SOME SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SRIVASTAVA-ATTIYA OPERATOR

SHAHID KHAN, NAZAR KHAN, SAQIB HUSSAIN, QAZI ZAHOOR AHMAD, MUHAMMAD ASAD ZAIGHUM

Abstract. In this paper, we introduce certain new subclasses of bi-univalent functions in open unit disk associated with the Srivastava-Attiya operator. We obtain coefficient bounds $|a_2|$ and $|a_3|$ for the functions belonging to these new classes.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in E) \quad (1.1)$$

which are analytic in the open unit disk $E = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Further, by $\mathcal{S}$, we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $E$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*(\beta)$ of starlike functions of order $\beta$ ($0 \leq \beta < 1$) if the following condition is satisfied:

$$\Re \left( \frac{zf'(z)}{f(z)} - \beta \right) > 0 \quad (z \in E).$$

Moreover, a function $f \in \mathcal{A}$ is in the class $\mathcal{C}(\beta)$ of convex functions of order $\beta$ ($0 \leq \beta < 1$) if the following condition is satisfied:

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \beta \right) > 0 \quad (z \in E).$$

For two analytic functions $f$ given by (1.1) and $g$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in E).$$

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Their convolution (Hadamard product) is defined by

\[(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.2)\]

It is well known that every univalent function \(f\) has an inverse \(f^{-1}\), defined by

\[f^{-1}(f(z)) = z \quad (z \in E)\]

and

\[f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4})\]

where

\[f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots\]

A function \(f \in A\) is said to be bi-univalent in \(E\) if both \(f\) and \(f^{-1}\) are univalent in \(E\). The class of all such functions is denoted by \(\Sigma\).

The work of Srivastava et al. [10] essentially revived the investigation of various subclasses of the bi-univalent function class \(\Sigma\) in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [10], several different subclasses of the bi-univalent function class \(\Sigma\) were introduced and studied analogously by many authors (see, for example, [2], [5], [11], [12], [13], [15] and [16]), but only non-sharp estimates on the initial coefficients \(|a_2|\) and \(|a_3|\) in the Taylor-Maclaurin expansion (1.1) were obtained in these recent papers.

Furthermore, generalized Hurwitz-Lerch Zeta function \(\phi(u, b, z)\) is defined by

\[\phi(\mu, b, z) = \sum_{n=0}^{\infty} \frac{z^n}{(n + b)^\mu},\]

\[= b^{-\mu} + \frac{z}{(1 + b)^\mu} + \sum_{n=2}^{\infty} \frac{z^n}{(n + b)^\mu},\]

where \(b \in \mathbb{C}\) with \(b \neq 0, -1, -2, \ldots, \mu \in \mathbb{C}, \ \Re(\mu) > 1\) and \(z \in E\).

Using Hurwitz-Lerch zeta functions with the convolution of an analytic functions, Srivastava and Attiya [14] introduced a family of linear operators \(J_{\mu,b} : A \rightarrow A\) as:

\[J_{\mu,b} f(z) = G_{\mu,b} * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1 + b}{n + b}\right)^\mu a_n z^n, \quad (1.3)\]

where \(b \in \mathbb{C}\) with \(b \neq 0, -1, -2, \ldots, \mu \in \mathbb{C}, z \in E\) and \(G_{\mu,b} \in A\) given by

\[G_{\mu,b} = (1 + b)^\mu \left[\phi(\mu, b, z) - b^{-\mu}\right],\]

\[= z + \sum_{n=2}^{\infty} \left(\frac{1 + b}{n + b}\right)^\mu z^n. \quad (1.4)\]

The following recursive relation can easily be obtained by using (1.3) and (1.4)

\[z [J_{\mu,b} f(z)]' = (1 + b)J_{\mu-1,b} f(z) - bJ_{\mu,b} f(z).\]

**Remark 1:** \(J_0,b\) and \(J_{-\mu,b}\) give the identity and inverse operator of \(J_{\mu,b}\) respectively.

**Remark 2:** Srivastava-Attiya operator defined in (1.3) generalizes many known operators for example:
(i) For \( \mu = 1 \) and \( b = 0 \), (1.3) reduces to the well-known operator defined earlier by Alexander [1].

ii) For \( \mu = 1 \) and \( b = 1 \), (1.3) reduces to the well-known operator defined by Libera [8].

(iii) For \( \mu = 1 \) and \( b = \gamma > -1, \gamma \in \mathbb{N} \), (1.3) reduces to the Bernardi integral operator defined by Bernardi [3].

