THE $D_\theta$-CLASSICAL ORTHOGONAL POLYNOMIALS

(COMMUNICATED BY FRANCISCO MARCELLAN)

ATEF ALAYA

Abstract. The $D_\theta$-classical orthogonal polynomial sequences are defined through the $D_\theta$-Hahn’s property: sequences that are orthogonal together with their $D_\theta$-first derivative, where $D_\theta(p) = p' + \theta \cdot p$, for all $p \in \mathbb{C}[X]$. We characterize them by means of a functional equation, a $D_\theta$-second order linear differential equation, the first and the second structure relations. A $D_\theta$-classical orthogonal sequence is especially a $D$-Laguerre-Hahn sequence of class less than or equal to two. A complete classification of the $D_\theta$-classical sequences is obtained. The functional equation coefficients, the structure relations coefficients, the three-term recurrence relation coefficients and the class are whenever given.

1. INTRODUCTION

Lot of works dealing with orthogonal polynomials mention the very classical ones: continuous (Hermite, Laguerre, Bessel and Jacobi), discrete (Charlier, Meixner, Krawtchouk and Hahn) or their analogues with respect to lowering difference or differential operators (Hahn operator, Delta operator, Dunkl operator, etc.) [14], [18], [19], [22], [26], [30], [32], [36], [37], [49], [51], [55], and [11], [13], [15], [35], [56].

Rodrigues formula, Hahn property, Bochner condition, first and second structure relations, and Pearson equation are important common tools to characterize and construct these polynomial sequences [1], [4], [6], [7], [8], [14], [18], [22], [26], [30], [32], [36], [37], [49], [50], [51], [57], [63], [20]. In fact, using these tools, different unified presentations of classical orthogonal polynomials are done in the literature either for continuous case, discrete case or their analogues: through an algebraic approach [47], [49], a functional approach [26], [39], a distributional approach [27], [53], a hypergeometrical approach [9], [43], a difference calculus approach [19], a matrix approach [63], etc...[44]

Besides, different natural procedures are used to build new orthogonal polynomial sequences (see [2], [5], [10], [13], [18], [21], [23], [24], [25], [38], [40], [41], [48], [49], [50], [56].

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and [24]. They start by using some classical polynomials and yield a lot of polynomial sequences but generally with no apparent link between them. In front of the accumulation of the obtained results of such procedures, the need of structure and classification of orthogonal polynomial sequences was natural. Among others, the papers [9], [12], [23], [25], [31], [44], [34], [35], [41], [43], [45] and [46] provide sketches in this direction highlighting the so-called semiclassical polynomial sequences, and where classical orthogonal polynomials are seen as semiclassical ones of class zero. In the same context, in [33] the authors emphasize the hypergeometric character rather than the sequences therein. An algebraic approach is also presented in [47] for the so-called Laguerre-Hahn polynomials generalizing the semiclassical case. Some early tries worthy of being evoked too in the same object are [59], [60] and [61].

All these works usually give orthogonal polynomials generalizing (by their properties and their characterizations) the classical orthogonal polynomials, but generally it is difficult to explicitly construct them except under further assumptions as the symmetry [2], [3], [16], [17], [29], [52], [57], [58] and [62].

The aim of this work is to pick up orthogonal polynomial sequences under a lowering operator denoted by \( D_u \), generalizing the standard derivative \( D = \frac{d}{dx} \). This operator was first introduced to see all Laguerre-Hahn polynomials of class zero, build in [16], as the unique solutions of \( D_u \) Hahn’s property. In [42], the authors define the \( D_u \)-semiclassical polynomials and classify them using the notion of the class. Here, we expose the \( D_u \)-classical orthogonal polynomials by means of Hahn’s property with respect to the operator \( D_u \). In particular, through an algebraic approach, we state several characterizations of them as a natural extension of the corresponding properties for the very classical ones (generalizing the Hahn property, the Pearson equation, the second order linear differential equation and the structure relations). It is also shown that such polynomial sequences are \( D \)-Laguerre-Hahn sequences of class at most two, and that are \( D_u \)-semiclassical sequences of class zero. Finally, by solving a nonlinear system fulfilled by the corresponding three-term recurrence relation coefficients, we give explicitly all \( D_u \)-classical orthogonal polynomial sequences.

2. BACKGROUND

Let \( P \) be the vector space of polynomials with coefficients in \( \mathbb{C} \) and \( P' \) its dual. For \( u \in P' \), \( (u, p) \) means the action of the form (linear functional) \( u \) over the polynomial \( p \). In particular, \( (u)_n = (u, x^n) \), \( n \geq 0 \), are the moments of \( u \). When \( (u)_0 = 1 \), then \( u \) is said to be normalized.

We set \( P'_0 := \{ u \in P' \mid (u)_0 \neq -n, \ n \geq 1 \} \).

Let us define the following operations on \( P' \):

For all \( c \in \mathbb{C} \), \( p, q \in P \), and \( u, v \in P' \):

\[
\langle qu, p \rangle = \langle u, qp \rangle, \quad \langle u', p \rangle = -\langle u, p' \rangle,
\]

\[
\langle \delta_c, p \rangle = p(c), \quad \delta_c : \text{Dirac delta at } c, \ (\delta := \delta_0),
\]

\[
\langle uv, p \rangle = \langle v, up \rangle, \text{ where } (up)(x) = \langle u, \frac{xp(x) - yp(y)}{x - y} \rangle,
\]

\[
\langle (x - c)^{-1}u, p \rangle = \langle u, \theta_c(p) \rangle = \langle u, \frac{p(x) - p(c)}{x - c} \rangle,
\]
where \( \langle u_y, \cdot \rangle \) means the action of \( u \) over the polynomial on the \( y \)-variable.

Now, let \( \{ B_n \}_{n \geq 0} \) be a monic polynomial sequence (MPS), with \( \deg B_n = n \), and let \( \{ w_n \}_{n \geq 0} \) be its dual sequence defined by \( \langle w_n, B_m \rangle = \delta_{n,m} \), \( n, m \geq 0 \), where \( \delta_{n,m} \) is the Kronecker delta. Note that \( w_0 \) is normalized and it is said to be the canonical form of \( \{ B_n \}_{n \geq 0} \). The following Lemmas are helpful for the sequel.

**Lemma 2.1.** \cite{47} \cite{19} Let \( \{ w_n \}_{n \geq 0} \) denotes the dual sequence of a given MPS \( \{ B_n \}_{n \geq 0} \). For any \( u \in \mathbb{P}' \), and any integer \( p \geq 1 \), the following statements are equivalent

(a) \( \langle u, B_{p-1} \rangle \neq 0 \), and \( \langle u, B_n \rangle = 0 \), \( n \geq p \).

(b) \( \exists \lambda_\mu \in \mathbb{C}, 0 \leq \mu \leq p - 1, \lambda_{p-1} \neq 0 \) such that \( u = \sum_{\mu = 0}^{p-1} \lambda_\mu w_\mu \).

**Lemma 2.2.** \cite{2} \cite{10} Let \( u, v \in \mathbb{P}' \), \( f, g \in \mathbb{P} \), and \( (a, b, c) \in \mathbb{C}^3 \) with \( a \neq 0 \), we have

\[
\delta u = u, \quad uv = vu, \quad f(uv) = (fu)v + x(u\theta_0 f)(x)v,
\]

(2.1)

\[
(fu) = fu' + f' u, \quad f(\tau_b u) = \tau_b ((\tau_{-b} f)u), \quad f(h_a u) = h_a ((h_a f)u),
\]

\[
x^{-1}(uv) = (x^{-1} u)v = u(x^{-1} v), \quad (x - c)^{-1}(x - c)^{-1} u = u,
\]

\[
(x - c)^{-1}(x - c)^{-1}(x - c)^{-1} u = u - (u_0 \delta_c, (x - c)^{-1}(x - c)^{-1} f(u) = f(c)((x - c)^{-1} u) + (\theta_{-c} f)u - \langle u, \theta_{-c} f \rangle \delta_c,
\]

\[
\langle u, f g \rangle(x) = \langle (fu)g \rangle(x) + xg(x)(u\theta_0 f)(x),
\]

\[
\langle \theta_0 (fg) \rangle(x) = \langle (\theta_0 f)g \rangle(x) + f(0)(\theta_0 g)(x).
\]

(2.2)

Here the shift \( \tau_b \), and the dilation \( h_a \) are respectively defined by

\[
\langle \tau_b u, f \rangle = \langle u, \tau_{-b} f \rangle = \langle u, f(x + b) \rangle, \quad \text{and} \quad \langle h_a u, f \rangle = \langle u, h_a f \rangle = \langle u, f(ax) \rangle.
\]

**Lemma 2.3.** \cite{2} \cite{10} For all \( u, v \in \mathbb{P}' \), and \( f, g \in \mathbb{P} \), the following formulas hold

\[
\langle u\theta_0 (fg) \rangle(x) = g(x)(u\theta_0 f)(x) + \langle fu, \theta_0 g \rangle(x),
\]

(2.4)

\[
f(x^{-1} u) = x^{-1}(fu) + \langle u, \theta_0 f \rangle \delta,
\]

(2.5)

\[
f(x^{-1}(uv)) = x^{-1}(uf(v)) + (u\theta_0 f)(x)u, \quad f^2 u^2 = (fu)^2 + 2xf(x)(u\theta_0 f)(x)u,
\]

\[
\langle u^2, \theta_0 (fg) \rangle = \langle u, f(u\theta_0 g) + g(u\theta_0 f) \rangle.
\]

(2.6)

**Remark.** \cite{2} \cite{10} A form \( u \) has an inverse \( u^{-1} \) (i.e. \( uu^{-1} = \delta \)), if and only if \( \langle u, 0 \rangle \neq 0 \).

A MPS \( \{ B_n \}_{n \geq 0} \) is called orthogonal (MOPS) with respect to a form \( w \), if \( \langle w, B_n B_m \rangle = 0 \), \( n \neq m \), and \( \langle w, B_n^2 \rangle \neq 0 \), \( n \geq 0 \). In this case, \( w \) is said to be regular (quasi-definite). Necessarily \( w = \langle w_0 \rangle w_0 \), with \( \langle w_0 \rangle \neq 0 \). In the sequel, we shall take any regular form \( w \) normalized. Hence \( w = w_0 \).

**Definition 2.4.** \cite{16} \cite{42} A nonzero form \( w \) is said to be weakly regular, if for any polynomial \( A \) such that \( Aw = 0 \), then \( A = 0 \).

**Lemma 2.5.** \cite{12} A regular form is weakly regular.

**Proposition 2.6.** \cite{16} \cite{47} \cite{19} Let \( \{ P_n \}_{n \geq 0} \) and \( \{ Q_n \}_{n \geq 0} \) be two MOPS with respect to \( u \) and \( v \) respectively, and \( A, B \) are two polynomials with \( \deg(A) = s \), and \( \deg(B) = t \). The following assertions are equivalent:

(a) \( Au = Bv \).
(b) \( A(x)Q_n(x) = \sum_{\nu=0}^{n+s} \lambda_{n,\nu} P_\nu(x), \ n \geq t \) with \( \lambda_{n,n-t} \neq 0, n \geq t \).

**Proposition 2.7.** [46-47] A MPS \( \{B_n\}_{n \geq 0} \) with dual sequence \( \{w_n\}_{n \geq 0} \) is orthogonal, if and only if one of the following statements holds:

(a) \( w_n = \langle w, B_n^2 \rangle^{-1} B_n w_0, n \geq 0 \).

(b) \( \{B_n\}_{n \geq 0} \) satisfies the three-term recurrence relation (TTRR)

\[
\begin{aligned}
B_0(x) &= 1, & B_1(x) &= x - \beta_0, \\
B_{n+2}(x) &= (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \ n \geq 0,
\end{aligned}
\]  

(2.7)

with \( \beta_n \in \mathbb{C} \) and \( \gamma_{n+1} \neq 0, n \geq 0 \).

(c) There exist two complex number sequences \( \{\beta_n\}_{n \geq 0} \) and \( \{\gamma_{n+1}\}_{n \geq 0} \), such that

\[
\gamma_{n+1}w_{n+1} = (x - \beta_n)w_n - w_{n-1}, \ \gamma_{n+1} \neq 0, \ n \geq 0, \quad (w_{-1} = 0).
\]  

(2.8)

**Definition 2.8.** Let \( w \) be a normalized form. For any MPS \( \{B_n\}_{n \geq 0} \), we define the associated sequence of the first kind with respect to \( w \), denoted by \( \{B_n^{(1)}(w)\}_{n \geq 0} \) as follows (see [1, 10])

\[
B_n^{(1)}(w)(x) = \langle w, B_{n+1}(x) - B_{n+1}(\xi) \rangle, \ n \geq 0.
\]

Once \( \{B_n\}_{n \geq 0} \) is orthogonal with respect to \( w_0 \) and fulfills (2.7), then we have:

**Proposition 2.9.** [2] The sequence \( \{B_n^{(1)}(w)\}_{n \geq 0} \) is orthogonal with respect to \( \varpi \), if and only if:

\[
\begin{aligned}
\langle w, B_n \rangle &= 0, \ n \geq 3, \\
\langle w, B_2 \rangle &\neq \gamma_2.
\end{aligned}
\]  

(2.9)

In this case, we have \( w = Aw_0 \) with \( A(x) = \frac{\langle w, B_2 \rangle}{\gamma_1 \gamma_2} B_2(x) + \frac{\langle w, B_1 \rangle}{\gamma_1} B_1(x) + 1 \).

Besides, the sequence \( \{B_n^{(1)}(w)\}_{n \geq 0} \) verifies the following TTRR

\[
\begin{aligned}
B_0^{(1)}(w)(x) &= 1, & B_1^{(1)}(w)(x) &= x - \beta_0^{(1)}, \\
B_{n+2}^{(1)}(w)(x) &= (x - \beta_{n+1}^{(1)})B_{n+1}^{(1)}(w)(x) - \gamma_{n+1}^{(1)}B_n^{(1)}(w)(x), \ n \geq 0,
\end{aligned}
\]  

(2.10)

where

\[
\begin{aligned}
\beta_0^{(1)} &= \beta_1 - \langle w, B_1 \rangle, & \beta_n^{(1)} &= \beta_{n+1}, \ n \geq 1, \\
\gamma_1^{(1)} &= \gamma_2 - \langle w, B_2 \rangle, & \gamma_{n+1}^{(1)} &= \gamma_{n+2}, \ n \geq 1.
\end{aligned}
\]  

(2.11)

(2.12)

Furthermore, the form \( \varpi \) fulfills

\[
(Aw_0)\varpi = x \frac{B_1}{\gamma_1} w_0.
\]  

(2.13)

**Proof.** From (2.7), one has

\[
B_{n+3}(x) - B_{n+3}(y) = (x - \beta_{n+2})(B_{n+2}(x) - B_{n+2}(y)) - \\
\gamma_{n+2}(B_{n+1}(x) - B_{n+1}(y)) + (x - y)B_{n+2}(y), \ n \geq 0,
\]

and

\[
B_2(x) - B_2(y) = (x - \beta_1)(B_1(x) - B_1(y)) + (x - y)B_1(y).
\]
Then \( \{B_n^{(1)}(w)\}_{n \geq 0} \) fulfills
\[
\begin{cases}
B_0^{(1)}(w)(x) = 1, & B_1^{(1)}(w)(x) = x - \beta_1 + \langle w, B_1 \rangle, \\
B_{n+2}^{(1)}(w)(x) = (x - \beta_{n+2})B_{n+1}^{(1)}(w)(x) - \gamma_{n+2}B_n^{(1)}(w)(x) + \\
\langle w, B_{n+2} \rangle, & n \geq 0.
\end{cases}
\]

By virtue of Favard’s theorem [46, 56], \( \{B_n^{(1)}(w)\}_{n \geq 0} \) is orthogonal if and only if
\[
\langle w, B_n \rangle = 0, \quad n \geq 3, \quad \text{and} \quad \langle w, B_2 \rangle \neq \gamma_2. \tag{2.14}
\]

The relations (2.10) − (2.12) hold.
Since we have (2.14), the expression of \( A \) is obtained by applying Lemma 2.1 to \( w \) and taking into account the assertion (a) of Proposition 2.7. On the other hand:
\[
\langle w, B_n^{(1)}(w) \rangle = \delta_{n,0}, \quad n \geq 0
\]
\[
\langle w, w_0 B_{n+1} \rangle = \langle w_1, B_{n+1} \rangle, \quad n \geq 0
\]
\[
\langle x^{-1}(w w) - w_1, B_n \rangle = 0, \quad n \geq 1.
\]

Besides \( \langle x^{-1}(w w) - w_1, B_n \rangle = 0 \). Hence (2.13) holds. \( \Box \)

**Remark.** When \( w = w_0 \), the conditions (2.9) are satisfied. Hence, if we set \( B_n^{(1)} = B_n^{(1)}(w_0) \), then \( \{B_n^{(1)}\}_{n \geq 0} \) verifies the following shifted TTRR
\[
\begin{cases}
B_0^{(1)}(x) = 1, & B_1^{(1)}(x) = x - \beta_1, \\
B_{n+2}^{(1)}(x) = (x - \beta_{n+2})B_{n+1}^{(1)}(x) - \gamma_{n+2}B_n^{(1)}(x), & n \geq 0.
\end{cases}
\]

Finally, if we denote \( \{w_n^{(1)}\}_{n \geq 0} \) the dual sequence of \( \{B_n^{(1)}\}_{n \geq 0} \), thus \( \gamma_1 w_0^{(1)} = -x^2 w_0^{-1} \).

