A NEW APPLICATION OF GENERALIZED ALMOST INCREASING SEQUENCES

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Abstract. In the present paper, a general theorem dealing with $|A, p_n; \delta|_k$ summability factors of infinite series has been proved by using almost increasing sequence. This theorem also includes some known and new results.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$, and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots$$

(1.1)

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty,$$

(1.2)

where

$$\Delta A_n(s) = A_n(s) - A_{n-1}(s).$$

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

(1.3)

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

(1.4)
defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\bar{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [5]). The series 
\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty,
\]
and it is said to be summable \(|\bar{N}, p_n|_k\), \(k \geq 1\), if (see [2])
\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |\Delta A_n(s)|^k < \infty.
\]
If we take \(a_{nv} = \frac{p_n}{p_0}\) and \(\delta = 0\), then \(|A, p_n; \delta|_k\) summability reduces to \(|\bar{N}, p_n|_k\) summability. Also, if we take \(\delta = 0\), then \(|A, p_n; \delta|_k\) summability reduces to \(|A, p_n|_k\) summability (see [9]). In the special case \(\delta = 0\) and \(p_n = 1\) for all \(n\), \(|A, p_n; \delta|_k\) summability is the same as \(|A|_k\) summability. Furthermore, if we take \(a_{nv} = \frac{p_n}{p_0}\), then \(|A, p_n; \delta|_k\) summability is the same as \(|\bar{N}, p_n|_k\) summability.

Before stating the main theorem we must first introduce some further notations. Given a normal matrix \(A = (a_{nv})\), we associate two lower semimatrices \(\bar{A} = (\bar{a}_{nv})\) and \(\hat{A} = (\hat{a}_{nv})\) as follows:
\[
\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...
\]
and
\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ...
\]
It may be noted that \(\bar{A}\) and \(\hat{A}\) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have
\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v
\]
and
\[
\hat{A} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v.
\]

### 2. Known Result

In [3], Bor has proved the following theorem for \(|\bar{N}, p_n|_k\) summability factors of infinite series.

**Theorem 2.1.** Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that
\[
|\Delta \lambda_n| \leq \beta_n, \quad \beta_n \to 0 \quad \text{as} \quad n \to \infty,
\]
\[
\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty,
\]
and
\[
\lambda_n \mid X_n = O(1).
\]
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If
\[ \sum_{v=1}^{\infty} \frac{|t_v|^k}{v} = O(X_n) \quad \text{as} \quad n \to \infty, \]  
(2.5)

where \((t_n)\) is the \(n\)th \((C,1)\) mean of the sequence \((na_n)\), and \((p_n)\) is a sequence such that
\[ P_n = O(np_n), \]  
(2.6)
\[ P_n \Delta p_n = O(p_n p_{n+1}), \]  
(2.7)

then the series \(\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}\) is summable \(|\bar{N}, p_n|_k\), \(k \geq 1\).

3. MAIN RESULT

The aim of this paper is to generalize Theorem 2.1 for \(|A,p_n;\delta|_k\) summability by using almost increasing sequence. For this we need the concept of an almost increasing sequence. A positive sequence \((b_n)\) is said to be almost increasing if there exist a positive increasing sequence \((c_n)\) and two positive constants A and B such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say \(b_n = ne^{-n}\).

Now, we shall prove the following theorem.

**Theorem 3.1.** Let \((X_n)\) be an almost increasing sequence. The conditions \((2.1)-(2.4)\) and \((2.6)-(2.7)\) of Theorem 2.1 and the conditions
\[ \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k-1} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \]  
(3.1)
\[ \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta v \hat{a}_{nv}| = O \left( \left( \frac{P_n}{p_n} \right)^{\delta k-1} \right) \quad \text{as} \quad m \to \infty, \]  
(3.2)