(iv) For \( \mu = \sigma > 0 \) and \( b = 1 \), (1.3) reduces to Jung–Kim–Srivastava integral operator [6].

The object of the present work is to introduce two new subclasses of the function class \( \Sigma \) and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in these new subclasses of the function class \( \Sigma \) using the technique of Srivastava et al. [10] (see, also [7]).

Here we recall a lemma which we will use in our main results.

**Lemma 1** [9]. If \( h \in P \), then \( |c_n| \leq 2 \) for each \( n \), where \( P \) is the family of all functions \( h \), analytic in \( E \), for which \( \Re(h(z)) > 0, z \in E \), where

\[ h(z) = 1 + c_1z + c_2z^2 + \ldots, z \in E. \]

2. **Coefficient bounds for the function class \( M_{\Sigma}(\mu, b, \alpha, \lambda) \)**

**Definition 1.** A function \( f \) defined by (1.1) is said to be in the class \( M_{\Sigma}(\mu, b, \alpha, \lambda) \) if the following condition are satisfied:

\[ |\arg \left( \frac{z [J_{\mu,b}f(z)]'}{(1-\lambda)z + \lambda J_{\mu,b}f(z)} \right) | < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; 0 \leq \lambda \leq 1; z \in E, \quad (2.1) \]

and

\[ |\arg \left( \frac{w [J_{\mu,b}g(w)]'}{(1-\lambda)w + \lambda J_{\mu,b}g(w)} \right) | < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; 0 \leq \lambda \leq 1; w \in E, \quad (2.2) \]

where the function \( g \) is given by

\[ g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \quad (2.3) \]

That is, the extension of \( f^{-1} \) to \( E \).

Special Cases:

i) For \( \mu = 0, \lambda = 0 \) and \( b = 0 \) in (2.1) and (2.2) we have the class \( M_{\Sigma}(0,0,\alpha,0) = H_\Sigma^\alpha \), defined by Srivastava et al. [10].

ii) For \( \mu = 0, \lambda = 1 \) and \( b = 0 \) in (2.1) and (2.2) we have the class \( M_{\Sigma}(0,0,\alpha,1) = \delta_\Sigma^\alpha \), defined by Brannan and Taha [4].

**Theorem 1.** Let the function \( f \) defined by (1.1) be in the class \( M_{\Sigma}(\mu, b, \alpha, \lambda) \) (\( 0 < \alpha \leq 1; 0 \leq \lambda \leq 1 \)). Then

\[ |a_2| \leq \frac{2\alpha}{\left\{ \left( 2\alpha(\lambda^2 - 2\lambda) - (\alpha - 1)(2 - \lambda)^2 \right) \left( \frac{1+b}{2+b} \right)^{2\mu} + 2\alpha(3 - \lambda) \left( \frac{1+b}{3+b} \right)^\mu \right\} }, \quad (2.4) \]

\[ |a_3| \leq \frac{4\alpha^2}{(2 - \lambda)^2 \left( \frac{1+b}{2+b} \right)^{2\mu}} + \frac{2\alpha}{(3 - \lambda) \left( \frac{1+b}{3+b} \right)^\mu}. \quad (2.5) \]
Proof. From (2.1) and (2.2) we have
\[
\frac{z [J_{\mu,b}(z)]'}{(1 - \lambda)z + \lambda J_{\mu,b}(z)} = [p(z)]^\alpha, \tag{2.6}
\]
\[
\frac{w [J_{\mu,b}(w)]'}{(1 - \lambda)w + \lambda J_{\mu,b}(w)} = [q(w)], \tag{2.7}
\]
where \(p(z)\) and \(q(w)\) have the following forms:
\[
p(z) = 1 + p_1 z + p_2 z^2 + \ldots, \tag{2.8}
\]
and
\[
q(w) = 1 + q_1 w + q_2 w^2 + \ldots. \tag{2.9}
\]
Now, equating the coefficients in (2.6) and (2.7), we have
\[
(2 - \lambda) \left( 1 + \frac{b_1}{2 + b} \right) a_2 = \alpha p_1, \tag{2.10}
\]
\[
(\lambda^2 - 2\lambda) \left( 1 + \frac{b_1}{2 + b} \right)^2 a_3 + (3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) a_2 = \frac{1}{2} \left[ \alpha (\alpha - 1) p_1^2 + 2\alpha p_2 \right], \tag{2.11}
\]
\[
- (2 - \lambda) \left( 1 + \frac{b_1}{2 + b} \right) a_2 = \alpha q_1, \tag{2.12}
\]
and
\[
(\lambda^2 - 2\lambda) \left( 1 + \frac{b_1}{2 + b} \right)^2 a_2 + (3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) (2a_2^2 - a_3) = \frac{1}{2} \left[ \alpha (\alpha - 1) q_1^2 + 2\alpha q_2 \right]. \tag{2.13}
\]
From (2.10) and (2.12), we have
\[
2(2 - \lambda) \left( 1 + \frac{b_1}{2 + b} \right) a_2 = \alpha^2 (p_1^2 + q_1^2), \tag{2.14}
\]
and
\[
p_1 = -q_1. \tag{2.15}
\]
From (2.11), (2.13), (2.14) and (2.15), we have
\[
\left\{ \begin{array}{l}
2\alpha (\lambda^2 - 2\lambda) - (\alpha - 1)(2 - \lambda)^2 \left( 1 + \frac{b_1}{2 + b} \right)^2 + 2\alpha (3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) \\
\end{array} \right\} a_2^2 = \alpha^2 (p_2 + q_2) \tag{2.16}
\]
Applying Lemma 1 on (2.16), we have
\[
|a_2| \leq \frac{2\alpha}{\sqrt{\left\{ \begin{array}{l}
2\alpha (\lambda^2 - 2\lambda) - (\alpha - 1)(2 - \lambda)^2 \left( 1 + \frac{b_1}{2 + b} \right)^2 + 2\alpha (3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) \\
\end{array} \right\}}. \]