The shifted MPS \( \{\tilde{B}_n\}_{n \geq 0} \) of a given MPS \( \{B_n\}_{n \geq 0} \) is given by:
\[
\tilde{B}_n(x) = a^{-n}B_n(ax + b), \quad n \geq 0, \quad (a, b) \in \mathbb{C}^* \times \mathbb{C}. \tag{2.15}
\]

A shift preserves the orthogonality. Precisely, if \( \{B_n\}_{n \geq 0} \) is a MOPS with respect to \( w_0 \) and fulfills (2.7), then the sequence \( \{\tilde{B}_n\}_{n \geq 0} \) defined by (2.15) is orthogonal with respect to
\[
\tilde{w}_0 = (h_{a^{-1}} \circ \tau_{-b})w_0 \tag{2.16}
\]

and one has
\[
\begin{cases}
\tilde{B}_0(x) = 1, & \tilde{B}_1(x) = x - \tilde{\beta}_0, \\
\tilde{B}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{B}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{B}_n(x), & n \geq 0,
\end{cases}
\]
with \( \tilde{\beta}_n = \beta_n - \frac{b}{a} \), and \( \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2} \).

**Definition 2.10.** Let \( v \in \mathbb{P}^t \), and \( \sigma \) a nonnegative integer. A MPS \( \{B_n\}_{n \geq 0} \) is said to be quasi-orthogonal of order \( \sigma \) with respect to \( v \), if
\[
\langle v, x^m B_n \rangle = 0, \quad 0 \leq m \leq n - \sigma - 1, \quad n \geq \sigma + 1, \tag{2.17}
\]
\[
\exists r \geq \sigma, \quad \langle v, x^{-r} B_r \rangle \neq 0.
\]

A MPS \( \{B_n\}_{n \geq 0} \) is said to be strictly quasi-orthogonal of order \( \sigma \), with respect to \( v \), if it satisfies (2.17), and \( \langle v, x^{n-\sigma} B_n \rangle \neq 0, n \geq \sigma \).
Remark.
1. The strict quasi-orthogonality of order zero is orthogonality.
2. A MOPS \( \{B_n\}_{n \geq 0} \) with respect to its canonical form \( w_0 \) is (strictly) quasi-orthogonal of order \( \sigma \) with respect to a form \( v \), if and only if there exists a (unique) polynomial \( \phi \) of degree \( \sigma \) such that \( v = \phi w_0 \).

From now onwards, the symbol \( D \) means the usual derivative operator; i.e. \( D = \frac{d}{dx} \).

**Definition 2.11.** \([3, 16, 23, 42]\) Let \( \{B_n\}_{n \geq 0} \) be a MOPS with respect to \( w_0 \). We say that \( w_0 \) is a \( D \)-Laguerre-Hahn form if it fulfills a functional equation of type

\[
(\Phi(w_0) + B(x^{-1}w_0^2)) = 0, \tag{2.18}
\]

with \((\Phi, \Psi, B) \in \mathbb{P}^3, \Phi \) monic.

The sequence \( \{B_n\}_{n \geq 0} \) is also called \( D \)-Laguerre-Hahn. The minimum value of all \( \max \{\deg \Psi - 1, \max (\deg \Phi, \deg B - 2)\} \), for each triplet \((\Phi, \Psi, B)\) satisfying (2.18), is called the class of \( w_0 \). If \( w_0 \) is of class \( s \), then the sequence \( \{B_n\}_{n \geq 0} \) is said to be of class \( s \).

Between many characterizations of a \( D \)-Laguerre-Hahn MOPS, we mention the following.

**Proposition 2.12.** \([3, 16, 23, 42]\) Let \( \{B_n\}_{n \geq 0} \) be a MOPS with respect to \( w_0 \). The following assertions are equivalent:

(a) \( \{B_n\}_{n \geq 0} \) is \( D \)-Laguerre-Hahn.

(b) \( \{B_n\}_{n \geq 0} \) fulfills the so-called structure relation:

\[
\Phi(x)B_{n+1}(x) - B(x)B_n(z) = \sum_{k=n-s}^{n+d} \theta_{n,k}B_k(x), \quad n \geq s + 1, \quad \theta_{n,n-s} \neq 0 \tag{2.19}
\]

with \( d = \sup(t, r) \) and \( s = \sup(p - 1, d - 2) \), and where \( t, r \) and \( p \) are respectively the degrees of \( \Phi, B \) and \( \Psi \).

When \( B = 0 \) in (2.19), the sequence \( \{B_n\}_{n \geq 0} \) (resp. \( w_0 \)) is called \( D \)-semiclassical.\([17]\). A \( D \)-semiclassical MOPS (resp. form) of class zero is known as \( D \)-classical one.

We recall that for all \( u \in \mathbb{P}' \), we define the linear derivative operator \( D_u \) (see \([42]\))

\[
D_u : \mathbb{P} \longrightarrow \mathbb{P} \quad \quad p \longrightarrow D_u(p) = p' + u\theta_0 p,
\]

or equivalently \( D_u(p)(x) = p'(x) + \langle u, \frac{p(x) - p(y)}{x - y} \rangle, \quad p \in \mathbb{P} \). In particular, we have

\[
D_u(x^n) = (n + (u)_0) x^{n-1} + \sum_{\nu=0}^{n-2} (u)_{n-\nu-1} x^{\nu}, \quad n \geq 2, \quad D_u(x) = (u)_0 + 1, \quad D_u(1) = 0
\]

and

\[
D_u(w) = w' - x^{-1}(uw), \quad w \in \mathbb{P}', \quad \langle D_u(w), p \rangle = -\langle w, D_u(p) \rangle, \quad p \in \mathbb{P}. \tag{2.20}
\]

The first \( D_u \)-derivative MPS of a MPS \( \{B_n\}_{n \geq 0} \) is denoted by \( \{B_n^{[1]}(\cdot; u)\}_{n \geq 0} \). Thus

\[
B_n^{[1]}(x; u) = (n + (u)_0 + 1)^{-1} D_u(B_{n+1})(x), \quad n \geq 0. \tag{2.21}
\]
If we denote by \( \{w_n\}_{n \geq 0} \) (resp. \( \{w_n^{[1]}(u)\}_{n \geq 0} \)) the dual sequence of \( \{B_n\}_{n \geq 0} \) (resp. \( \{B_n^{[1]}(;u)\}_{n \geq 0} \)), then \(^{[12]}\):

\[
D_u(w_n^{[1]}(u)) = -(n + (u)_0 + 1)w_{n+1}, \quad n \geq 0. \tag{2.22}
\]

In what follows, we need the following formulas \(^{[12]}\):

\[
D_u(fg)(x) = D_u(f)(x)g(x) + f(x)D_u(g)(x) + \left((u\theta_0)(fg))(x) - (u\theta_0f)(x)g(x) - (u\theta_0g)(x)f(x),
\]

\[
D_u(fv) = fD_u(v) + D_u(f)v + (v\theta_0f)(x)u - (u\theta_0f)(x)v. \tag{2.24}
\]

Besides, with some straightforward calculations, the following formulas hold.

**Proposition 2.13.** For all \( w \in P', \) \( p \in \mathbb{P}, \) and \( (a, b) \in \mathbb{C} \backslash \{0\} \times \mathbb{C}, \) we have:

\[
D_u(\tau_b p) = \tau_b D_{\tau_{-1}u}(p),
\]

\[
D_u(\tau_b w) = \tau_b D_{\tau_{-1}u}(w), \tag{2.25}
\]

\[
D_u(h_n p) = ah_n D_{h_{n-1}u}(p), \tag{2.26}
\]

\[
D_u(h_n w) = a^{-1}h_n D_{h_{n-1}u}(w). \tag{2.27}
\]

In \(^{[12]}\), the authors studied the so-called \( D_u \)-semiclassical sequences. In particular, they showed that the defined sequences are special \( D \)-Laguerre Hahn ones. In this work we shall elaborate the \( D_u \)-classical case. In section 3, we start by defining a \( D_u \)-classical sequence through the \( D_u \)-Hahn’s property (see definition 3.1). Then, we prove four characterizations that generalize the standard ones in the usual \( D \)-classical case (Hermite, Laguerre, Bessel and Jacobi); the Pearson’s equation, the second order linear differential equation, the first and the second structure relations. We show particularly that any \( D_u \)-classical orthogonal polynomial sequence is, in sense of \(^{[12]}\), a \( D_u \)-semiclassical sequence of class zero. Hence, it is a \( D \)-Laguerre-Hahn’s sequence of class \( s \) at most 2. In section 4, we establish and solve in detail the nonlinear system fulfilled by the corresponding three-term recurrence relation coefficients. This allows us to give explicitly the functional equation coefficients and precise the class \( s \).

## 3. The \( D_u \)-classical sequences

**Definition 3.1.** Let \( u \in P'_0 \). We say that a MPS \( \{B_n\}_{n \geq 0} \) is \( D_u \)-classical sequence, if it is orthogonal together with its first \( D_u \)-derivative sequence \( \{B_n^{[1]}(;u)\}_{n \geq 0} \) given by \(^{[2.27]}\) (The \( D_u \)-Hahn’s property).

A regular form that its corresponding MPS is \( D_u \)-classical is also said to be \( D_u \)-classical. Thus, for any \( D_u \)-classical MPS we have:

\[
\begin{align*}
B_0(x) &= 1, \quad B_1(x) = x - \beta_0, \\
B_{n+2}(x) &= (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0,
\end{align*}
\]

and

\[
\begin{align*}
B_0^{[1]}(x;u) &= 1, \quad B_1^{[1]}(x;u) = x - \tilde{\beta}_0, \\
B_{n+2}^{[1]}(x;u) &= (x - \tilde{\beta}_{n+1})B_{n+1}^{[1]}(x;u) - \tilde{\gamma}_{n+1}B_n^{[1]}(x;u), \quad n \geq 0.
\end{align*}
\]

In the sequel, we denote by \( \{w_n\}_{n \geq 0} \) (resp. \( \{w_n^{[1]}(u)\}_{n \geq 0} \)) the dual sequence of \( \{B_n\}_{n \geq 0} \) (resp. \( \{B_n^{[1]}(;u)\}_{n \geq 0} \)). Also, the letter \( u \) will usually denote an element of \( P'_0 \).
Unless otherwise stated, and in order to simplify the notation, we will write $w_n^{[1]}$ instead of $w_n^{[1]}(u)$ and $B_n^{[1]}$ instead of $B_n^{[1]}(x; u)$.

3.1. The functional equation fulfilled by $w_0$.

**Lemma 3.2.** If $\{B_n\}_{n \geq 0}$ is a $D_u$-classical sequence, then there exist two nonzero polynomials $\Phi$ and $\hat{B}$, $\Phi$ monic, $\deg \Phi \leq 2$, and $\deg \hat{B} \leq 2$, such that

\[
\begin{align*}
\kappa &\,\Phi w_0, \quad \kappa \text{ is a normalizing factor,} \\
u &\,= \hat{B} w_0,
\end{align*}
\]

with

\[
\kappa \Phi(x) = e B_2(x) + f B_1(x) + B_0(x),
\]

where

\[
e = \frac{1}{\gamma_1 \gamma_2} \left\{ \gamma_2 + ((u)_0 + 2) \left( \gamma_3 + (\beta_2 - \hat{\beta}_1)^2 \right) - ((u)_0 + 4) \hat{\gamma}_2 \right\},
\]

\[
f = \frac{1}{\gamma_1} \left[ \beta_1 + ((u)_0 + 2) \beta_2 - ((u)_0 + 3) \hat{\beta}_1 \right],
\]

and

\[
\hat{B}(x) = a B_2(x) + b B_1(x) + (u)_0 B_0(x),
\]

with

\[
a = \frac{1}{\gamma_1 \gamma_2} \left\{ (u)_0 \gamma_2 - ((u)_0 + 2) \left( \gamma_3 + \hat{\gamma}_1 + (\beta_2 - \hat{\beta}_1)^2 \right) + ((u)_0 + 4) \hat{\gamma}_2 \right\},
\]

\[
b = \frac{1}{\gamma_1} \left[ (u)_0 \beta_1 - ((u)_0 + 2) \beta_2 + ((u)_0 + 3) \hat{\beta}_1 - ((u)_0 + 1) \hat{\beta}_0 \right].
\]

Furthermore, we have two additional conditions:

\[
\begin{align*}
\left\{ \begin{array}{l}
((u)_0 + 2) \gamma_3 \gamma_4 - 2((u)_0 + 3) \gamma_4 \hat{\gamma}_2 + ((u)_0 + 4) \hat{\gamma}_2 \hat{\gamma}_3 = 0, \\
((u)_0 + 2) \gamma_3 (\beta_2 + \beta_3 - 2 \hat{\beta}_1) - ((u)_0 + 3) \hat{\gamma}_2 (2 \beta_3 - \beta_1 - \beta_2) = 0.
\end{array} \right.
\]

\[(3.7)\]

**Proof.** From (2.10), we have $\hat{\gamma}_{n+1} w_n^{[1]} = (x - \hat{\beta}_n) w_n^{[1]} - w_{n-1}^{[1]}$, $n \geq 0$, $(w_{-1}^{[1]} = 0)$. Applying $D_u$ in both hand sides, we obtain thanks to (2.21) and (2.22)

\[
w_n^{[1]} = - (n + (u)_0 + 2) \hat{\gamma}_{n+1} w_{n+2} + (n + (u)_0 + 1) (x - \hat{\beta}_n) w_{n+1} - (n + (u)_0) w_n, n \geq 1,
\]

\[(3.8)\]

\[
w_n^{[1]} + u - ((u)_0 + 1) (x - \hat{\beta}_n) w_1 + \hat{\gamma}_1 (u)_0 + 2) w_2 = 0.
\]

Taking into account assertion (a) of the Proposition 2.7, the relation (3.8) becomes

\[
B_n^{[1]} w_0^{[1]} = Z_{n+2} w_0, \quad n \geq 1,
\]

\[(3.10)\]

with $Z_{n+2}(x) = (w_0^{[1]} B_{n+2}^{[1]} - (n + (u)_0 + 2) \hat{\gamma}_{n+1} B_{n+2}(x) + (n + (u)_0 + 1) \hat{\gamma}_1 (u)_0 + 2) w_2 = 0$.

In particular, one has:

\[
\begin{align*}
B_1^{[1]} w_0^{[1]} &\,= Z_3 w_0, \\
B_2^{[1]} w_0^{[1]} &\,= Z_4 w_0.
\end{align*}
\]

Since $B_2^{[1]}(x) = (x - \hat{\beta}_1) B_1^{[1]}(x) - \hat{\gamma}_1$, then $w_0^{[1]} = \kappa \Phi w_0$, where $\kappa$ is a normalizing factor and $\kappa \Phi(x) = \hat{\gamma}_1^{-1} ((x - \hat{\beta}_1) Z_3(x) - Z_4(x))$. With help of the expressions of...
Substituting (3.3) in (3.10), we obtain thanks to Lemma 2.5
\[ \kappa B_n^{[1]}(x) \Phi(x) = Z_{n+2}(x), \quad n \geq 1. \] (3.11)

The analysis of degrees in latter equation shows that \( \deg \Phi \leq 2 \).

Substituting (3.3) in (3.9), and taking into account the assertion (a) of Proposition 2.7, one has \( u = \hat{B} w_0 \), where
\[ \hat{B}(x) = -\kappa \Phi(x) + ((u)_0 + 1)\gamma_1^{-1}(x - \tilde{\beta}_0)B_1(x) - ((u)_0 + 2)\gamma_1^{-1}\gamma_2^{-1}B_2(x). \]

Finally, the expression (3.6) is obtained using (3.5) and (3.1). \( \square \)

**Remark.**

1. From (3.3) and Remark 2.10, any \( \mathbf{D}_u \)-classical sequence \( \{B_n\}_{n \geq 0} \) is (strictly) quasi-orthogonal of order equal to \( \deg \Phi \) with respect to \( w_0^{[1]} \).

   Besides, when \( \hat{B} \neq 0 \), it is also (strictly) quasi-orthogonal of order equal to \( \deg \hat{B} \) with respect to \( u \).

