1}),
\]
which implies
\[ F\left(\frac{1}{2} G_d(a_n, a_{n+1}), \varphi\left(\frac{1}{2} G_d(a_n, a_{n+1})\right)\right) = \frac{1}{2} G_d(a_n, a_{n+1}). \]

By the property of $F$, we get
\[ \frac{1}{2} G_d(a_n, a_{n+1}) = 0 \quad \text{or} \quad \varphi\left(\frac{1}{2} G_d(a_n, a_{n+1})\right) = 0. \]

Then,
\[ \frac{1}{2} G_d(a_n, a_{n+1}) = 0. \]

It is a contradiction because $a_{n+1} \neq a_n$. In third case, if
\[ W(a_{n-1}, a_n) = \frac{1}{2} G_d(a_{n-1}, a_n), \]
then, we have
\[
G_d(a_n, a_{n+1}, a_{n+1}) \leq F\left(\frac{1}{2} G_d(a_{n-1}, a_n), \varphi\left(\frac{1}{2} G_d(a_{n-1}, a_n)\right)\right) \\
\leq \frac{1}{2} G_d(a_{n-1}, a_n) \\
\leq G_d(a_{n-1}, a_n), \quad (2.4)
\]

In fourth case, if
\[ W(a_{n-1}, a_n) = G_d(a_{n-1}, a_{n+1}, a_{n+1}), \]
then,
\[
G_d(a_n, a_{n+1}, a_{n+1}) \leq F\left(\frac{1}{2} G_d(a_{n-1}, a_{n+1}, a_{n+1}), \varphi\left(\frac{1}{2} G_d(a_{n-1}, a_{n+1}, a_{n+1})\right)\right) \\
\leq \frac{1}{2} G_d(a_{n-1}, a_{n+1}, a_{n+1}) \\
\leq \frac{G_d(a_{n-1}, a_n) + G_d(a_n, a_{n+1}, a_{n+1})}{2} \\
\leq G_d(a_n, a_{n+1}, a_{n+1}), \quad (2.5)
\]

\[ G_d(a_n, a_{n+1}, a_{n+1}) \leq G_d(a_{n-1}, a_n). \]
Hence, by combining (2.4) and (2.5), we have
\[ G_d(a_n, a_{n+1}, a_{n+1}) \leq G_d(a_{n-1}, a_n) \to d. \]

Now, by inequality (2.3) with \( n \to \infty \), we have
\[ d \leq F(d, \varphi(d)), \]
then,
\[ d = 0 \quad \text{or} \quad \varphi(d) = 0. \]

So, we have
\[ \lim_{n \to \infty} G_d(a_n, a_{n+1}, a_{n+1}) = 0. \]

We shall show that \( \{a_n\} \) is a \( G_d \)-Cauchy sequence. Suppose that \( \{a_{2n}\} \) is not a \( G_d \)-Cauchy sequence and from Lemma 1.7, there exists \( \epsilon > 0 \) such that
\[ G_d(a_{m_k+1}, a_{n_k+1}, a_{n_k+1}) \leq F(W(a_{m_k}, a_{n_k}, a_{n_k}), \varphi(W(a_{m_k}, a_{n_k}, a_{n_k}))). \quad (2.6) \]

Now, by using (2.6) as \( k \to \infty \), then
\[ \epsilon \leq F(\epsilon, \varphi(\epsilon)) \leq \epsilon. \]

By the property of \( F \), we get
\[ \epsilon = 0 \quad \text{or} \quad \varphi(\epsilon) = 0. \]

Then, \( \epsilon = 0 \), which is a contradiction. This proves that \( \{a_{2n}\} \) is a \( G_d \)-Cauchy sequence and hence \( \{a_n\} \) is a \( G_d \)-Cauchy sequence. So, we have
\[ G_d(a_n, a_m, a_m) \to 0, \quad \text{as} \quad n \to \infty. \]

Therefore, Picard sequence \( \{a_n\} \) is Cauchy sequence in \( X \). Hence, \( a_n \to a \) as
\[ n \to \infty. \quad \text{In general it is clear that}, \]
\[ \lim_{n \to \infty} G_d(a_n, a, a) = \lim_{n \to \infty} G_d(a, a_n, a_n) = 0. \quad (2.7) \]

To check either \( a \in X \) is a fixed point of \( T \) or not, we consider
\[ G_d(a, Ta, Ta) \leq G_d(a, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, Ta, Ta) \]
\[ \leq G_d(a, a_{n+1}, a_{n+1}) + F(W(a_n, a, a), \varphi(W(a_n, a, a))). \quad (2.6) \]