This gives the bound on \(|a_2|\) as given in (2.4).

Next, to find the bound on \(|a_3|\), by subtracting (2.13) from (2.11), we have
\[
2(3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) a_3 - 2(3 - \lambda) \left( 1 + \frac{b_1}{3 + b} \right) a_2^2 = \alpha (p_2 - q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 - q_1^2) \tag{2.17}
\]
From (2.14), (2.15) and (2.17), we have

\[ a_3 = \left[ \frac{\alpha^2 p_3^2}{(2-\lambda)^2 (1+\frac{1}{2+\gamma})^2} + \frac{\alpha (p_2 - q_2)}{2(3-\lambda) \left( 1+\frac{1}{2+\gamma} \right)^2} \right] \]  \hspace{1cm} (2.18)

Applying Lemma 1 once again on (2.18) for the coefficients \( p_2 \) and \( q_2 \), we have

\[ |a_3| \leq \frac{4\alpha^2}{(2-\lambda)^2 (1+\frac{1}{2+\gamma})^2} + \frac{2\alpha}{(3-\lambda) \left( 1+\frac{1}{2+\gamma} \right)^2}. \]

This completes the proof.

For \( \lambda = 0, \mu = 0 \) and \( b = 0 \) in Theorem 1 we have the following corollary due to Srivastava et al. \([\text{II}]\).

**Corollary 1.** Let \( f \) given by (1.1) be in the class \( \mathcal{H}_\Sigma^0 \). Then

\[ |a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 2}} \quad \text{and} \quad |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}. \]

**Corollary 2.** Let the function \( f \) defined by (1.1) be in the class \( M_\Sigma(1,0,\alpha,\lambda) \) for \( 0 < \alpha \leq 1; 0 \leq \lambda \leq 1 \). Then

\[ |a_2| \leq \frac{2\alpha}{\sqrt{\{\lambda^2(\alpha + 1) - 4(\alpha + \lambda - 1)\}} \frac{1}{4} + \frac{\alpha}{2}(3-\lambda)}, \quad |a_3| \leq \frac{4\alpha^2}{\frac{1}{4}(2-\lambda)^2} + \frac{2\alpha}{\frac{1}{2}(3-\lambda)}.
\]

**Corollary 3.** Let the function \( f \) defined by (1.1) be in the class \( M_\Sigma(1,1,\alpha,\lambda) \) for \( 0 < \alpha \leq 1; 0 \leq \lambda \leq 1 \). Then

\[ |a_2| \leq \frac{2\alpha}{\sqrt{\{\lambda^2(\alpha + 1) - 4(\alpha + \lambda - 1)\}} \frac{1}{4} + \alpha(3-\lambda)}, \quad |a_3| \leq \frac{4\alpha^2}{\frac{1}{3}(2-\lambda)^2} + \frac{2\alpha}{\frac{1}{2}(3-\lambda)}.
\]