2. Assume \( u \neq 0 \). From (3.3) and Lemma 2.3, the form \( u \) is weakly regular.

3. From (3.3) and (3.4), one has \( \kappa \Phi u = B w_0^{[1]} \). Therefore, when \( \hat{B} \neq 0 \), the orthogonal sequence \( \{B_n^{[1]}\}_{n \geq 0} \) is quasi-orthogonal of order equal to \( \deg \hat{B} \) with respect to the weakly-regular form \( v = \Phi u \).

   We conclude that for any nonzero \( u \in \mathbb{P}_n \), a \( \mathbf{D}_u \)-classical MPS \( \{B_n\}_{n \geq 0} \) is in particular a \( \mathbf{D}_u \)-semiclassical sequence (see [42], Definition 2.7, p7).

**Definition 3.3.** Let \( w_0 \) be a regular form with \( (w)_0 \neq -n, \quad n \geq 1, \) and \( \Phi, \Psi, \hat{B} \) are three nonzero polynomials, \( \Phi \) monic, \( \deg \Phi \leq 2 \), \( \deg \Psi = 1 \) and \( \deg \hat{B} \leq 2 \). The triplet \( (\Phi, \Psi, \hat{B}) \) is said to be admissible with respect to \( w_0 \), if we have
\[ \Psi'(0) - \frac{1}{2} \{ \Phi''(0)n + \langle w_0, \Phi \rangle \hat{B}''(0) \} \neq 0, \quad n \geq 1. \]

**Remark.** When \( \hat{B} = 0 \), we recognize the usual notion of admissibility of a pair of polynomials [39].

**Proposition 3.4.** A MPS \( \{B_n\}_{n \geq 0} \) (with respect to \( w_0 \)) is \( \mathbf{D}_u \)-classical, if and only if there exists a triplet \( (\Phi, \Psi, \hat{B}) \) admissible with respect to \( w_0 \), such that \( u = \hat{B} w_0 \), and \( w_0 \) is a solution of the functional equation
\[ \mathbf{D}_u (\Phi w_0) + \Psi w_0 = 0. \] (3.12)

**Proof.** Applying \( \mathbf{D}_u \) in both hand sides of (3.3) and using (2.22), we obtain the functional equation (3.12), with \( \Psi(x) = \frac{(u)_0 + 1}{\kappa \gamma_1} B_1(x) \). From Lemma 3.2, there exists a polynomial \( \hat{B} \) with \( \deg \hat{B} \leq 2 \) such that \( u = \hat{B} w_0 \).

On the other hand, using the relations \( \langle w_0, B_n^2 \rangle w_n = B_n w_0 \) and \( \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle w_n^{[1]} = B_n^{[1]} w_n^{[1]} \) in (2.22), we get:
\[ -\kappa \Psi B_n^{[1]} w_0 + \mathbf{D}_u (B_n^{[1]} w_0^{[1]} + (w_0^{[1]} \theta_0^0 B_1^{[n]}) u - (u_0 B_n^{[1]}) w_0^{[1]} =
\]
\[ -(n + (u)_0 + 1) \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle \langle w_0, B_{n+1} \rangle B_{n+1} w_0, \quad n \geq 0. \]
Taking into account Lemma 2.5, the fact that \( \gamma_1 = \langle w_0, B_1^2 \rangle \), \( \kappa^{-1} = \langle w_0, \Phi \rangle \), and the regularity of \( w_0 \), we conclude that:

\[
\Phi(x)D_u^2(B_{n+1})(x) - \Psi(x)D_u(B_{n+1})(x) = (3.13)
\]

\[-(n + (u_0 + 1)^2 \frac{\langle w_0, \Phi \rangle \langle w_0^{[1]} \rangle \langle B_{n+1}^{[1]} \rangle^2}{\langle w_0, B_{n+1}^2 \rangle} B_{n+1}(x) - \Delta(B_{n+1}; \Phi, \hat{B}, w_0)(x), \ n \geq 0,
\]

with \( \Delta(p; \Phi, \hat{B}, w_0) = \langle (\Phi w_0) \theta_0 D_{\hat{B}w_0}(p) \rangle \hat{B} - \langle (\hat{B}w_0) \theta_0 D_{\hat{B}w_0}(p) \rangle \Phi \), \( \forall p \in \mathbb{P} \).

In fact \((3.13)\) is a linear \( D_u \)-differential equation of second order, hence we obtain a second necessary condition taking with each element of the sequence \( \{B_n\}_{n \geq 0} \).

We later prove that it is sufficient, too.

Examination of degrees in \((3.13)\) proves that

\[
\Psi'(0) - \frac{1}{2} \{\Phi''(0) n + \langle w_0, \Phi \rangle \hat{B}''(0)\} = (n + (u_0 + 1)^2 \frac{\langle w_0, \Phi \rangle \langle w_0^{[1]} \rangle \langle B_{n+1}^{[1]} \rangle^2}{\langle w_0, B_{n+1}^2 \rangle} \neq 0, \ n \geq 1.
\]

Thus, the condition is necessary.

Let us prove that it is sufficient. We shall first establish that \( w_0^{[1]} = \kappa \Phi w_0 \). Indeed, we have \( D_u(\Phi w_0) + \Psi w_0 = 0 \) and \( \Psi(x) = \Psi'(0)B_1(x) + \langle w_0, \Psi \rangle \). But \( \langle w_0, \Psi \rangle = \langle \Psi w_0, 1 \rangle = -\langle D_u(\Phi w_0), 1 \rangle = 0 \). Thus \( D_u(\Phi w_0) + \Psi'(0)w_1 = 0 \), or \( D_u(\Phi w_0) - \Psi'(0)\kappa D_u(w_0^{[1]}) = 0 \). Hence \( w_0^{[1]} = \kappa \Phi w_0 \), where \( \kappa = \frac{(u_0 + 1)}{\Psi'(0)\kappa} \).

Now, we will prove that \( \{B_n^{[1]}\}_{n \geq 0} \) is orthogonal with respect to \( w_0^{[1]} = \kappa \Phi w_0 \).

Indeed, one has

\[
\langle w_0^{[1]}, x^m B_n^{[1]} \rangle = (n + (u_0 + 1)^{-1}\langle w_0^{[1]}, x^m D_u(B_{n+1}) \rangle = (n + (u_0 + 1)^{-1}\langle w_0^{[1]}, D_u(x^m B_{n+1}) - D_u(x^m B_{n+1}) \rangle
\]

\[-\{((B_{n+1} + u_0 x^m) - (u_0 x^m)) B_{n+1} \}
\]

\[= (n + (u_0 + 1)^{-1}\langle w_0^{[1]}, D_u(x^m B_{n+1}) \rangle
\]

\[= (n + (u_0 + 1)^{-1}\langle w_0^{[1]}, -m x^m B_{n+1} - (B_{n+1} + u_0 x^m) \rangle
\]

\[= (n + (u_0 + 1)^{-1}\langle w_0^{[1]}, -m x^m B_{n+1} - (B_{n+1} + u_0 x^m) \rangle
\]

by virtue of \((2.23), (3.3) \) and \((2.3)\). But

\[
\langle w_0^{[1]}, \langle B_{n+1} + u_0 x^m \rangle \rangle = \langle B_{n+1} + u_0 x^m \rangle = \langle w_0, \hat{B}(w_0^{[1]} \theta_0 x^m) \rangle B_{n+1} \rangle.
\]

Then,

\[
\langle w_0^{[1]}, x^m B_n^{[1]} \rangle = (n + (u_0 + 1)^{-1}\{(-D_u(w_0^{[1]}), x^m B_{n+1})
\]

\[-\langle w_0, \{m \kappa x^m - \hat{B}(w_0^{[1]} \theta_0 x^m) \} B_{n+1} \rangle
\]

\[= (n + (u_0 + 1)^{-1}\langle w_0, \{m \kappa x^m - \hat{B}(w_0^{[1]} \theta_0 x^m) \} B_{n+1} \rangle
\]

\[= (n + (u_0 + 1)^{-1}\langle w_0, \{m \kappa x^m - \hat{B}(w_0^{[1]} \theta_0 x^m) \} B_{n+1} \rangle.
\]

Therefore, \( \langle w_0^{[1]}, x^m B_n^{[1]} \rangle = 0 \), \( 0 \leq m \leq n - 1, \ n \geq 1 \). Besides, for \( m = n \):

\[
\langle w_0^{[1]}, x^n B_n^{[1]} \rangle = \frac{\Psi'(0) - \frac{1}{2} \{\Phi''(0) n + \langle w_0, \Phi \rangle \hat{B}''(0)\}}{\langle w_0, \Phi \rangle(n + (u_0 + 1)^{-1}\langle w_0, x^n B_{n+1}(x) \rangle \neq 0,
\]
for all integer \( n \geq 1 \).

As a consequence, a \( D_u \)-classical form is a particular \( D_u \)-Laguerre-Hahn one:

**Theorem 3.5.** Let \( w_0 \) be a regular form, \( \Phi \) be a monic polynomial, and \( \hat{B} \) a polynomial such that \( \langle w_0, \hat{B} \rangle \neq -n, n \geq 1 \). We set \( u = \hat{B}w_0 \), then the following assertions are equivalent:

(a) There exists a polynomial \( \Psi \), \( \deg \Psi \leq 1 \) such that

\[
D_u(\Phi w_0) + \Psi w_0 = 0.
\]  

(b) There exists a polynomial \( \Psi_1 \), such that

\[
(\Phi w_0)' + \Psi_1 w_0 - (\Phi \hat{B})(x^{-1}w_0^2) = 0.
\]  

In this case:

\[
\Psi_1(x) = \Psi(x) + \Phi(x)(w_0\theta_0(\hat{B}))(x) + \hat{B}(x)(w_0\theta_0(\Phi))(x).
\]  

**Proof.** (a) \( \Rightarrow \) (b). From (2.20) one has \( D_u(\Phi w_0) = (\Phi w_0)' - x^{-1}(u(\Phi w_0)) \). But (2.1) and the hypothesis give

\[
u(\Phi w_0) = \Phi(uw_0) - x(w_0\theta_0(\Phi))(x)u = \Phi((\hat{B}w_0)w_0) - x(w_0\theta_0(\Phi))(x)\hat{B}w_0
\]

\[
= \Phi(\hat{B}w_0^2 - x(w_0\theta_0(\hat{B}))(x)w_0) - x(w_0\theta_0(\Phi))(x)\hat{B}w_0
\]

\[
= \Phi(\hat{B}w_0^2 - Fw_0),
\]

where

\[
F(x) = \Phi(x)(w_0\theta_0(\hat{B}))(x) + \hat{B}(x)(w_0\theta_0(\Phi))(x).
\]

Using (2.5) and (2.2), it follows that

\[
x^{-1}(u(\Phi w_0)) = (\Phi \hat{B})x^{-1}w_0^2 - Fw_0 - \left[\langle w_0^2, \theta_0(\Phi \hat{B}) \rangle - \langle w_0, F \rangle \right] \delta.
\]

Thus, (3.14) becomes

\[
(\Phi w_0)' + (F + \Psi)w_0 - (\Phi \hat{B})(x^{-1}w_0^2) + \left(\langle w_0^2, \theta_0(\Phi \hat{B}) \rangle - \langle w_0, F \rangle \right) \delta = 0.
\]

Hence (3.15) and (3.16) hold taking into account (2.6).

(b) \( \Rightarrow \) (a). Let us consider the polynomial \( F \) given by (3.17).

Setting \( u = \hat{B}w_0 \) and \( \Psi = \Psi_1 - F \), we have

\[
D_u(\Phi w_0) + \Psi w_0 = (\Phi w_0)' + \Psi_1 w_0 - x^{-1}(u(\Phi w_0)) - Fw_0
\]

\[
= (\Phi \hat{B})(x^{-1}w_0^2) - x^{-1}(u(\Phi w_0)) - Fw_0.
\]

But, calculations done above (in the proof of (a) \( \Rightarrow \) (b)) prove that

\[
u(\Phi w_0) = (\Phi \hat{B}w_0^2 - Fw_0,
\]

Therefore,

\[
x^{-1}(u(\Phi w_0)) = x^{-1}\left(\Phi \hat{B}w_0^2 - Fw_0 + \langle w_0, F \rangle \delta,\right.
\]

\[
= x^{-1}\left(\langle x\theta_0(\Phi \hat{B}) + \Phi(0)\hat{B}(0) \rangle w_0^2 \right) - Fw_0 + \langle w_0, F \rangle \delta,
\]

\[
= \theta_0(\Phi \hat{B})w_0^2 - \langle w_0^2, \theta_0(\Phi \hat{B}) \rangle \delta + \Phi(0)\hat{B}(0)x^{-1}w_0^2 - Fw_0 + \langle w_0, F \rangle \delta
\]

\[
= \theta_0(\Phi \hat{B})w_0^2 + \Phi(0)\hat{B}(0)x^{-1}w_0^2 - Fw_0 = (\Phi \hat{B})(x^{-1}w_0^2) - Fw_0.
\]

Thus the desired result. \( \square \)
Proposition 3.6. Let \( \{B_n\}_{n \geq 0} \) be a \( \mathbf{D}_u \)-classical sequence (with respect to \( w_0 \)) fulfilling (3.12) with \( u = \tilde{B} w_0 \). The sequence \( \{\tilde{B}_n\}_{n \geq 0} \) defined by (2.15) is \( \mathbf{D}_{\tilde{u}} \)-classical and it fulfills
\[
\mathbf{D}_{\tilde{u}}(\tilde{\Phi} \tilde{w}_0) + \tilde{\Psi} \tilde{w}_0 = 0, \quad \text{and} \quad \tilde{u} = \tilde{B} \tilde{w}_0 = h_{u-1} \circ \tau_{-b} u,
\]
where \( \tilde{B}(x) = \tilde{B}(ax + b) \), \( \tilde{w}_0 = (h_{u-1} \circ \tau_{-b}) w_0 \), \( \tilde{\Phi}(x) = a^{-t} \Phi(ax + b) \), \( \tilde{\Psi}(x) = a^{1-t} \Psi(ax + b) \), and \( t = \deg(\Phi) \).

Proof. Using (2.16), we have
\[
\Phi(x) w_0 = \Phi(x)(\tau_b \circ h_a) \tilde{w}_0 = \tau_b \left( (\tau_{-b} \Phi)(h_a \tilde{w}_0) \right) = \tau_b \circ h_a \left( (h_a \circ \tau_{-b}) \Phi \tilde{w}_0 \right) = \tau_b \circ h_a \left[ \Phi(ax + b) \tilde{w}_0 \right].
\]

Thanks to (2.25) and (2.27), we obtain
\[
\mathbf{D}_u(\Phi w_0) = \tau_b \mathbf{D}_{\tau_{-b} u} h_a \left( \Phi(ax + b) \tilde{w}_0 \right) = a^{-1} \tau_b \circ h_a \mathbf{D}_{h_{u-1} \circ \tau_{-b} u}(\Phi(ax + b) \tilde{w}_0).
\]

From (3.12), we get (3.18). The triplet \( (\tilde{\Phi}, \tilde{\Psi}, \tilde{B}) \) is admissible since for any integer \( n \geq 1 \), one has:
\[
\tilde{\Psi}'(0) - \frac{1}{2} \left\{ \tilde{\Phi}''(0) n + (\tilde{w}_0, \tilde{\Phi}) \tilde{\Phi}''(0) \right\} = a^{2-t} \left\{ \Psi'(0) - \frac{1}{2} \left\{ \Phi''(0) n + (w_0, \Phi) \tilde{B}''(0) \right\} \right\} \neq 0.
\]

Proposition 3.7. A \( \mathbf{D}_u \)-classical form \( (u \neq 0) \) fulfilling (3.15) is a \( \mathbf{D} \)-Laguerre-Hahn form of class \( \max(\deg(\Psi_1) - 1, \deg(\tilde{B}) + \deg(\Phi) - 2) \).