From (2.2),
\[ W(a_n, a, a) = \frac{1}{2} \max \{G_d(a, T^2a_n, Ta), G_d(Ta_n, T^2a_n, Ta), G_d(a_n, Ta_n, a), \]
\[ G_d(a_n, Ta_n, a), G_d(a, T^2a_n, Ta), G_d(a, Ta_n, Ta), \]
\[ G_d(Ta_n, T^2a_n, Ta), G_d(a, Ta_n, Ta), G_d(a, a, a), \]
\[ G_d(a_n, Ta_n, Ta_n), G_d(a, Ta, Ta), G_d(a, Ta, Ta), \]
\[ G_d(a_n, Ta, Ta), G_d(a, Ta, Ta), G_d(a, Ta_n, Ta_n) \} \]
\[ W(a_n, a, a) = \frac{1}{2} \max \{G_d(a, a_{n+2}, Ta), G_d(a_{n+1}, a_{n+2}, Ta), G_d(a_n, a_{n+1}, a), \]
\[ G_d(a_n, a_{n+1}, a), G_d(a, a_{n+2}, Ta), G_d(a, a_{n+1}, Ta), \]
\[ G_d(a_{n+1}, a_{n+2}, Ta), G_d(a, a_{n+1}, Ta), G_d(a_n, a, a), \]
\[ G_d(a_n, a_{n+1}, a_{n+1}), G_d(a, Ta, Ta), G_d(a, Ta, Ta), \]
\[ G_d(a_n, Ta, Ta), G_d(a, Ta, Ta), G_d(a, a_{n+1}, a_{n+1}) \} \]
\[ W(a, a, a) = \frac{1}{2} \max\{G_d(a, a, a), G_d(a, a+1, a), G_d(a, a+1, a), G_d(a, a+1, a), G_d(a, a+1, a), \} \]

After applying limit \( n \to \infty \), by (2.8), for every selection of \( W(a, a, a) \) from (2.9) and by using the fact that \( G_d \) is symmetry, we get

\[ G_d(a, Ta, Ta) \leq F(G_d(a, Ta, Ta), \phi(G_d(a, Ta, Ta))). \]

By the property of \( F \), we get

\[ G_d(a, Ta, Ta) = 0 \quad \text{or} \quad \phi(G_d(a, Ta, Ta)) = 0. \]

That is

\[ G_d(a, Ta, Ta) = 0. \]

Hence, \( Ta = a \) where \( a \in X \) is a fixed point for \( T \). For uniqueness of fixed point, consider \( a, b \in X \) be two distinct fixed points. So consider the relation,

\[ G_d(a, b, b) = G_d(Ta, Tb, Tb) \]

From (2.2),

\[ W(a, b, b) = \frac{1}{2} \max\{G_d(a, b, b), G_d(b, a, b), G_d(a, a, b), G_d(b, b, b)\}. \]

Also,

\[ G_d(a, a, a) \leq G_d(a, b, b), \]
\[ G_d(b, b, b) \leq G_d(a, b, b), \]
\[ G_d(a, a, b) \leq G_d(a, b, b), \]

and

\[ G_d(b, a, a) \leq G_d(a, b, b). \]

Hence, (2.11) gives

\[ W(a, b, b) = \frac{1}{2} G_d(a, b, b). \]

From (2.8),

\[ G_d(a, b, b) \leq F(\frac{1}{2}(G_d(a, b, b), \phi(\frac{1}{2}(G_d(a, b, b))\), \]
\[ \leq \frac{1}{2}(G_d(a, b, b), \]

which implies

\[ G_d(a, b, b) = 0. \quad \text{or} \quad \phi(G_d(a, b, b)) = 0. \]

That is

\[ G_d(a, b, b) = 0. \]

It is a contradiction to our assumption, that is \( a \neq b \). So our supposition is wrong. Hence, \( a \in X \) is a unique fixed point for \( T \).

**Example 2.2:** Let \( X = \{0, 1, 2, 3, 4\} \), and \( G_d : X \times X \times X \to X \), be a mapping defined by,

\[ G_d(a, b, c) = \max\{a, b, c\} \quad \text{for all} \quad a, b, c \in X \]
then, \((X, G_d)\) is a complete dislocated \(G_d\)-metric space. Let, \(T : X \to X\) be defined by,

\[
T x = \begin{cases} 
0 & \text{if } x \in \{0, 1, 2\} \\
1 & \text{if } x \in \{3, 4\}
\end{cases}
\]

and

\[
F(s, t) = s - t \quad \text{for all } s, t \geq 0.
\]

Take \(\varphi(t) = \frac{t}{5}\) for all \(t \geq 0\).