**Corollary 4.** Let the function \( f \) defined by (1.1) be in the class \( M_\Sigma(1,\gamma,\alpha,\lambda) \) for \( 0 < \alpha \leq 1; 0 \leq \lambda \leq 1 \). Then

\[ |a_2| \leq \frac{2\alpha}{\sqrt{\{\lambda^2(\alpha + 1) - 4(\alpha + \lambda - 1)\}} \left( 1+\frac{\gamma}{2+\gamma} \right)^2} + \frac{2\alpha}{\lambda^2(\alpha + 1) - 4(\alpha + \lambda - 1)} \left( 1+\frac{\gamma}{2+\gamma} \right)^2 + \frac{2\alpha}{\lambda^2(\alpha + 1) - 4(\alpha + \lambda - 1)} \left( 1+\frac{\gamma}{2+\gamma} \right)^2.
\]

3. **Coefficient bounds for the function class \( M_\Sigma(\mu, b, \beta, \lambda) \)

**Definition 2.** A function \( f \) defined by (1.1) is said to be in the class \( M_\Sigma(\mu, b, \beta, \lambda) \) if the following condition is satisfied:

\[ \Re \left( \frac{z [J_{\mu,b}f(z)]'}{(1-\lambda)z + \lambda J_{\mu,b}f(z)} \right) \geq \beta, \quad 0 \leq \beta < 1; 0 \leq \lambda \leq 1; z \in E, \] \hspace{1cm} (3.1)

and

\[ \Re \left( \frac{w [J_{\mu,b}g(w)]'}{(1-\lambda)w + \lambda J_{\mu,b}g(w)} \right) \geq \beta, \quad 0 \leq \beta < 1; 0 \leq \lambda \leq 1; w \in E, \] \hspace{1cm} (3.2)

where the function \( g \) is given in (2.3).

Special Cases:
i) For $\mu = b = \lambda = 0$, (3.1) and (3.2) reduced to the class $\mathcal{H}_2(\beta)$ defined by Srivastava et.al [10].

ii) For $\mu = b = 0$ and $\lambda = 1$, (3.1) reduced to the well-known starlike function of order $\beta$, see [9].

**Theorem 2.** Let $f \in \mathcal{A}$ defined by (1.1) be in the class $M_{\Sigma}(\mu, b, \beta, \lambda)$ for $0 \leq \beta < 1; 0 \leq \lambda \leq 1$.

Then

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{\sqrt{\left\{(\lambda^2 - 2\lambda) \left(\frac{1+b}{2+b}\right)^{2\mu} + (3-\lambda) \left(\frac{1+b}{3+b}\right)^{\mu}\right\}}},$$

(3.3)

$$|a_3| \leq (1-\beta) \left\{\frac{4(1-\beta)}{(2-\lambda) \left(\frac{1+b}{2+b}\right)^{2\mu}} + \frac{2}{(3-\lambda) \left(\frac{1+b}{3+b}\right)^{\mu}}\right\}.$$  

(3.4)

**Proof.** From (3.1) and (3.2), we have

$$z \left[ J_{\mu,b} f(z) \right]' = \beta + (1-\beta)p(z),$$

(3.5)

$$w \left[ J_{\mu,b} f(w) \right]' = \beta + (1-\beta)q(w),$$

(3.6)

where $p(z)$ and $q(w)$ are given in (2.8) and (2.9) respectively. Equating the coefficients in (3.5) and (3.6), we obtain

$$\left(2-\lambda\right) \left(\frac{1+b}{2+b}\right)^{\mu} a_2 = (1-\beta)p_1,$$

(3.7)

$$\left(\lambda^2 - 2\lambda\right) \left(\frac{1+b}{2+b}\right)^{2\mu} a_2^2 + (3-\lambda) \left(\frac{1+b}{3+b}\right)^{\mu} a_3 = (1-\beta)p_2,$$

(3.8)

$$- \left(2-\lambda\right) \left(\frac{1+b}{2+b}\right)^{\mu} a_2 = (1-\beta)q_1,$$

(3.9)

and

$$\left(\lambda^2 - 2\lambda\right) \left(\frac{1+b}{2+b}\right)^{2\mu} a_2^2 + (3-\lambda) \left(\frac{1+b}{3+b}\right)^{\mu} \left(2a_2 - a_3\right) = (1-\beta)q_2.$$  

(3.10)

From (3.7) and (3.9), we have

$$\left(2(2-\lambda)^2 \left(\frac{1+b}{2+b}\right)^{2\mu}\ a_2^2 = (1-\beta)^2 \left(p_1^2 + q_1^2\right),$$

(3.11)

and

$$p_1 = -q_1.$$  

(3.12)

Adding (3.8) and (3.10), we have

$$\left\{2(\lambda^2 - 2\lambda) \left(\frac{1+b}{2+b}\right)^{2\mu} + 2(3-\lambda) \left(\frac{1+b}{3+b}\right)^{\mu}\right\} a_2^2 = (1-\beta)(p_2 + q_2).$$  

(3.13)
Applying Lemma 1 on (3.13), we have
\[ |a_2| \leq \sqrt{2(1 - \beta)} \sqrt{\left\{ (\lambda^2 - 2\lambda) \left( \frac{1 + b}{3 + b} \right)^{2\mu} + (3 - \lambda) \left( \frac{1 + b}{3 + b} \right)^{\mu} \right\}}. \]
This gives the bound on \(|a_2|\) as given in (3.3).