Proof. Let \( w_0 \) be a \( \mathbf{D}_u \)-Laguerre-Hahn form of class \( s \), thus
\[
s \leq \max \left\{ \deg(\Psi_1) - 1, \deg(\Phi) + \deg(\tilde{B}) - 2 \right\}
\]
\[
\leq \max \left\{ \deg(\Psi_1) - 1, \deg(\Phi) + \deg(\tilde{B}) - 2 \right\}.
\]

Note that \( \tilde{B} \neq 0 \) since \( u \neq 0 \). Necessarily \( s = \max \left\{ \deg(\Psi_1) - 1, \deg(\Phi) + \deg(\tilde{B}) - 2 \right\} \). If not, the equation (3.15) will be simplified. Then, there exists a root \( c \) of \( \Phi \) such that if we write \( \Phi(x) = (x - c) \theta_c(\Phi)(x) \), we obtain
\[
\left( \theta_c(\Phi) w_0 \right)' + \left( \theta_c(\Psi_1) + \theta_c^2(\Phi) \right) w_0 - \theta_c(\Phi \tilde{B})(x^{-1} w_0^2) = 0.
\]

From Theorem 3.5, the form \( w_0 \) fulfills
\[
\mathbf{D}_u(\theta_c(\Phi) w_0) + \Psi_2 w_0 = 0, \tag{3.19}
\]
where
\[
\Psi_2(x) = \theta_c(\Psi_1) + \theta_c^2(\Phi) - \theta_c(\Phi)(w_0 \theta_c(\Phi))(x) - \tilde{B}(x) \left( \theta_c(\Phi \tilde{B})(x^{-1} w_0^2) \right)(x). \tag{3.20}
\]

On the other hand, computing \( \mathbf{D}_u (x - c) \theta_c(\Phi) w_0 \) gives, taking into account the regularity of \( w_0 \), and the relations (3.12), (3.19) and (3.20):
\[(x - c)\Psi_2(x) = \Psi(x) + ((u_0 + 1)\theta_c(\Phi) + (\theta_c(\Phi)w_0)\theta_0(x - c))\tilde{B}(x) - (u\theta_0(x - c))(\theta_c(\Phi)(x)).\]

In particular, one has \(\text{deg}(\Psi_2) \leq 1\). So, we can write \(\Psi_2(x) = aB_1(x) + b\), with \(a, b \in \mathbb{C}\), and \(a \neq 0\). Applying \(B_0\) in both hand sides of \((3.19)\) gives \(b = 0\). Therefore \(\Psi_2(x) = aB_1(x)\). Similarly we get \(\Psi(x) = \tilde{a}B_1(x)\) with \(\tilde{a} \neq 0\). Hence, \(w_0\) fulfills two functional equations \((3.12)\) and \((3.19)\). Therefore we have \(D_u( (\Phi - \tilde{a}\theta_c(\Phi))w_0 ) = 0\). So \(a\Phi - \tilde{a}\theta_c(\Phi) = 0\). Examination of degrees proves that this is not possible. \(\square\)

### 3.2. The second order linear differential equation.

**Proposition 3.9.** A MOPS \(\{B_n\}_{n \geq 0}\) with respect to \(w_0\) is \(\mathcal{D}_a\)-classical if and only if there exist three polynomials \(\Phi, \Psi, \tilde{B}\), \(\Phi\) monic, \(\text{deg} \Phi \leq 2\), \(\text{deg} \Psi = 1\) \(\text{deg} \tilde{B} \leq 2\), and a complex number sequence \(\{\lambda_n\}_{n \geq 0}\), \(\lambda_n \neq 0\), \(n \geq 0\), such that \(u = \tilde{B}w_0\) and that \(B_n\) for \(n \geq 0\) satisfies

\[
\Phi(x)D_u(B_{n+1})(x) - \Psi(x)D_u(B_{n+1})(x) = \lambda_n B_{n+1}(x) - \Delta(B_{n+1}; \Phi, \tilde{B}, w_0)(x), \quad n \geq 0,
\]

with \(\Delta(p; \Phi, \tilde{B}, w_0) = [(\Phi w_0)\theta_0D_{\tilde{B}w_0}(p)]\tilde{B} - [(\tilde{B}w_0)\theta_0 D_{\tilde{B}w_0}(p)]\Phi, \forall p \in \mathcal{P}\).

**Proof.**

We have seen that the condition is necessary (see \((3.13)\) ), where

\[
\lambda_n = -\frac{(n + (u_0 + 1))^2 \langle w_0, \Phi \rangle \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle}{\langle w_0, B_{n+1}^2 \rangle}, \quad n \geq 0.
\]

Conversely, the examination of leading coefficients in both hand sides of \((3.21)\) gives

\[
\frac{1}{2} \{\Phi''(0)n + \langle w_0, \Phi \rangle \tilde{B}''(0)\} - \Psi'(0) = (n + (u_0 + 1)^{-1}\lambda_n \neq 0, \quad n \geq 1.
\]

Thus \((\Phi, \Psi, \tilde{B})\) is admissible. Besides, using \((3.21)\) we get for \(n \geq 0\):

\[
\langle w_0, \Phi D_u^{[2]}(B_{n+1}) - \Psi D_u(B_{n+1}) + \Delta(B_{n+1}; \Phi, \tilde{B}, w_0) \rangle = \frac{\lambda_n \langle w_0, B_{n+1} \rangle}{n + (u_0 + 1)^{-1}} = 0.
\]

But \(\langle w_0, \Phi D_u^{[2]}(B_{n+1}) - \Psi D_u(B_{n+1}) + \Delta(B_{n+1}; \Phi, \tilde{B}, w_0) \rangle = (n + (u_0 + 1)^{-1}D_u \left[ D_u(\Phi w_0) + \Psi w_0 \right], B_{n+1}) \geq 0.
\]

Furthermore, it is easy to see that \(\langle D_u \left[ D_u(\Phi w_0) + \Psi w_0 \right], B_{n+1} \rangle = 0\). Hence

\(\langle D_u \left[ D_u(\Phi w_0) + \Psi w_0 \right], B_{n+1} \rangle = 0, \quad n \geq 0\). This implies that \(\langle D_u \left[ D_u(\Phi w_0) + \Psi w_0 \right], B_{n+1} \rangle = 0\). Then \(D_u(\Phi w_0) + \Psi w_0 = 0\). This ends the proof. \(\square\)

### 3.3. The first structure relation.

**Proposition 3.9.** A MOPS \(\{B_n\}_{n \geq 0}\) is \(\mathcal{D}_a\)-classical if and only if there exists a monic polynomial \(\Phi\) with \(\text{deg}(\Phi) = t \leq 2\), such that

\[
\Phi(x)B_{n+1}^{[1]}(x) = \sum_{\nu = n}^{n+t} \lambda_{n,\nu}B_{\nu}(x), \quad \lambda_{n,\nu} \neq 0, \quad n \geq 0,
\]

where \(\{B_n^{[1]}\}_{n \geq 0}\) is defined by \((2.21)\).
Proof. The condition is necessary. Indeed, the regular forms $w_0$ and $w_0^{[1]}$ verify $w_0^{[1]} = \kappa \Phi w_0$. Thanks to Proposition 2.6, this is equivalent to (3.22). The condition is sufficient. Indeed, applying $w_0$ in both hand sides of (3.22) gives $\langle \Phi w_0, B_0^{[1]} \rangle = 0$, $n \geq 1$, and $\langle \Phi w_0, B_0^{[1]} \rangle = \lambda_{0,0} \neq 0$. Lemma 2.1 implies that $w_0^{[1]} = \kappa \Phi w_0$ where $\kappa = \langle w_0, \Phi \rangle^{-1} = \lambda_{0,0}^{-1}$. Then
\[
\langle w_0^{[1]}, B_n^{[1]} B_m^{[1]} \rangle = \kappa \langle w_0, \Phi B_n^{[1]} B_m^{[1]} \rangle
\]
\[
= \kappa \sum_{\nu=0}^{n+t} \lambda_{n,\nu} w_0, B_\nu B_m^{[1]} = \left\{ \begin{array}{ll}
0, & m < n, n \geq 1, \\
\kappa \lambda_{n,n} w_0, B_n^{[1]} & m = n.
\end{array} \right.
\]
The sequence $\{B_n^{[1]}\}_{n\geq0}$ is orthogonal with respect to $w_0^{[1]}$. \qed

Remark.

(a) From (3.11), (3.5) and (3.1), we can write the structure relation (3.22) as given by Al-Salam and Chihara in the $D$-classical case [7]:
\[
\Phi(x) B_n^{[1]}(x) = \{X_n + Y_n x\} B_{n+1}(x) - \gamma_n T_n B_n(x), \ n \geq 0,
\]
where for $n \geq 1$:
\[
X_n = -\kappa^{-1} \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle \left[ \frac{(n + (u)0 + 1) \tilde{\beta}_n}{w_0, B_{n+1}^2} - \frac{(n + (u)0 + 2) \beta_{n+1} \gamma_{n+1}}{w_0, B_{n+2}^2} \right],
\]
\[
X_0 = \kappa^{-1} (f - \beta_1 e),
\]
\[
Y_n = -\kappa^{-1} \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle \left[ \frac{(n + (u)0 + 1) \tilde{\beta}_n}{w_0, B_{n+1}^2} - \frac{(n + (u)0 + 2) \gamma_{n+1}}{w_0, B_{n+2}^2} \right],
\]
\[
Y_0 = \kappa^{-1} e,
\]
\[
T_n = -\kappa^{-1} \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle \left[ \frac{(n + (u)0)}{w_0, B_{n+1}^2} + \frac{(n + (u)0 + 2) \gamma_{n+1}}{w_0, B_{n+2}^2} \right],
\]
\[
T_0 = \kappa^{-1} (e - \gamma_1^{-1} (2(u)0 + 3)).
\]
Using the orthogonality of $\{B_n\}_{n\geq0}$ in (3.23), we get (3.22). In particular:
\[
\lambda_{n,n} = \gamma_{n+1} (Y_n - T_n) = \kappa^{-1} \langle w_0^{[1]}, (B_n^{[1]})^2 \rangle \left( 2n + 2(u)0 + 1 \right) \neq 0, \ n \geq 1,
\]
and $\lambda_{0,0} = \frac{(2(u)0 + 3)}{\kappa \gamma_1} \neq 0$.

Thus, the condition $2n + 2(u)0 + 1 \neq 0, \ n \geq 1$ holds.

(b) In fact, the condition $Y_n - T_n \neq 0, \ n \geq 0$ is a consequence of the regularity of $w_0$. Indeed, let $\{B_n\}_{n\geq0}$ be a MOPS with respect to $w_0$, and fulfilling (3.23). Applying $w_0$ in both hand sides of (3.23) we get $\lambda w_0^{[1]} = \Phi w_0$, with $\lambda = \langle w_0, \Phi \rangle$. Since $w_0$ is regular, we have $\lambda \neq 0$.

Besides, from the orthogonality of $\{B_n\}_{n\geq0}$, and thanks to (3.23), one has:
For $n \geq 0$, and $0 \leq m \leq n$:
\[
\langle w_0^{[1]}, B_n^{[1]} B_m^{[1]} \rangle = \lambda^{-1} \langle \Phi w_0, B_n^{[1]} B_m^{[1]} \rangle
\]
\[
= \lambda^{-1} \langle w_0, \{X_n + Y_n x\} B_{n+1}(x) - \gamma_{n+1} T_n B_n(x) \rangle B_m^{[1]} \rangle
\]
\[
= \left\{ \begin{array}{ll}
0, & m < n, n \geq 1, \\
\gamma_{n+1} (Y_n - T_n) \langle w_0, B_n^{[1]} \rangle \langle w_0, B_n^{[1]} \rangle & m = n.
\end{array} \right.
\]
We must have $Y_n - T_n \neq 0$, $n \geq 0$. If not, there exists an integer $n_0 \geq 0$ such that $Y_{n_0} - T_{n_0} = 0$. Then $\langle w_0^{[1]}, (B_{n_0})^2 \rangle = 0$. Thanks to (3.29), this implies $\langle B_{n_0} w_0^{[1]}, B_{n_0} \rangle = 0, n \geq 0$. Hence $B_{n_0} w_0^{[1]} = 0$. But $\lambda w_0^{[1]} = \Phi w_0$. Then $B_{n_0} \Phi w_0 = 0$. This contradicts the regularity of $w_0$.

**Proposition 3.10.** A MOPS $\{B_n\}_{n \geq 0}$ is $D_u$-classical if and only if there exist two polynomials $\Phi$ and $\hat{B}$, $\Phi$ monic, $\deg(\Phi) \leq 2$, $\deg(\hat{B}) \leq 2$ and $\langle w_0, \hat{B} \rangle = 1$, such that

$$
\Phi(x)B_{n+1}'(x) - B(x)B_n'(x) = q_3(x; n)B_{n+1}(x) + F_n B_n(x), \quad n \geq 0,
$$

where $q_3(\cdot; n), n \geq 0$, is a polynomial $\deg(q_3(\cdot; n)) \leq 3, n \geq 0$ and

$$
B(x) = -\Phi(x)\hat{B}(x).
$$

In this case, we have:

$$
q_3(x; n) = \Phi(x)\{w_0\theta_0 \hat{B}(x)\} + (n + (u)_0 + 1)\{X_n + Y_n x\}, \quad n \geq 0,
$$

$$
F_n(x) = -(n + (u)_0 + 1)\gamma_{n+1} T_n + \gamma_2^{-1}(w_0, \hat{B} B_2) \delta_{n,0}, \quad n \geq 0.
$$

In particular, the sequence $\{B_n\}_{n \geq 0}$ is a $D$-Laguerre-Hahn sequence.

**Proof.** The condition is necessary. Thanks to Remark 3.3 (a), and (3.1), we have:

$$
\Phi(x)B_{n+1}'(x) + \Phi(x)(u\theta_0 B_{n+1}) = (n + (u)_0 + 1)\{X_n + Y_n x\} B_{n+1}(x)
$$

$$
-(n + (u)_0 + 1)\gamma_{n+1} T_n B_n(x), \quad n \geq 0.
$$

Since $u = \hat{B} w_0$, we get:

$$
(u\theta_0 B_{n+1})(x) = \langle w_0, \hat{B}(y) (B_{n+1}(x) - B_{n+1}(y)) \rangle
$$

$$
= \hat{B}(x) B_n'(x) - \langle w_0 \theta_0 \hat{B}(x) B_{n+1}(x) \rangle
$$

$$
+ \langle w_0, \left( \frac{\hat{B}(x) - \hat{B}(y)}{x - y} \right) B_{n+1}(y) \rangle,
$$

$$
= \hat{B}(x) B_n'(x) - \langle w_0 \theta_0 \hat{B}(x) B_{n+1}(x) \rangle
$$

$$
- \gamma_2^{-1}(w_0, \hat{B} B_2) \delta_{n,0}, \quad n \geq 0.
$$

Substituting (3.29) in (3.28), the equations (3.24)-(3.27) hold.

The condition is sufficient. Indeed, we set $u = \hat{B} w_0$. Using (3.29) and (3.25) in (3.24), gives, for $n \geq 0$:

$$
\Phi(x)B_{n+1}'(x) + \Phi(x) [(u\theta_0 B_{n+1})(x) - (u\theta_0 \hat{B}) B_{n+1}(x)] = q_3(x, n) B_{n+1}(x) + F_n B_n(x),
$$

which can be written as $\Phi(x)B_{n+1}'(x) = A_1(x, n) B_{n+1}(x) - \gamma_{n+1} T_n B_n(x), \quad n \geq 0$, where

$$
T_n = -(n + (u)_0 + 1) \gamma_{n+1} F_n, \quad n \geq 0, \quad \text{and} \quad A_1(x, n) = (n + (u)_0 + 1) \gamma_{n+1} F_n, \quad n \geq 0.
$$

Comparing the degrees, we get $\deg(A_1(\cdot, n)) \leq 1$, $n \geq 0$. Thus, we can write $A_1(x, n) = X_n + Y_n x, \quad n \geq 0$. From Remark 3.3 (b), necessarily $Y_n - T_n \neq 0$, $n \geq 0$. Taking into account the first part of Remark 3.3 and Proposition 3.9, the desired result holds. \qed
3.4. The second structure relation.

**Proposition 3.11.** A MOPS \( \{B_n\}_{n \geq 0} \) is \( D_u \)-classical if and only if there exist an integer \( 0 \leq t \leq 2 \) and a polynomial \( B \), \( \deg(B) \leq 2 \), such that \( u = Bu_0 \) and:

\[
B_n(x) = \sum_{\nu = n-t}^{n} \lambda_{n,\nu} B_{\nu}^{[1]}(x), \quad \lambda_{n,n-t} \neq 0, \quad n \geq t, \tag{3.30}
\]

where \( \{B_{n}^{[1]}\}_{n \geq 0} \) is defined by (2.21).

**Proof.** The condition is necessary. Indeed, we have \( u_0^{[1]} = \kappa \Phi u_0 \). From Proposition 2.6, this is equivalent to (3.30). Besides, from Lemma 3.2 one has \( u = Bu_0 \).

The condition is also sufficient. Indeed, the relation (3.30) can be written as:

\[
B_n(x) = B_n^{[1]}(x) + a_n B_{n-1}^{[1]}(x) + b_n B_{n-2}^{[1]}(x), \quad n \geq 0, \tag{3.31}
\]

with \( a_0 = b_0 = b_1 = 0 \).