Case I: If \(a = 0, b = 1,\) and \(c = 2,\) then

\[
G_d(T a, T b, T c) = \max\{0, 0, 0\} = 0.
\]

Moreover

\[
W(a, b, c) = \frac{1}{2} \max\{1, 0, 1, 2, 1, 0, 2, 2, 0, 1, 2, 0, 1, 2\} = \frac{2}{2} = 1.
\]

Therefore

\[
F(W(a, b, c), \varphi(W(a, b, c))) = F(1, \varphi(1)) = F(1, \frac{1}{5}) = 1 - \frac{1}{5} = \frac{4}{5}.
\]

Thus

\[
G_d(T a, T b, T c) = 0 < \frac{4}{5} = F(W(a, b, c), \varphi(W(a, b, c))),
\]

that is, (2.1) holds.

Case II: If \(a = 0, b = 1,\) and \(c = 3,\) then

\[
G_d(T a, T b, T c) = \max\{0, 0, 1\} = 1.
\]

Moreover

\[
W(a, b, c) = \frac{1}{3} \max\{1, 0, 1, 3, 3, 1, 1, 3, 3, 0, 1, 3, 0, 1\} = \frac{2}{3}.
\]

Therefore

\[
F(W(a, b, c), \varphi(W(a, b, c))) = F(\frac{3}{2}, \varphi(\frac{3}{2})) = F(\frac{3}{2}, \frac{3}{10}) = \frac{3}{2} \cdot \frac{3}{10} = \frac{6}{5}.
\]
Thus
\[ G_d(Ta, Tb, Tc) = 1 < \frac{6}{5} = F(W(a, b, c), \varphi(W(a, b, c))), \]
that is, (2.1) holds.

Case III: If \( a = 1, b = 1, \) and \( c = 1, \) then
\[ G_d(Ta, Tb, Tc) = \max\{0, 0, 0\} = 0. \]

Moreover
\[ W(a, b, c) = \frac{1}{2} \max\{1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1\} = \frac{1}{2} \]
Therefore
\[ F(W(a, b, c), \varphi(W(a, b, c))) = F\left(\frac{1}{2}, \varphi\left(\frac{1}{2}\right)\right) \]
\[ = F\left(\frac{1}{2}, \frac{1}{10}\right) \]
\[ = \frac{1}{2} - \frac{1}{10} \]
\[ = \frac{2}{5}. \]

Thus
\[ G_d(Ta, Tb, Tc) = 0 < \frac{2}{5} = F(W(a, b, c), \varphi(W(a, b, c))), \]
that is, (2.1) holds.

It is clear from above cases, the contractive condition of Theorem 2.1 holds and similarly for other cases. Therefore, \( 0 \in X, \) is a fixed point for \( T, \) such that \( T0 = 0. \)

In Theorem 2.1, \( W(a, b, c) \) contains 15 elements. Hence many corollaries can be constructed by taking different subsets of \( W(a, b, c). \) Some of them are given below.

**Corollary 2.3:** Let \((X, G_d)\) be a complete dislocated \( G_d \)-metric space, let \( T : X \rightarrow X \) be a mapping satisfying
\[ G_d(Ta, Tb, Tc) \leq F\left(\frac{1}{2}G_d(b, T^2a, Tb), \varphi\left(\frac{1}{2}G_d(b, T^2a, Tb)\right)\right) \]
for all \( a, b, c \in X, \) where \( \varphi \in \Phi_u, \) and \( F \) is a \( C \) class function. Then, there exists a unique fixed point \( a \in X \) such that \( Ta = a. \)

**Corollary 2.4:** Let \((X, G_d)\) be a complete dislocated \( G_d \)-metric space, let \( T : X \rightarrow X \) be a mapping satisfying
\[ G_d(Ta, Tb, Tc) \leq F\left(\frac{1}{2}G_d(Ta, T^2a, Tb), \varphi\left(\frac{1}{2}G_d(Ta, T^2a, Tb)\right)\right) \]
for all \( a, b, c \in X, \) where \( \varphi \in \Phi_u, \) and \( F \) is a \( C \) class function. Then, there exists a unique fixed point \( a \in X \) such that \( Ta = a. \)

**Corollary 2.5:** Let \((X, G_d)\) be a complete dislocated \( G_d \)-metric space, let \( T : X \rightarrow X \) be a mapping satisfying
\[ G_d(Ta, Tb, Tc) \leq F\left(\frac{1}{2}G_d(a, Ta, b), \varphi\left(\frac{1}{2}G_d(a, Ta, b)\right)\right) \]
for all \( a, b, c \in X, \) where \( \varphi \in \Phi_u, \) and \( F \) is a \( C \) class function. Then, there exists a unique fixed point \( a \in X \) such that \( Ta = a. \)
for all $a, b, c \in X$, where $\varphi \in \Phi_u$, and $F$ is a $C$ class function. Then, there exists a unique fixed point $a \in X$ such that $Ta = a$.

**Competing Interest:**
The authors declare that they have no competing interest.

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