Next, in order to find the bound on \(|a_3|\), by subtracting (3.10) from (3.8), we have
\[ 2(3 - \lambda) \left( \frac{1 + b}{3 + b} \right)^{\mu} a_3 - 2(3 - \lambda) \left( \frac{1 + b}{3 + b} \right)^{\mu} a_2^2 = (1 - \beta)(p_2 - q_2) \quad (3.14) \]
Substitution the value of \(a_2^2\) from (3.11) in (3.14), we have
\[ a_3 = \left( \frac{(1 - \beta)^2 (p_2^2 + q_2^2)}{2(2 - \lambda)^2 \left( \frac{1 + b}{3 + b} \right)^{2\mu}} \right) + \frac{(1 - \beta)(p_2 - q_2)}{2(3 - \lambda) \left( \frac{1 + b}{3 + b} \right)^{\mu}} \quad (3.15) \]
Applying Lemma 1 on (3.15) for the coefficient \(p_2\) and \(q_2\), we have
\[ |a_3| \leq (1 - \beta) \left\{ \frac{4(1 - \beta)}{(2 - \lambda)^2 \left( \frac{1 + b}{3 + b} \right)^{2\mu}} + \frac{2}{(3 - \lambda) \left( \frac{1 + b}{3 + b} \right)^{\mu}} \right\}. \]
This completes the proof.

**Corollary 5** [10]. Let \(f(z)\) be given by (1.1) be in the function class \(H_{\Sigma}(\beta)\) \((0 \leq \beta < 1)\). Then
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{3}} \quad \text{and} \quad |a_3| \leq \frac{(1 - \beta)(5 - 3\beta)}{3}. \]

**Corollary 6.** Let the function \(f(z)\) defined by (1.1) be in the class \(M_{\Sigma}(0, 0, \beta, 1)\) \((0 \leq \beta < 1)\). Then
\[ |a_2| \leq \sqrt{2(1 - \beta)} \quad \text{and} \quad |a_3| \leq (1 - \beta)(5 - 4\beta). \]

**Corollary 7.** Let the function \(f(z)\) defined by (1.1) be in the class \(M_{\Sigma}(1, 1, \beta, \lambda)\) \((0 \leq \beta < 1; 0 \leq \lambda \leq 1)\). Then
\[ |a_2| \leq \frac{\sqrt{2(1 - \beta)}}{\sqrt{\frac{4}{5} \{ (\lambda^2 - 2\lambda) + \frac{1}{2} (3 - \lambda) \}}} \]
\[ |a_3| \leq (1 - \beta) \left\{ \frac{4(1 - \beta)}{5(2 - \lambda)^2} + \frac{2}{5(3 - \lambda)} \right\}. \]

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References


SHAHID KHAN
DEPARTMENT OF MATHEMATICS RIPHAH INTERNATIONAL UNIVERSITY ISLAMABAD, PAKISTAN.
E-mail address: shahidmath761@gmail.com

NAZAR KHAN
DEPARTMENT OF MATHEMATICS ABBOTTABAD UNIVERSITY OF SCIENCE AND TECHNOLOGY, ABBOTTABAD, PAKISTAN.
E-mail address: nazarmaths@gmail.com

SAQIB HUSSAIN
DEPARTMENT OF MATHEMATICS COMSATS INSTITUTE OF INFORMATION TECHNOLOGY ABBOTTABAD, PAKISTAN.
E-mail address: saqib.math@yahoo.com

QAZI ZAHOOR AHMAD
DEPARTMENT OF MATHEMATICS ABBOTTABAD UNIVERSITY OF SCIENCE AND TECHNOLOGY, ABBOTTABAD, PAKISTAN.
E-mail address: zahoorqazi15@gmail.com

MUHAMMAD ASAD ZAIGHUM
DEPARTMENT OF MATHEMATICS RIPHAH INTERNATIONAL UNIVERSITY ISLAMABAD, PAKISTAN.
E-mail address: asadzaighum@gmail.com