On the other hand, applying \( D_u \) in both hand sides of (3.1) gives

\[(n + (u_0) + 2)B_{n+1}^{[1]}(x) = (n + (u_0) + 1)(x - \beta_{n+1})B_n^{[1]}(x) - (n + (u_0))\gamma_{n+1}B_{n-1}^{[1]}(x) + (u, B_{n+1}) + B_{n+1}(x), \quad n \geq 0.\]

By (3.31), the fact that \( u = Bu_0 \), and the orthogonality of \( \{B_n\}_{n \geq 0} \), we have:

\[
\begin{cases}
B_0^{[1]}(x) = 1, & B_1^{[1]}(x) = x - \beta_0, \\
B_{n+2}^{[1]}(x) = (x - \beta_{n+1})B_n^{[1]}(x) - \gamma_{n+1}B_{n-1}^{[1]}(x), \quad n \geq 0,
\end{cases} \tag{3.32}
\]

with \( \beta_0 = \beta_1 = -\frac{a_1 + (u, B_1)}{(u_0) + 1}, \quad \beta_{n+1} = \beta_{n+2} = \frac{(n + (u_0) + 2)^{-1}a_{n+2}}{n + (u_0) + 2}, \quad n \geq 0 \)

\[
\gamma_1 = -\frac{b_2 + ((u_0) + 1)\gamma_2 - (u, B_2)}{(u_0) + 2}, \quad \gamma_{n+1} = -\frac{b_{n+2} + ((n + (u_0) + 1)\gamma_{n+2}}{n + (u_0) + 2}, \quad n \geq 1.
\]

Necessarily, \( \gamma_n \neq 0, \quad n \geq 1 \), otherwise there exists an integer \( N \geq 1 \) such that \( \gamma_N = 0 \). Thanks to (3.32), there exists \( c \in \mathbb{C} \) such that \( B_{n+1}^{[1]}(c) = 0 \).

Then, \( B_{n}^{[1]}(c) = 0, \quad n \geq N \). This is absurd because if not, the relation (3.31) implies that \( B_n(c) = 0, \quad n \geq N \). Thus the sequence \( \{B_n\}_{n \geq 0} \) is not orthogonal. \( \square \)

**Proposition 3.12.** We suppose that \( \{B_n\}_{n \geq 0} \) is \( D_u \)-classical, and fulfills (3.1) and (3.2). Then \( \{B_n\}_{n \geq 0} \) verifies:

\[
B_n(x) = B_n^{[1]}(x) + a_n B_{n-1}^{[1]}(x) + b_n B_{n-2}^{[1]}(x), \quad n \geq 1, \tag{3.33}
\]

with \( a_1 = (u_0 + 1)(\beta_1 - \beta_0) - (u, B_1), \quad a_n = (n + (u_0))\beta_n - \beta_{n-1}, \quad n \geq 2, \)

and \( b_1 = 0, \quad b_2 = ((u_0) + 1)\gamma_2 - ((u_0) + 2)\gamma_1 - (u, B_2), \quad b_n = (n + (u_0) + 1)\gamma_n - (n + (u_0))\gamma_{n-1}, \quad n \geq 3. \)

**Proof.** Taking the derivative in (3.1), then using (2.4), we get:

\[
(n + (u_0) + 2)B_{n+1}^{[1]}(x) = (n + (u_0) + 1)(x - \beta_{n+1})B_n^{[1]}(x) \tag{3.34}
\]

\[
-(n + (u_0))\gamma_{n+1}B_{n-1}^{[1]}(x) + (u, B_{n+1}) + B_{n+1}(x), \quad n \geq 0.
\]

But from (3.2), where \( n \to n - 1 \), one has:

\[
x B_{n+1}^{[1]}(x) = B_{n+1}^{[1]}(x) + \beta_n B_n^{[1]}(x) + \gamma_n B_{n-1}^{[1]}(x), \quad n \geq 0. \tag{3.35}
\]
Using (3.33) in (3.34), we obtain
\[ B_{n+1}(x) = B_{n+1}^{[1]}(x) + (n + (u_0 + 1)(\beta_{n+1} - \beta_n))B_{n+1}^{[1]}(x) + \left((n + (u_0)\gamma_{n+1} - (n + (u_0 + 1)\gamma_n)B_{n+1}^{[1]}(x) - \langle u, B_{n+1} \rangle, n \geq 0. \]

The relation (3.33) holds since we have \( \langle u, B_n \rangle = \langle w_0, \tilde{B}B_n \rangle = 0, \ n \geq 3, \) and \( B_n^{[1]} = 0, \ n \geq 1. \)

3.5. The functional equation fulfilled by \( w_0^{[1]} \).

Proposition 3.13. The form \( w_0^{[1]} \) satisfies:
\[ (\Phi w_0^{[1]})' + \tilde{\Psi} w_0^{[1]} - \kappa^{-1} \tilde{B} \left( x^{-1} (w_0^{[1]})^2 \right) + a \tilde{B}(0)\delta = 0, \]
where \( \tilde{\Psi}(x) = H - \Phi - 2\tilde{B}(w_0\theta_0\Phi)(x), \ H := F - \Psi, \) and \( a = 2\langle w_0^{[1]}, (w_0\theta_0\Phi) - \kappa w_0^{[1]} \theta_0(\Phi^2) \rangle. \)

Proof. Multiplying both hand sides of (3.15) by \( \kappa \Phi, \) and taking into account the identity \( w_0^{[1]} = \kappa \Phi w_0, \) we get
\[ (\Phi w_0^{[1]})' + (H - \Phi')w_0^{[1]} - \kappa \tilde{B}\Phi^2(x^{-1}w_0^{[1]}) = 0. \] (3.36)

On the other hand, we have
\[ \Phi^2(x^{-1}w_0^{[1]}) = x^{-1}(\Phi^2w_0^{[1]}) + \langle w_0^{[1]}, \theta_0(\Phi^2) \rangle \delta, \] (3.37)
and \( \Phi^2w_0^{[1]} = (\Phi w_0)^2 + 2x\Phi(x)(w_0\theta_0\Phi)(x)w_0. \) Then (3.37) will be written, thanks to \( w_0^{[1]} = \kappa \Phi w_0, \) as follows
\[ \Phi^2(x^{-1}w_0^{[1]}) = \kappa^2 x^{-1} \left( w_0^{[1]} \right)^2 + 2\kappa^{-1}x^{-1}(x(w_0\theta_0\Phi)(x)w_0^{[1]}) + \langle w_0^{[1]}, \theta_0(\Phi^2) \rangle \delta = 0. \]

With the help of the last equation, the relation (3.36) gives the desired result. \( \Box \)

Remark. Another expression for \( a \) is: \( a = \kappa \langle w_0, w_0(\theta_0\Phi)^2 - 2x(w_0\theta_0\Phi)(x)(\theta_0\Phi)(x) \rangle. \)

3.6. The nonlinear system fulfilled by \( \beta_n, \ \gamma_{n+1}, \ \beta_n \) and \( \gamma_{n+1}. \)

On the one hand, the comparison of constant terms in the equation \( B_n^{[1]}(x) = ((u_0 + 2) - (B_2^{[1]}(x) + (u_0\theta_0B_2)(x)), \) gives
\[ ((u_0 + 2)\beta_0 = (u_0 + 1)\beta_1 + \alpha_0 - \langle u, B_1 \rangle. \] (3.38)

On the other hand, multiplying both hand sides of (3.33) by \( x, \) and taking into account (3.2), we have
\[ xB_{n+1}(x) = B_{n+2}^{[1]}(x) + \left\{ \beta_{n+1} + a_{n+1} \right\} B_{n+1}^{[1]}(x) + \left\{ \gamma_{n+1} + \beta_{n+1}a_{n+1} + b_{n+1} \right\} B_n^{[1]}(x) + \left\{ a_{n+1}\gamma_n + b_{n+1}\beta_n + a_{n+1}\beta_n - 1 \right\} B_{n-1}^{[1]}(x) + \left\{ a_{n+1}\gamma_n + b_{n+1}\beta_n - 1 \right\} B_{n-1}^{[1]}(x) - \langle u, B_{n+1} \rangle B_n^{[1]}(x) - \langle u, B_{n+1} \rangle \beta_0, \ n \geq 1. \]

Using (3.1) and (3.33), we get
\[ xB_{n+1}(x) = B_{n+2}^{[1]}(x) + \left\{ a_{n+1} + \beta_{n+1} \right\} B_{n+1}^{[1]}(x) + \left\{ b_{n+1}\beta_{n+1} + a_{n+1}\gamma_{n+1} \right\} B_{n-1}^{[1]}(x) - \langle u, B_{n+1} \rangle B_n^{[1]}(x) + \left\{ a_{n+1}\gamma_n + b_{n+1}\beta_n - 1 \right\} B_{n-1}^{[1]}(x) + \left\{ a_{n+1}\gamma_n + b_{n+1}\beta_n - 1 \right\} B_{n-1}^{[1]}(x) - \langle u, B_{n+1} \rangle \beta_0, \ n \geq 1. \]
Comparing (3.39) and (3.40) gives, thanks to (3.38), the following system:

\[
(n + (u)0 + 2)\beta_n - (n + (u)0)\beta_{n-1} = (n + (u)0 + 1)\beta_{n+1} - (n + (u)0 - 1)\beta_n + \langle u, B1 \rangle \delta_{n,1} + ((u)0\beta_0 - \langle u, B1 \rangle)\delta_{n,0}, \quad n \geq 0. \quad (\beta - 1 = 0)
\]

\[
(n + (u)0 + 3)\tilde{\gamma}_{n+1} - (n + (u)0 + 1)\tilde{\gamma}_n = (n + (u)0 + 1)\gamma_{n+2} - (n + (u)0 - 1)\gamma_{n+1} + (n + (u)0 + 1)(\beta_{n+1} - \beta_n)^2 + \langle u, B2 \rangle \delta_{n,1} - [(u, B2) + \langle u, B1 \rangle(\beta_1 - \tilde{\beta}_0) - \langle u, B1 \rangle \delta_{n,0}, \quad n \geq 0. \quad (\tilde{\gamma}_0 = 0)
\]

\[
(n + (u)0 + 3)\tilde{\gamma}_{n+2}\tilde{\gamma}_{n+1} - 2(n + (u)0 + 2)\gamma_{n+3}\tilde{\gamma}_{n+1} + (n + (u)0 + 1)\gamma_{n+3}\tilde{\gamma}_{n+2} \quad (3.42)
\]

\[-\gamma_3(u, B2)\delta_{n,0} = 0, \quad n \geq 0.
\]

**Remark.** The first additional condition (3.7) is none other than (3.42), with \( n = 1. \)

4. **Canonical cases**

If we set:

\[
\tilde{\gamma}_n = \frac{(n + (u)0)\vartheta_n}{n + (u)0 + 1}\gamma_{n+1}, \quad \text{with } \vartheta_n \neq 0, \quad n \geq 1,
\]

then, (3.42) gives

\[
\vartheta_{n+2}\vartheta_{n+1} - 2\vartheta_{n+1} + 1 = 0, \quad n \geq 1,
\]

\[
\vartheta_2\vartheta_1 - 2\vartheta_1 + 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 = 0. \quad (4.3)
\]

Since (4.2) is a Riccati equation, we consider \( \vartheta_n = \frac{\xi_{n+1}}{\xi_n}, \quad n \geq 1, \quad \xi_n \neq 0, \quad n \geq 1. \)

Thus \( \xi_{n+3} - 2\xi_{n+2} + \xi_{n+1} = 0, \quad n \geq 1. \) So, the general solution is \( \xi_n = an + b, \quad n \geq 2, \)

where \( (a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}. \) We deduce that \( \vartheta_n = \frac{a(n + 1) + b}{an + b}, \quad n \geq 2. \) Therefore, the relation (4.3) gives

\[
\left[ 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 \right] \xi_1 = a + b.
\]

Two cases appear:

(A) \( 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 = 0. \) Consequently, \( \vartheta_n = \frac{n}{n - 1}, \quad n \geq 2 \) and \( \vartheta_1 \) is arbitrary.

(B) \( 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 \neq 0. \) Here we have:

\[
\vartheta_n = \frac{a(n + 1) + b}{an + b}, \quad n \geq 2, \quad \text{and } \vartheta_1 = \frac{2a + b}{a + b} \left[ 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 \right].
\]

We have to distinguish two subcases:

**B1.** \( a = 0. \) Then \( \vartheta_n = 1, \quad n \geq 2, \) and \( \vartheta_1 = 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2. \)

**B2.** \( a \neq 0. \) If we set \( \rho = \frac{b}{a}, \) we can write \( \vartheta_n = \frac{n + \rho + 1}{n + \rho}, \quad \rho \neq -n, \quad n \geq 2, \) and \( \vartheta_1 = \frac{\rho + 2}{\rho + 1} \left[ 1 - \frac{\langle u, B2 \rangle}{\langle (u)0 + 1 \rangle}\gamma_2 \right], \quad \rho \neq -1. \)

Note that when \( \rho \rightarrow \infty, \) we reorganize the subcase B1.
Remark. Taking into account (3.41), the second additional condition (3.4) becomes (4.1), with \( n = 1 \).

Remark. Before solving the above system, we shall give update expressions of polynomials \( \Phi(x) \) and \( \hat{B}(x) \). Indeed, it is easy to see that:

\[
\kappa \Phi(x) = \frac{\langle w_0, B_2 \rangle}{\gamma_1 \gamma_2} B_2(x) + \frac{\langle w_0, B_1 \rangle}{\gamma_1} B_1(x) + 1.
\]

But from (3.33), one has \( \langle w_0, B_2 \rangle = b_2 = ((u)_0 + 1) \gamma_2 - ((u)_0 + 2) \gamma_1 - \langle u, B_2 \rangle \).

Using (4.1), we obtain \( \langle w_0, B_2 \rangle = ((u)_0 + 1) \gamma_2(1 - \vartheta_1) - \langle u, B_2 \rangle \). We also have \( \langle w_0, B_1 \rangle = a_1 \). Using (3.38), gives \( \langle w_0, B_1 \rangle = \frac{((u)_0 + 1) \beta - \langle u, B_1 \rangle}{(u)_0 + 2} \).

For \( \hat{B}(x) \), we should remark that from (3.7) and (3.8), we get \( e = \kappa, a + \kappa = (\langle u, B_0 \rangle + (\beta_0 - \hat{\beta}_0) \vartheta_1 - \langle u, B_1 \rangle \).

Finally, it is clear that:

\[
\hat{B}(x) = \frac{\langle u, B_2 \rangle}{\gamma_1 \gamma_2} B_2(x) + \frac{\langle u, B_1 \rangle}{\gamma_1} B_1(x) + \langle u \rangle_0. \tag{4.4}
\]

Case (A). The system (3.41) becomes:

\[
(n + (u)_0 + 2) \hat{\beta}_n - (n + (u)_0) \hat{\beta}_{n-1} = (n + (u)_0 + 1) \beta_{n+1} - (n + (u)_0 - 1) \beta_n + \langle u, B_1 \rangle \delta_{n, 0}, n \geq 0. \tag{4.5}
\]

\[
\frac{(2n + (u)_0 + 3)}{n(n + (u)_0 + 2)} \gamma_{n+2} = \frac{(2n + (u)_0 - 1)}{(n - 1)(n + (u)_0 + 1)} \gamma_{n+1} + (\beta_{n+1} - \hat{\beta}_n)^2, n \geq 2.
\]

\[
\gamma_3 = \frac{(u)_0 + 3}{(u)_0 + 2((u)_0 + 5)} \left[ (u)_0 + 1 \right] \gamma_2 + ((u)_0 + 2) \beta_2 - \hat{\beta}_1)^2
\]

\[
\gamma_2 = \frac{(u)_0 + 2}{(u)_0 + 1((u)_0 + 3)} \gamma_1 + (u)_0 + 1(\beta_1 - \hat{\beta}_0)^2 - (\beta_1 - \hat{\beta}_0)(u, B_1) \right]
\]

\[
(n + 1) \hat{\beta}_{n+1} - (n - 1) \hat{\beta}_n = (n + 2) \beta_{n+2} - n \beta_{n+1} + \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle \right] \delta_{n, 0}, n \geq 0. \tag{4.7}
\]

Subtracting (4.7) from (4.5), one has \( \hat{\beta}_n - \frac{(u)_0}{(u)_0 + 2} \beta_{n+1} = \beta_{n-1} - \frac{(u)_0}{(u)_0 + 2} \beta_n, n \geq 2, \)

then by iteration \( \hat{\beta}_n = \frac{(u)_0}{(u)_0 + 2} \beta_{n+1} + \hat{\beta}_1 - \frac{(u)_0}{(u)_0 + 2} \beta_2, n \geq 1 \). Inserting this equation in (4.5), we get

\[
(2n + (u)_0 + 2) \beta_{n+1} - (2n + (u)_0 - 2) \beta_n = 2 \left( ((u)_0 + 2) \hat{\beta}_1 - (u)_0 \beta_2 \right), n \geq 2.
\]

Therefore, \( \beta_n = d + \frac{(u)_0 + 2}{(2n + (u)_0)(2n + (u)_0 - 2)} \left[ ((u)_0 + 2 - \sigma) \beta_0 + ((u)_0 + 2 + \sigma) \beta_1 - \frac{\sigma}{(u)_0 + 1} \langle u, B_1 \rangle \right], \sigma = \vartheta_1^{-1}, \)

n \geq 2, where

\[
d = \frac{1}{2((u)_0 + 2)} \left[ ((u)_0 + 2 - \sigma) \beta_0 + ((u)_0 + 2 + \sigma) \beta_1 - \frac{\sigma}{(u)_0 + 1} \langle u, B_1 \rangle \right], \sigma = \vartheta_1^{-1}, \]
and \( \theta = \beta_1 + (u_0 + 1)\beta_0 + \langle u, B_1 \rangle \).

Hence, \( \beta_n = d + \frac{(u_0)[(u_0 + 2)d - \theta]}{(2n + (u_0))(2n + (u_0) + 2)} \), \( n \geq 1 \). Thus, iterating (4.6) gives:

\[
\gamma_{n+1} = \frac{(n-1)(n + (u_0) + 1)}{(2n + (u_0) - 1)(2n + (u_0) + 1)} \left\{ -\frac{[(u_0) + 2)d - \theta]^2}{(2n + (u_0)^2} + \frac{((u_0) + 3)\gamma_3 + [(u_0) + 2)d - \theta]^2}{(u_0) + 4} \right\}, \quad n \geq 2.
\]

Moreover, we have \( \gamma_3 = \frac{(u_0) + 3}{(u_0) + 2} \left[ \frac{(u_0) + 2}{\sigma} \gamma_2 + \frac{4[(u_0) + 2)d - \theta]^2}{(u_0) + 2)\gamma_3} \right] \), \( \gamma_2 = \frac{(u_0) + 2}{(u_0) + 3} \left\{ \gamma_1 + \frac{\theta - (u_0) + 2)(\beta_0)}{(u_0) + 2} \left[ \frac{(u_0 + 1)[\theta - (u_0) + 2)(\beta_0)]}{(u_0) + 2} + \langle u, B_1 \rangle \right] \right\} \).

The above system is completely solved.

Note that in this case, the polynomial \( \Phi(x) \) is of degree 2 given by

\[
\Phi(x) = (x - d)^2 - \frac{\mu}{4}, \quad \text{where} \quad \mu = 4\left\{ (\beta_0 - d)^2 + \frac{(u_0) + 2) + 1}{(u_0) + 1} \right\}. \quad (4.8)
\]

Since \( \kappa = -\frac{(u_0) + 1}{\sigma \gamma_1} \), we get \( \Psi(x) = -\sigma(x - \beta_0) \), and \( \tilde{B}(x) = \frac{(u_0) + 1}{\gamma_1} B_2(x) + \frac{\langle u, B_1 \rangle}{\gamma_1} B_1(x) + (u_0) \).

Depending on the number of roots of \( \Phi(x) \), two subcases appear:

A1. \( \Phi \) has a double root (\( \mu = 0 \)). In this case \( \Phi(x) = (x - d)^2 \). Through a shift, we can choose 0 as a root. Then we assume that \( d = 0 \). So \( \Phi(x) = x^2 \).

A2. \( \Phi \) has two different roots (\( \mu \neq 0 \)). Upper to an affine transformation, we can assume that \( \Phi(x) = x^2 - 1 \). This amounts to take \( d = 0 \) and \( \mu = 4 \).

In the subcase \( A1 \), we have \( d = \mu = 0 \). Then

\[
\left\{ \begin{array}{l}
\langle u, B_1 \rangle = \frac{(u_0) + 1}{(u_0) + 2 - \sigma)\beta_0 + ((u_0) + 2 + \sigma)\beta_1}, \\
\beta_0^2 + \frac{(u_0) + 2) + 1}{(u_0) + 1} \gamma_1 = 0.
\end{array} \right. \quad (4.9)
\]

Necessarily, \( \sigma \neq -(u_0) + 1 \) and then \( \beta_0 \neq 0 \). Otherwise, (4.9) implies \( \beta_1 = -\langle u, B_1 \rangle \), and \( \theta = 0 \). Thus, \( \gamma_3 = 0 \) which is impossible. Consequently, \( \gamma_1 = \frac{-(u_0) + 1)\beta_0^2}{(u_0) + 1} \). If \( \sigma = 2\alpha \), then a dilation allows us to assume that \( \beta_0 = -\alpha^{-1} \).

Hence, \( \Psi(x) = -2(\alpha x + 1) \) and \( \gamma_1 = -\frac{(u_0) + 1}{\alpha^2((u_0) + 2\alpha + 1)} \). Taking into account (4.9), we get the results summarized in Table 1. In particular, thanks to Proposition 3.7, the sequence \( \{B_n\}_{n \geq 0} \) is a \( \mathbf{D} \)-Laguerre-Hahn sequence of class \( s = 2 \).

For the subcase \( A2 \), the relation (4.9) remains valid. Besides, we have \( \frac{(u_0) + 2) + 1}{(u_0) + 1} \gamma_1 = 1 - \beta_0^2 \). According to the value of \( \beta_0 \), we give explicit the corresponding coefficients (See Table 1). In particular, thanks to Proposition 3.7, the sequence \( \{B_n\}_{n \geq 0} \) is a \( \mathbf{D} \)-Laguerre-Hahn sequence of class \( s = 2 \).
### Table 1. The $D_{n}$-Classical Polynomials - Case (A)

<table>
<thead>
<tr>
<th>$A_1$: $\Phi(x) = x^2$</th>
<th>$A_2$: $\Phi(x) = x^2 - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0 = \pm 1$</td>
<td>$\beta_0 \neq \pm 1$</td>
</tr>
</tbody>
</table>

**Case (A)**: $(u, B_2) = ((u_0) + 1)\gamma_2$, $(u_0) \neq -n$, $n \geq 1$

<table>
<thead>
<tr>
<th>$\Phi(x) = -2(ax + 1)$,</th>
<th>$\Phi(x) = (u_0 + 1)(x - \beta_0)$,</th>
<th>$\Phi(x) = -\sigma(x - \beta_0)$, $\sigma \neq 0$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0 = \frac{-a}{\beta_1}$, $\alpha \neq 0$,</td>
<td>$\beta_0 = \frac{a}{\beta_1}$, $\beta_1 \in \mathbb{C}$,</td>
<td>$\beta_0 = \frac{\sigma a}{\beta_1}$, $\sigma \neq 0$,</td>
</tr>
<tr>
<td>$\beta_1 = \frac{(u_0) + 1)((u_0) + 2) + 2a^2\theta}{(u_0) + 2 + 2a + 1}$, $(u_0) + 2 + 1 \neq 0$,</td>
<td>$\beta_1 = \frac{2(2u_0 + (u_0) + 2)\beta_0}{2(2(u_0) + 1)}$, $n \geq 2$,</td>
<td>$\beta_1 = \frac{2a + (u_0) + 2}{\beta_0}$, $n \geq 2$,</td>
</tr>
<tr>
<td>$\gamma_1 = -\frac{a}{(u_0) + 2 + 1}$, $\beta_1 \neq 0$,</td>
<td>$\gamma_1 = \frac{2(2u_0 + 1)}{(u_0) + 3}$, $n \geq 2$,</td>
<td>$\gamma_1 = \frac{2}{\beta_0}((u_0) + 1)$, $n \geq 2$,</td>
</tr>
<tr>
<td>$\gamma_2 = (u_0) + 2((u_0) + 2)\gamma_1$, $\theta \neq 0$,</td>
<td>$\gamma_2 = \frac{2(2u_0 + 1)}{(u_0) + 3}$, $n \geq 2$,</td>
<td>$\gamma_2 = \frac{2a + 2}{\beta_0}$, $n \geq 2$,</td>
</tr>
<tr>
<td>$\gamma_{n+1} = \frac{2(2u_0 + 1) - (u_0) + 3 + (u_0) + 2 + 1}{(u_0) + 3}$, $n \geq 2$,</td>
<td>$\gamma_{n+1} = \frac{2(2u_0 + 1) - (u_0) + 3 + (u_0) + 2 + 1}{(u_0) + 3}$, $n \geq 2$,</td>
<td>$\gamma_{n+1} = \frac{2a + 2}{\beta_0}$, $n \geq 2$,</td>
</tr>
</tbody>
</table>

**$\tilde{B}(x)$**

- $-\alpha(x + \alpha(x + 2 + 1))^2$
- $+((u_0) + \alpha(\theta + 2))x + (\theta + 2)\theta$

**$\Psi_1(x)$**

- $-2\alpha(\theta + 2) + 1)x^2 + (u_0)x + \theta$
- $-2\alpha(\theta + 2) + 1)x^2 (u_0)x + \theta$

| $\tilde{B}(x) = -\frac{1}{\beta_0} \left\{ ((u_0) + 1)x^2 \right\} - \frac{1}{\beta_0} \left\{ (1 + (u_0) + 1)x^2 \right\}$ |
|-------------------------|-------------------------|
| $\Psi_1(x) = \frac{1}{\gamma_1} \left( (u_0) + 1 \right)^x - 2(2(u_0) + 1) + 1 \right) \Psi_1(x) = \frac{1}{\gamma_1} \left( (u_0) + 1 \right)^x$ |

| $\tilde{B}(x) = \frac{1}{\beta_0} \left\{ ((u_0) + 1)x^2 \right\}$ |
|-------------------------|-------------------------|
| $\Psi_1(x) = \frac{1}{\gamma_1} \left( (u_0) + 1 \right)^x - 2(2(u_0) + 1) + 1 \right) \Psi_1(x) = \frac{1}{\gamma_1} \left( (u_0) + 1 \right)^x$ |
Case (B).

The subcase $B_1$. Here $(u, B_2) = ((u)_0 + 1)(1 - \vartheta_1)\gamma_2$. The following system holds

$$(n + (u)_0 + 2)\beta_n - (n + (u)_0)\beta_{n+1} = (n + (u)_0 + 1)\beta_{n+1} - (n + (u)_0 - 1)\beta_n, \ n \geq 2.$$  

(4.10)

$$(u)_0 + 3)\beta_1 - ((u)_0 + 1)\beta_0 = ((u)_0 + 2)\beta_2 - (u)_0 \beta_1 + \langle u, B_1 \rangle.$$  

(4.11)

$$\frac{\gamma_{n+2}}{n + (u)_0 + 2} - \frac{\gamma_{n+1}}{n + (u)_0 + 1} = (\beta_{n+1} - \beta_n)^2, \ n \geq 1.$$  

(4.12)

$$\hat{\beta}_{n+1} - \hat{\beta}_n = \beta_{n+2} - \beta_{n+1}, \ n \geq 1.$$  

(4.13)

Using (4.12) in (4.10), we get $\hat{\beta}_n = -\frac{1}{2}\beta_{n+2} + \frac{3}{2}\beta_{n+1}, n \geq 1$. Thus

$$\beta_n = (\beta_4 - \beta_2)(n - 2) + \beta_2, n \geq 2,$$  

so

$$\hat{\beta}_n = \frac{2n - 3}{2}(\beta_3 - \beta_2) + \beta_2, \ n \geq 1.$$  

(4.14)

Thanks to (4.11), one has

$$\hat{\beta}_0 = -\frac{(u)_0 + 3}{2((u)_0 + 1)} \beta_3 + \frac{(u)_0 + 5}{2((u)_0 + 1)} \beta_2 + \frac{(u)_0}{(u)_0 + 1} \beta_1 - \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle.$$  

(4.15)

comparing (3.38) and (4.15) we obtain

$$-\frac{(u)_0 + 2)((u)_0 + 3)}{2} \beta_3 + \frac{(u)_0 + 2)((u)_0 + 5)}{2} \beta_2 = \beta_1 + (u)_0 \beta_0 + \langle u, B_1 \rangle.$$  

(4.16)

Thanks to (4.14), the relation (4.13) gives $\beta_3 = \beta_2 - \frac{2c}{\vartheta_1}$, where

$$c = \frac{1}{(u)_0 + 2} \left\{ \beta_0 - \beta_1 - \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle \right\}.$$  

(4.17)

Equation (4.16) becomes $\beta_2 = \beta_1 + \frac{c}{\vartheta_1} \left[ ((u)_0 + 1)\vartheta_1 - ((u)_0 + 3) \right]$. So $\beta_3 = \beta_1 + \frac{c}{\vartheta_1} \left[ ((u)_0 + 1)\vartheta_1 - ((u)_0 + 5) \right]$, which gives $\beta_n = -\frac{c}{\vartheta_1} \left[ 2n + (u)_0 - 1 \right] - c + \beta_0 + \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle, \ n \geq 2. \ \text{From (4.17), we have}\ \beta_1 = -(u)_0 \beta_0 + \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle.$ Also, we have $\hat{\beta}_n = -\frac{c}{\vartheta_1} \left[ 2n + (u)_0 \right] - c + \beta_0 + \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle, n \geq 1. \ \text{From (3.38), we obtain}\ \hat{\beta}_0 = -((u)_0 + 1)c + \beta_0$. Using the new expressions of $\beta_n$ and $\hat{\beta}_n$, we get for $n \geq 1$:

$$\left\{ \begin{array}{l}
\gamma_{n+1} = \frac{(n + (u)_0 + 1)}{(n + (u)_0 + 1)\vartheta_1} \gamma_1 + c((u)_0 + 1) \left[ c - \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle \right] + \frac{c^2}{\vartheta_1} \langle (u)_0 + 1 \rangle (n - 1), \\
\tilde{\gamma}_{n+1} = \frac{(n + (u)_0 + 1)}{(n + (u)_0 + 1)\vartheta_1} \tilde{\gamma}_1 + c((u)_0 + 1) \left[ c - \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle \right] + \frac{c^2}{\vartheta_1} \langle (u)_0 + 1 \rangle n, \\
\tilde{\gamma}_1 = \gamma_1 + ((u)_0 + 1)c \left[ c - \frac{1}{(u)_0 + 1} \langle u, B_1 \rangle \right].
\end{array} \right.$$

The system is completely solved. Note that in this case, $\deg \Phi \leq 1$ and $\kappa \Phi(x) = -\frac{(u)_0 + 1}{\gamma_1} x + \frac{((u)_0 + 1)c\beta_0}{\gamma_1} + 1$. 

\[ \]
If \( \deg(\Phi) = 0 \) \((c=0)\), then \( \Phi(x) = 1 \). After a suitable affine transformation, we can suppose that \( \beta_0 = -\frac{1}{(u_0)+1} (u,B_1) := \lambda \) and \( \gamma_1 = \frac{(u_0)+1}{2} \rho \), where \( \rho := \vartheta_1 \).

From \( 4.4 \) and \( (3.16) \), we get the expressions of \( B(x) \) and \( \Psi_1(x) \). In particular, we recognize here the \( D \)-Laguerre-Hahn sequence of class zero, nonsingular, of Hermite type \([2,16]\) (see first column in Table 2).

If \( \deg(\Phi) = 1, \ (c \neq 0) \), then \( \kappa = -\frac{((u_0)+1)c}{\gamma_1} \), \( \Phi(x) = x - \frac{\gamma_1}{((u_0)+1)c} - \beta_0 \) and \( \Psi(x) = -\frac{1}{c} (x - \beta_0) \). By means of a suitable affine transformation, we can choose \( \beta_0 \) and \( c \) such that \( \frac{\gamma_1}{((u_0)+1)c} + \beta_0 = 0 \), and \( c = -\vartheta_1 \). Thus, \( \Phi(x) = x \) and \( \Psi(x) = \frac{1}{\vartheta_1} (x - \beta_0) \). Putting \( \alpha = \vartheta_1 - ((u_0)+2) + \beta_0 + \frac{1}{(u_0)+1} (u,B_1) \), we get the expressions written in second column of Table 2.

In particular: If \( \vartheta_1 = 1 \), according to Proposition \( 3.7 \), the obtained sequence is the \( D \)-Laguerre-Hahn sequence of class zero, nonsingular, of Laguerre type \([2,16]\). If \( \vartheta_1 \neq 1 \), we recognize the perturbated sequence of order one of the \( D \)-Laguerre-Hahn sequence of class zero, nonsingular, of Laguerre type \([2,16]\). It is a \( D \)-Laguerre-Hahn sequence of class \( s = 1 \).

<table>
<thead>
<tr>
<th>Subcase ( B_1 : \vartheta_1 := 1 - \frac{(u_0,B_2)}{(u_0)+1} ; \vartheta_2 \neq 0 ) ( , (u_0) \neq -n, n \geq 1 )</th>
<th>( \Phi(x) = \frac{x}{\vartheta_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Psi(x) = -\frac{1}{\rho} (x - \lambda), \rho \neq 0 ), ( \beta_0 \neq 0, )</td>
<td>( \Psi(x) = \frac{1}{\vartheta_1} (x - \beta_0) )</td>
</tr>
<tr>
<td>( \beta_0 = -\frac{1}{(u_0)+1} (u,B_1) := \lambda, )</td>
<td>( \beta_0 := \alpha + 2(u_0)+1 + \lambda, )</td>
</tr>
<tr>
<td>( \beta_n = 0, n \geq 1, )</td>
<td>( \beta_n := (u_0)+1 \rho, \rho = \vartheta_1 )</td>
</tr>
<tr>
<td>( \gamma_1 = \frac{(u_0)+1}{2} \rho, \rho = \vartheta_1 )</td>
<td>( \gamma_1 = \beta_0 \vartheta_1 ((u_0)+1) )</td>
</tr>
<tr>
<td>( \gamma_{n+1} = \frac{n + (u_0)+1}{2}, n \geq 1, )</td>
<td>( \gamma_{n+1} = (n + (u_0)+1)(\alpha + \vartheta_1 (u_0)+1), n \geq 1, \rho \neq -\vartheta_1 \rho )</td>
</tr>
<tr>
<td>( \beta_0 = \lambda, )</td>
<td>( \beta_0 = 2(u_0)+\alpha + 2 + (\vartheta_1 (u_0)+1) \vartheta_1 = 1 + \lambda, )</td>
</tr>
<tr>
<td>( \beta_n = (u_0)+1 \rho, \rho \neq 0, )</td>
<td>( \beta_n = 2(n + (u_0)+1) + \alpha + 2, n \geq 1, \rho \neq -\vartheta_1 \rho )</td>
</tr>
<tr>
<td>( \gamma_{n+1} = \frac{n + (u_0)+1}{2}, n \geq 2, )</td>
<td>( \gamma_{n+1} = (n + (u_0)+1)(\alpha + (u_0)+1), n \geq 2, \rho \neq -\alpha - (u_0)+1, \rho \geq 2, )</td>
</tr>
<tr>
<td>( \beta_0 = \lambda, )</td>
<td>( \beta_0 = 2(u_0)+\alpha + 2 + (\vartheta_1 (u_0)+1) \vartheta_1 = 1 + \lambda, )</td>
</tr>
<tr>
<td>( \beta_n = (u_0)+1 \rho, \rho \neq 0, )</td>
<td>( \beta_n = 2(n + (u_0)+1) + \alpha + 2, n \geq 1, \rho \neq -\vartheta_1 \rho )</td>
</tr>
<tr>
<td>( \gamma_{n+1} = \frac{n + (u_0)+1}{2}, n \geq 2, )</td>
<td>( \gamma_{n+1} = (n + (u_0)+1)(\alpha + (u_0)+1), n \geq 2, \rho \neq -\alpha - (u_0)+1, \rho \geq 2, )</td>
</tr>
</tbody>
</table>

**Table 2. The \( D \)-Classical Polynomials - Case (B)**
The subcase $B_2$. Here $\kappa = -\frac{(u_0 + 1)}{(\rho + 2)\gamma_1}$, and the polynomial $\Phi(x)$ is of degree 2 given by \[ (4.8), \]\ where
\[
\begin{align*}
\frac{d}{d} = \frac{1}{2}\left\{ \beta_0 + \beta_1 + \frac{\rho + 2}{(u_0 + 2)\gamma_1}\left[ \beta_1 - \beta_0 - \frac{1}{(u_0 + 1)}\langle u, B_1 \rangle \right] \right\},
\mu = 4\left\{ \beta_0 - d + \left[ \frac{(\rho + 2)}{((u_0 + 1)\gamma_1) + 1}\right] \right\}.
\end{align*}
\]
As in the previous cases, we analyse the situations $\Phi(x) = x^2$, and $\Phi(x) = x^2 - 1$. Particular, we usually suppose $d = 0$. Note that here $\Psi(x) = -\frac{\rho + 2}{u_1}B_1(x)$, and
\[
\tilde{B}(x) = \frac{(u_0 + 1)}{\gamma_1}\left\{ \left[ 1 - \frac{(\rho + 1)\partial_1}{\rho + 2} \right] x^2 + \left[ \frac{(\rho + 1)\partial_1}{\rho + 2} - 1 \right] (\beta_0 + \beta_1) + \frac{\langle u, B_1 \rangle}{(u_0 + 1)} \right\} x
\]
\[ + \left[ 1 - \frac{(\rho + 1)\partial_1}{\rho + 2} \right] (\beta_0\beta_1 - \gamma_1) + \frac{u_0\gamma_1}{(u_0 + 1)} - \frac{\beta_0\langle u, B_1 \rangle}{(u_0 + 1)}. \] \[ (4.19) \]
In particular, the sequence $\{B_n\}_{n \geq 2}$ is a D-Laguerre-Hahn sequence of class $s \leq 2$. Besides, the system \[ (3.41) \] becomes
\[
(n + (u_0 + 2))\beta_n - (n + (u_0))\beta_{n-1} = (n + (u_0 + 1))\beta_{n+1} - (n + (u_0 - 1))\beta_n, \quad n \geq 2.
\]
\[ (4.20) \]
\[
((u_0 + 3)\beta_1 - ((u_0 + 1)\beta_0 = ((u_0 + 2)\beta_2 - (u_0)\beta_1 + \langle u, B_1 \rangle).
\]
\[ (4.21) \]
\[
((u_0 + 2)\beta_0 = ((u_0 + 1)\beta_1 + \beta_0 - \langle u, B_1 \rangle).
\]
\[ (4.22) \]
\[
\frac{(2n + (u_0 + \rho + 4))}{(n + (u_0 + 2)(n + \rho + 1))}\gamma_{n+2} - \frac{(2n + (u_0 + \rho))}{(n + (u_0 + 1)(n + \rho))}\gamma_{n+1} = ((\beta_{n+1} - \beta_n)^2), \quad n \geq 2.
\]
\[ (4.23) \]
\[
\frac{(u_0 + \rho + 6)}{(u_0 + 3)(\rho + 2)}\gamma_3 - \frac{(u_0 + 1)}{(\rho + 2)}\gamma_2 + \frac{\gamma_1}{(u_0 + 1)} = (\beta_3 - \beta_2)^2.
\]
\[ (4.24) \]
\[
\frac{(u_0 + \rho + 4)}{(u_0 + 2)(\rho + 2)}\partial_1\gamma_2 - \frac{\gamma_1}{(u_0 + 1)} = (\beta_1 - \beta_0)(\beta_1 - \beta_0 - \frac{1}{(u_0 + 1)}\langle u, B_1 \rangle).
\]
\[ (4.25) \]
\[
(n + \rho + 2)\beta_{n+1} - (n + \rho)\beta_n = (n + \rho + 3)\beta_{n+2} - (n + \rho + 1)\beta_{n+1}, \quad n \geq 1.
\]
\[ (4.26) \]
\[
\partial_1\beta_1 - (1 - \frac{\partial_1}{\rho + 2})\beta_0 = \frac{(\rho + 3)\partial_1}{\rho + 2}\beta_2 - \beta_1 - \frac{1}{(u_0 + 1)}\langle u, B_1 \rangle.
\]
\[ (4.27) \]
Subtracting \[ (4.26) \] from \[ (4.20) \], we get
\[
((u_0 - \rho + 1)(\beta_{n+1} - \beta_n) = ((u_0 - \rho - 1)(\beta_{n+2} - \beta_{n+1}), \quad n \geq 1.
\]
\[ (4.28) \]
Two situations arise:

1. \[ B_{21} : \rho \neq (u_0) - 1 \]

Here $\beta_{n+2} - \beta_{n+1} = \xi(\beta_{n+1} - \beta_n)$, $n \geq 1$, where $\xi = \frac{(u_0 - \rho + 1)}{(u_0 - \rho - 1)}$. Then
\[
\beta_{n+1} = \xi\beta_n + \beta_2 - \xi\beta_1, \quad n \geq 1.
\]

Relation \[ (4.20) \] with $n \rightarrow n + 1$, gives
\[
-\frac{(2n + (u_0 + \rho + 5))}{((u_0) - (u_0 + 1))}\beta_{n+1} + \frac{(2n + (u_0 + \rho + 1))}{(u_0) - (u_0 + 1)}\beta_n = 2(\beta_2 - \xi\beta_1), \quad n \geq 1.
\]
\[ (4.29) \]
On the other hand, since $d = 0$, we have
\[
\langle u, B_1 \rangle = \langle ((u_0 + 2)\ell - (u_0 + 1))\beta_0 + ((u_0 + 2)\ell + (u_0 + 1))\beta_1, \quad (4.30) \]
where \( \ell := \frac{(w_0^0 + 1)\rho}{\theta^2} \). Then, (4.22) becomes

\[
\hat{\beta}_0 = -\ell \beta_1 + (1 - \ell) \beta_0.
\] (4.31)

Using (4.30) and (4.31), respectively in (4.21) and (4.27), gives

\[
((w_0^0 + 3)\hat{\beta}_1) = ((w_0^0 + 2)\beta_2 + \ell \beta_0 + (\ell + 1) \beta_1)
\] (4.32)

\[
(\rho + 2)\hat{\beta}_1 = (\rho + 3)\beta_2 + \ell \beta_0 + (\ell + 1) \beta_1
\] (4.33)

The difference and the sum of the two last equations give respectively:

\[
\beta_2 - \xi \hat{\beta}_1 = 0,
\] (4.34)

\[
((w_0^0 + \rho + 5)(\hat{\beta}_1 - \beta_2) = 2\ell \beta_0 + 2(1 + \ell) \beta_1.
\] (4.35)

Therefore,

\[
\hat{\beta}_1 = -((w_0^0 - \rho - 1)(\ell \beta_0 + 2(1 + \ell) \beta_1),
\] (4.36)

\[
\beta_2 = -((w_0^0 - \rho + 1)(\ell \beta_0 + 2(1 + \ell) \beta_1).
\] (4.37)

Back to (4.29) and taking into account (4.34), we discuss two situations:

- \( B_{21} : 2n + (w_0^0) + \rho + 1 \neq 0, \ \forall n \geq 1. \)

Iterating (4.29) gives

\[
\begin{cases}
\hat{\beta}_n = \frac{((u_0^0 + \rho + 3)(w_0^0 - \rho + 1)\ell \beta_0 + 2(1 + \ell) \beta_1)}{(2n + (u_0^0 + \rho + 1)(2n + (u_0^0 + \rho + 3))}, n \geq 1, \\
\beta_{n+1} = \frac{((u_0^0 + \rho + 3)(w_0^0 - \rho + 1)\ell \beta_0 + 2(1 + \ell) \beta_1)}{(2n + (u_0^0 + \rho + 1)(2n + (u_0^0 + \rho + 3))}, n \geq 1.
\end{cases}
\] (4.38)

Therefore,

\[
\beta_{n+1} - \hat{\beta}_n = -\frac{2((u_0^0 + \rho + 3)(\ell \beta_0 + 2(1 + \ell) \beta_1)}{(2n + (u_0^0 + \rho + 1)(2n + (u_0^0 + \rho + 3))}, n \geq 1.
\] (4.39)

In order to compute the coefficients \( \gamma_{n+1} \) and \( \tilde{\gamma}_{n+1} \), we study two possibilities:

- \( \Phi(x) = x^2 \). Taking \( \mu = 0 \) and using (4.30) in (4.18), we have \( \beta_0^2 + \gamma_1(1 + \frac{\ell}{\theta}) = 0. \)

Besides, from (4.31) and (4.30), one has

\[
\begin{cases}
\beta_1 - \beta_0 = (\ell - 1)\beta_0 + (\ell + 1) \beta_1, \\
\beta_1 - \beta_0 = -\hat{\beta}_0 = -\frac{\ell}{(u_0^0 + 1)\ell} (\beta_0 + \beta_1).
\end{cases}
\] (4.40)

Necessarily \( 1 + \frac{\ell}{\theta} \neq 0 \). Otherwise \( \beta_0 = 0 \). Then, from (4.31) we obtain \( \hat{\beta}_0 = \beta_0 = 0. \)

Therefore, equation (4.35) leads to \( \hat{\beta}_1 - \beta_2 = 0. \) Then (4.24) gives \( (u_0^0 + \rho + 6 = 0. \)

Taking \( n = 3 \) in (4.24), gives \( \gamma_5 = 0 \) which is absurd. We conclude that

\[
\gamma_1 = \frac{-\ell \beta_0^2}{1 + \ell} \neq 0.
\] (4.41)

In (4.25), necessarily \( (u_0^0 + \rho + 4 \neq 0. \) If not, one has \( \gamma_1 = -((u_0^0 + 1)(\beta_1 - \beta_0)(\beta_1 - \beta_0 - \hat{\beta}_0 - \frac{1}{u_0^0 + 1} (u, B_1)) \). Thanks to (4.40) and (4.41), we get \( (\ell \beta_0 + (1 + \ell) \beta_1)^2 = 0. \)

or equivalently \( \ell \beta_0 + (1 + \ell) \beta_1 = 0. \) Thus, (4.35) proves that \( \hat{\beta}_1 - \beta_2 = 0. \) Taking into account (4.34), we deduce \( \hat{\beta}_1 = \beta_2 = 0. \) So, from (4.39) we obtain \( \beta_{n+1} - \hat{\beta}_n = 0, \) \( n \geq 1. \) Equation (4.23), where \( n = 2, \) implies that \( \gamma_4 = 0 \) which is not possible. Once the expression of \( \gamma_2 \) is obtained, we easily deduce the rest of the three term recurrence coefficients (see first column in Table 3). In addition, we have

\[
\tilde{B}(x) = -\frac{1 + \ell}{\theta \gamma_0^0} \left\{ [(u_0^0 + 1 - (\rho + 1)\ell] x^2 +
\right\}.
\]
\[
\left\{\left[(u_0 + \rho + 3)\ell - 2((u_0 \rho + 1)]\beta_0 + ((u_0 + \rho + 3)\ell \beta_1\right\}x + \\
\left[(u_0 + 1 - (\rho + 1)\ell)\beta_0\beta_1 - \frac{\ell \beta_0^2((\mu + 1)\ell - 1)}{1 + \ell}\right].
\]

\[
\Psi_1(x) = -\frac{1 + \ell}{\ell \beta_0^2} \left\{\left[(u_0 + 1 - (\rho + 1)\ell\right]x^3 + \\
\left\{\left[(u_0 + 2)\ell - ((u_0 + 1)]\beta_0 + ((u_0 + 3)\ell \beta_1\right\}x^2 + \\
\left\{\left[(u_0 + \rho + 3)\ell - 2((u_0 + 1)]\beta_0^2 + \left[(u_0 + 1 + ((u_0 + 2)\ell)\beta_0\beta_1 - \frac{\ell \beta_0^2((\mu + 1)^2 - \ell - (w_0 - 1]}{1 + \ell}\right)\right\}x + \\
\left[(u_0 + 1 - (\rho + 1)\ell\right]\beta_0^2\beta_1 - \frac{\ell \beta_0^2((\mu + 1)^2 - \ell - (w_0 - 1]}{1 + \ell}\right]}
\]

\[\Phi(x) = x^2 - 1.\]

We note here that since too many subcases will appear later, we simply refer to (4.19) and (3.16) for computing the polynomials \(B(x)\) and \(\Psi_1(x)\). On the other hand, taking \(\mu = 4\) and using (4.30) in (4.18), we get \(\gamma_1(1 + \frac{1}{\ell}) = -\beta_0^2\).

We distinguish two subcases:

If \(\ell = -1\), relations (4.30), (4.31) and (4.40) are still valid. Besides, \(\beta_0 = \pm 1\) and \(\gamma_1 \in C \setminus \{0\}\). Thus, it is easy to obtain the corresponding coefficients given in the second column of Table 3.

If \(\ell \neq -1\), we have \(\gamma_1 = \frac{\ell(1 - \beta_0^2)}{\ell + 1}\), \(\beta_0 \neq \pm 1\) and relations (4.30), (4.31), (4.35) and (4.40). In order to explicit the coefficients \(\gamma_n\) and \(\hat{\gamma}_n\), we need to discuss many situations when solving equations (4.23)-(4.25). The results are summarized in the last column of Table 3.

\[B_{212} : \exists N \geq 1, \text{such that } 2N + (u_0 + \rho + 1 = 0. \text{ From (4.34), we get}\]

\((N + (u_0 + 1)\beta_0 - (N + (u_0)\beta_2 = 0. \] (4.42)

Therefore, subtracting (4.20) from (4.26), we obtain \(\tilde{\beta}_n = \frac{N + (u_0)}{N + (u_0 \rho + 1} \beta_n + 1, n \geq 1.\)

Then, from (4.20) we have \((n + 2)\beta_n + 1 - (n - N)\beta_n = 0, n \geq 1.\) Thus, the expressions of \(\beta_n\) and \(\tilde{\beta}_n\) hold as shown in Table 4.

For \(N \geq 3\), we have \(\beta_2 = 0.\) Thanks to (4.42), one has \(\tilde{\beta}_1 = 0. Consequently, taking into account (4.30), (4.31) and (4.32), we get

\[
\begin{align*}
\tilde{\beta}_0 &= \beta_0 + \beta_1, \beta_0 = (1 + \ell)\beta_0, \beta_1 = -\ell \beta_0, \beta_0 - \tilde{\beta}_0 = -\beta_0, \\
\langle u, B_1 \rangle &= -\langle (u_0)\ell + (u_0 + 1)\tilde{\beta}_0, \\
\beta_1 - \tilde{\beta}_0 - \frac{1}{\langle u_0, T \rangle} \langle u, B_1 \rangle &= -\frac{\ell}{\langle u_0, T \rangle} \langle u_0. \\
\end{align*}
\]

(4.43)

where \(\ell = \frac{((u_0 + 1)\beta_1}{2N + (u_0) - 1}.\) The coefficients \(\gamma_{n+1}\) satisfy the following system:

\[
\begin{align*}
\frac{(2n - 2N + 3)}{(n + (u_0 + 2)(n - 2N - (u_0) - 1)} \gamma_{n+2} - \frac{(2n - 2N - 1)}{(n + (u_0 + 1)(n - 2N - (u_0) - 1)} \gamma_{n+1} &= 0, \\
\text{where } n \geq 2, n \notin \{N - 1, N\}, \\
\frac{1}{(N + (u_0) + 1)^2} N_{n+1} + \frac{1}{(N + (u_0) + 1)(N + (u_0) + 2)} N_{n+2} &= \frac{3}{(N + (u_0) + 1)^2} N_{n+1} = -\beta_0^2, \\
\frac{1}{(N + (u_0) + 2)(N + (u_0) + 1)} N_{n+2} + \frac{1}{(N + (u_0) + 1)^2} N_{n+1} &= -\beta_0^2, \\
\frac{1}{(u_0 + 1)} N_{n+3} + \frac{1}{(2N - 5)(n + 2)} N_{n+2} + \frac{1}{(u_0 + 2)} N_{n+1} &= 0, \\
\frac{1}{(u_0 + 1)} N_{n+3} + \frac{1}{(2N - 4)(n + 2)} N_{n+2} + \frac{1}{(u_0 + 1)} N_{n+1} &= 0.
\end{align*}
\]

Remark that \(\ell \neq -1\), otherwise we obtain \(\gamma_3 = 0.\) If we assume that \(\mu = 0\), then thanks to (4.18) this is equivalent to \(\gamma_1 = -\beta_0^2\), which is impossible.

We conclude that necessarily \(\mu = 4\), or equivalently \(\gamma_1 = \frac{1}{(1 + \ell)^2}.\) The coefficients \(\gamma_n\) and \(\hat{\gamma}_n\) follow.
Thanks to (4.42), it is easy to see that equations (4.43) are still valid for \( N = 2 \). Besides, \( \beta_3 - \beta_2 = \hat{\beta}_1 - \beta_2 = -\frac{1}{(u_0+3)} \beta_2 \) and

\[
\begin{align*}
(2n-1) & \left[ \frac{n+(u_0)+2}{n+(u_0)-3} \right] \gamma_{n+2} - \frac{2n-5}{3} \gamma_{n+1} = 0, \ n \geq 3, \\
(2n-1) & \left[ \frac{1}{(u_0+2)(u_0)+4} \right] \gamma_3 - \frac{1}{(u_0+3)^2} \gamma_2 = -\left( \beta_2 - \hat{\beta}_1 \right)^2, \\
(2n-1) & \left[ \frac{1}{(u_0+2)^2} - \frac{1}{(u_0+3)^2} \right] \gamma_2 - \frac{1}{(u_0+2)^2} \gamma_1 = -(1+\ell) \ell \beta_0^2.
\end{align*}
\]

As in the case where \( N \geq 3 \), we prove that \( \ell \neq -1 \), and so \( \mu = 4 \). This leads to the expressions found in Table 4. Finally, if \( N = 1 \), the relations (4.30)-(4.37) give thanks to (4.42): \( \beta_1 - \beta_2 = \ell \beta_0 + (\ell + 1) \beta_1, \beta_1 - \hat{\beta}_0 = (\ell + 1) \beta_1 + (\ell + 1) \beta_1, \beta_1 - \hat{\beta}_0 - \frac{1}{(u_0+1)} \langle u, B_1 \rangle = -\frac{\ell}{(u_0+1)} \langle \beta_0 + \beta_1 \rangle \).

In particular, as done above, we obtain:

\[
\gamma_{n+1} = \frac{n+(u_0)+1}{(n+(u_0)-3)n+(u_0)-3}, \ n \geq 2, \quad \text{and} \quad \gamma_n = \frac{n+(u_0)(n-(u_0)-2)}{(n+(u_0)-1)(n+(u_0)-3)}, \ n \geq 2.
\]

We can also prove that necessarily \( \mu = 4 \). Hence, \( (1+\ell) \gamma_1 = (1+\beta^2_0) \). Discussing the cases \( \ell = -1 \) and \( \ell \neq -1 \), ends the computations.

- \( B_{22}^0 \): \( \rho = (u_0-1) \). Thanks to (4.28), we have \( \hat{\beta}_{n+1} = \hat{\beta}_1, n \geq 1 \). So, from (4.26) we get

\[
\beta_{n+1} = \beta_1 + \frac{(u_0+1)(u_0+2)(\beta_2 - \hat{\beta}_1)}{(n+(u_0)+1)(n+(u_0)+1)} \frac{1}{(u_0+2)(u_0+3)}, \ n \geq 1.
\]

Using (3.38) in (4.21) gives

\[
\hat{\beta}_1 = \frac{((u_0+2)^2 \beta_2 + \beta_2 + ((u_0+1)\beta_0 + (u_0, B_1))}{((u_0+2)(u_0+3))}. \quad \text{Hence, (4.27) leads to}
\]

\[
\beta_2 = \frac{((u_0+1)((u_0+3)-(u_0+4))\beta_1) - ((u_0+1)(u_0+3)-(u_0+4))\beta_1}{2((u_0+2)\beta_1)}
\]

therefore, \( \hat{\beta}_1 = \frac{((u_0+1)+(u_0+2)\beta_1) - ((u_0+1)(u_0+2)\beta_1)}{2((u_0+2)\beta_1)} \).

It is mentioned previously that \( d = 0 \), thus

\[
\langle u, B_1 \rangle = \frac{(u_0+2)\beta_1 - (u_0+1)\beta_0 + ((u_0+2)\beta_1 - ((u_0+1)\beta_0 + (u_0+1)\beta_1) \beta_1}. \quad (4.44)
\]

This implies \( \hat{\beta}_1 = 0 \) and \( \hat{\beta}_n = 0, n \geq 1 \). Always with the help of (4.44), we obtain: \( \beta_0 = (1-\vartheta_1)\beta_0 - \vartheta_1 \beta_1, \ \beta_{n+1} = -\frac{(u_0+1)}{(n+(u_0)+1)(n+(u_0))} \frac{\beta_1 + (1+\vartheta_1)\beta_1}{(n+(u_0)+1)(n+(u_0))}, n \geq 1, \ \beta_1 - \beta_0 = (\vartheta_1 - 1)\beta_0 + (1+\vartheta_1)\beta_1, \ \text{and} \beta_1 - \beta_0 - \frac{1}{(u_0+1)} \langle u, B_1 \rangle = -\frac{\vartheta_1}{(u_0+1)} \langle \beta_0 + \beta_1 \rangle \).

Then, for \( n \geq 2 \), we get

\[
\begin{align*}
\gamma_{n+1} &= \frac{(n+(u_0)+1)(n+(u_0)+1)}{(n+(u_0)+2)(n+(u_0)+1)} \gamma_{n+2} - \frac{(n+(u_0)+1)}{(n+(u_0)+2)(n+(u_0)+1)} \gamma_{n+1} = \frac{(u_0+1)^2 (\beta_1 + (1+\vartheta_1)\beta_1)^2}{(n+(u_0)+1)^2 (n+(u_0)+1)^2}, \\
\gamma_{n+1} &= \frac{(n+(u_0)+3)(u_0+1)}{(n+(u_0)+2)(u_0+1)} \gamma_{n+2} - \frac{(n+(u_0)+1)}{(n+(u_0)+2)(u_0+1)} \gamma_{n+1} = \frac{(u_0+1)^2 (\beta_1 + (1+\vartheta_1)\beta_1)^2}{(n+(u_0)+1)^2 (n+(u_0)+1)^2}, \\
\gamma_n &= \frac{(n+(u_0)+1)(n+(u_0)+1)}{(n+(u_0)+2)(n+(u_0)+1)} \gamma_{n+1} - \frac{(n+(u_0)+1)}{(n+(u_0)+2)(n+(u_0)+1)} \gamma_{n+1} = \frac{(u_0+1)^2 (\beta_1 + (1+\vartheta_1)\beta_1)^2}{(n+(u_0)+1)^2 (n+(u_0)+1)^2}, n \geq 3.
\end{align*}
\]
\[
\begin{align*}
\beta_0 & \in \mathbb{C} \setminus \{0\}, \beta_1 \in \mathbb{C}, \\
\beta_n &= \frac{-(2n + (\gamma_0 - \rho + 1))\beta_0}{(2n + (\gamma_0 - \rho + 1))\beta_0 + \rho + 3}, \quad n \geq 2, \\
\gamma_1 &= \frac{-6\beta_0}{1 + \ell}, \quad \beta_0 \neq 0, \\
\gamma_2 &= \frac{1 - (u_0 + 2\beta_0)^2}{1 + (u_0 + 2\beta_0)^2}, \quad \omega \neq 0, \quad \gamma_3 = \frac{-\omega + 2\beta_0^2}{1 + (u_0 + 2\beta_0)^2}, \\
\gamma_{n+1} &= \frac{\omega(n + 2\beta_0^2)(\rho + 4)}{(2n + (\gamma_0 + \rho + 2))(2n + (\gamma_0 + \rho + 2)(n + 1))}\gamma_n, \quad n \geq 2, \\
\gamma_0 &= \frac{-u_0 - (\gamma_0 + \rho + 1)^2u_0}{(2n + (\gamma_0 + \rho + 2)(2n + (\gamma_0 + \rho + 2)(n + 1))^2}, \quad n \geq 2.
\end{align*}
\]

\[
\begin{align*}
\beta_0 &= \pm 1, \beta_1 \in \mathbb{C}, \beta_0 = \beta_1 + 2\beta_0, \\
\beta_n &= \frac{(2n + (\gamma_0 + \rho + 1))(\rho + 1)}{(2n + (\gamma_0 + \rho + 1))(\rho + 1)\beta_0 + \rho + 3}, \quad n \geq 2, \\
\gamma_1 &= \frac{4(\gamma_0 + 3)(\rho + 2)}{(u_0 + 2)(\rho + 3)}, \quad \gamma_1 \in \mathbb{C} \setminus \{0\}, \quad \text{if } (u_0 - 1)^2 + 4(\gamma_0 + 3)(\rho + 2) \neq 0, \\
\gamma_2 &= \frac{(u_0 + 2)(\rho + 3)}{(u_0 + 2)(\rho + 3)}, \quad \text{if not}, \\
\gamma_{n+1} &= \frac{(2n + (\gamma_0 + \rho + 2))(2n + (\gamma_0 + \rho + 2)(n + 1))\gamma_n}{(2n + (\gamma_0 + \rho + 2))}, \quad n \geq 3, \\
\gamma_0 &= \frac{2n + (\gamma_0 + \rho + 2)(2n + (\gamma_0 + \rho + 2)(n + 1))}{(2n + (\gamma_0 + \rho + 2))}, \quad n \geq 3.
\end{align*}
\]
Table 4. The $D_{u}$-Classical Polynomials - Case (B)

Subcase $B_{21}$: \(1 - \frac{(u, B_2)}{(u_0 + 1)\gamma_2} \neq 0 \), \((u_0) \neq -n, n \geq 1\).

\[ B_{212}: \exists N \geq 1, 2N + (u_0) + \rho + 1 = 0. \]

\[ \theta_1 := 2^{N+1}(u_0-\gamma_1) - \frac{1 - \frac{(u, B_2)}{(u_0 + 1)\gamma_2}}{2N + (u_0) + 1} \neq 0, n \neq 1, \ell \geq 1, \ell := -(u_0 + 1)\gamma_1, \omega := \ell\beta_0 + (\ell + 1)\beta_1 \]

\[ \Phi(x) = x^2 - 1, \quad \Phi(x) = -\frac{(u_0 - \gamma_1)}{\beta_0}. \]

\[ \beta_{n+1} = 0, n \notin \{N - 1, N\}, n \geq 2, n \geq 1, \]
\[ \beta_{N+1} = 0, n \notin \{N - 1, N\}, n \geq 2, n \geq 1, \]
\[ \beta_{N+1} = 0, N = 1, n \geq 2, \]
\[ \beta_{N+1} \notin C, N = 1, \]
\[ \beta_1 \notin C, \]

\[ N = 1 \]
\[ N = 2 \]
\[ N = 3 \]

\[ \gamma_1 = \frac{\ell(1-\beta_0^2)}{1-\beta_0}, \beta_0 \neq \pm 1, \ell \neq 0, \]
\[ \gamma_2 = \frac{\ell(1-\beta_0^2)}{1-\beta_0}, \beta_0 \neq \pm 1, \ell \neq 0, \]
\[ \gamma_3 = \frac{(u_0 + 3)^2 - 2\beta_2}{2(u_0 + 3)} \]
\[ \gamma_{n+1} = \frac{(u_0 + 3)^2}{2} [(u_0 + 3) - \beta_2], \]
\[ \gamma_n = \frac{(u_0 + 3)^2}{2} (\frac{(u_0 + 3)^2}{2} - \beta_2), \]

\[ \gamma_1 = \frac{\ell(1-\beta_0^2)}{1-\beta_0}, \beta_0 \neq \pm 1, \ell \neq 0, \ell \neq -1, \]
\[ \gamma_2 = \frac{\ell(1-\beta_0^2)}{1-\beta_0}, \beta_0 \neq \pm 1, \ell \neq 0, \]
\[ \gamma_3 = \frac{\ell(1-\beta_0^2)}{1-\beta_0}, \beta_0 \neq \pm 1, \ell \neq 0, \]
\[ \gamma_{n+1} = \frac{(u_0 + 3)^2}{2} [(u_0 + 3) - \beta_2], \]
\[ \gamma_n = \frac{(u_0 + 3)^2}{2} (\frac{(u_0 + 3)^2}{2} - \beta_2), \]

\[ \gamma_1 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_2 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_3 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_{n+1} = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_n = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]

\[ \gamma_1 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_2 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_3 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_{n+1} = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_n = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]

\[ \gamma_1 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_2 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_3 = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_{n+1} = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
\[ \gamma_n = \frac{(2N + (u_0) + \rho + 1)}{2N + (u_0) + 1}, \]
Table 5. The $D_u$-Classical Polynomials - Case (B)

<table>
<thead>
<tr>
<th>Subcase $B_{22}$</th>
<th>$u$, $B_{−} \neq 0$, $(u)<em>0 \neq n$, $n \geq 0$, $ϕ_1 : \frac{[u_0+1]}{[u_0+1]}(1 - (u, B</em>{−})_{n+2})$, $ω := ϕ_1 β_0 + (1 + ϕ_1)β_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ϕ(x) = x^{2}$,</td>
<td>$ϕ(x) = x^{2} - 1$ and $ϕ_1 = -1$</td>
</tr>
<tr>
<td>$β_0 \neq 0$,</td>
<td>$β_0 = 1$, $β_0 \neq 1$, $β_0 \neq 0$.</td>
</tr>
</tbody>
</table>

$γ_1(1 - (u)_0) = 0$,

$γ_1 = \frac{u_0+2}{u_0+1}(ω_1^{−1})$, $ϕ_1 \neq -1$,

$\gamma_2 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$\gamma_n = \frac{(u+2)(u+1)}{(u+3)}(u+1)$, $ω_0 \neq 0$.

$\gamma_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_1 = \frac{1}{ω_1}(u+2)(u+1) - 2$. If $2(u)_0 + 3 = 0$:

$γ_1 \in C \setminus \{0\}$,

$γ_{n+1} = \frac{(n+1)(n+2)(u+2)}{(n+3)(n+4)}(2u+2)(u+1)$, $n \geq 2$.

$ϕ_2 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_3 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_4 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_5 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_6 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.

$ϕ_7 = \frac{(u+2)(u+1)}{(u+3)}(ω_1^{−1})$, $ω_0 \neq 0$.

$γ_n = \frac{(u+2)(u+1)}{(u+3)}(2n+2(u+1))$, $n \geq 2$.
Finally, note that if \( u = 0 \), we recognize the very classical orthogonal polynomial sequences. Precisely, we find Hermite and Laguerre polynomials respectively in the first and the second column of Table 2. The Bessel polynomials are in the first column of Table 3 and the Jacobi polynomials are obtained in the last case of the third column of Table 3.

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References


Atef Alaya
Department of Mathematics, College of Applied Sciences, Umm Al-Qura University, P.O Box(715), Makkah, Saudi Arabia.
E-mail address: aaalaya@uqu.edu.sa

Secondary: Atef Alaya
Département de mathématiques, Faculté des Sciences de Gabès, Université de Gabès, Cité Erradih 6072, Zrig, Gabès, Tunisie.
E-mail address: Atif.Alaya@fsg.rnu.tn