where \((t_n)\) as is in Theorem 2.1, are satisfied. If \(A = (a_{nv})\) is a positive normal matrix such that
\[ \pi_{nv} = 1, \quad n = 0, 1, ..., \]  
(3.3)
\[ a_{n-1,v} \geq a_{nv}, \quad \text{for} \quad n \geq v + 1, \]  
(3.4)
\[ a_{nn} = O \left( \frac{p_n}{P_n} \right), \]  
(3.5)
\[ |\hat{a}_{n,v+1}| = O(v |\Delta v (\hat{a}_{nv})|), \]  
(3.6)

then the series \(\sum_{n=1}^{\infty} a_n \frac{p_n \lambda_n}{np_n}\) is summable \(|A,p_n;\delta|_k\), \(k \geq 1\) and \(0 \leq \delta < 1/k\).

We need the following lemmas for the proof of Theorem 3.1.
Lemma 3.2. (6) If \((X_n)\) is an almost increasing sequence, then under the conditions (2.2)-(2.3) we have that
\[
nX_n\beta_n = O(1),
\]
(3.7)
\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]
(3.8)

Lemma 3.3. (3) If conditions (2.6) and (2.7) are satisfied, then we have
\[
\Delta \left( \frac{P_n}{n^2p_n} \right) = O \left( \frac{1}{n^2} \right).
\]
(3.9)

Lemma 3.4. (3) If conditions (2.1)-(2.4) are satisfied, then we have that
\[
\lambda_n = O(1),
\]
(3.10)
\[
\Delta \lambda_n = O \left( \frac{1}{n} \right).
\]
(3.11)

4. Proof of Theorem 3.1

Let \((I_n)\) denotes A-transform of the series \(\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n^p n} \). Then, we have by (1.9) and (1.10)
\[
\tilde{\Delta} I_n = \sum_{v=1}^{n} \frac{a_{nv} P_v \lambda_v}{v^2 p_v}.
\]

Applying Abel’s transformation to this sum, we get that
\[
\Delta I_n = \sum_{v=1}^{n} \frac{a_{nv} v a_v P_v \lambda_v}{v^2 p_v}
\]
\[= \sum_{v=1}^{n-1} \Delta v \left( \frac{a_{nv} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r
\]
\[= \sum_{v=1}^{n-1} \Delta v \left( \frac{a_{nv} P_v \lambda_v}{v^2 p_v} \right) (v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n
\]
\[= \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n + \sum_{v=1}^{n-1} \Delta v (\hat{a}_{nv}) \left( \frac{v+1}{v^2} \right) P_v \lambda_v t_v
\]
\[+ \sum_{v=1}^{n} \frac{a_{nv} P_v \lambda_v}{v^2 p_v} \Delta v t_v (v+1) + \sum_{v=1}^{n-1} \frac{a_{nv} \lambda_v+1}{v^2 p_v} \Delta \left( \frac{P_v}{v^2 p_v} \right) t_v (v+1)
\]
\[= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\]

Since
\[
|I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}|^k \leq 4^k (|I_{n,1}|^k + |I_{n,2}|^k + |I_{n,3}|^k + |I_{n,4}|^k)
\]
to complete the proof of Theorem 3.1, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k+k-1} |I_{n,r}|^k < \infty, \text{ for } r = 1, 2, 3, 4.
\]
(4.1)
First, by using Abel’s transformation, we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,1}|^k = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} a_{n,n} \left( \frac{P_n}{p_n} \right)^k |\lambda_n|^k |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} |\lambda_n|^{k-1} |\lambda_n| |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} |\lambda_n| |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{r=1}^{n} \left( \frac{P_r}{P_r} \right)^{\delta k - 1} |t_r|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m
\]

\[
= O(1) \text{ as } m \to \infty,
\]

by (2.1), (2.4), (2.6), (3.1), (3.5), (3.8), (3.10) and (3.11). Now, using the fact that \( P_v = O(vp_c) \) by (2.6), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) |\lambda_v| |t_v| \right)^k
\]

Now, applying Hölder’s inequality with indices \( k \) and \( k' \), where \( k > 1 \) and \( \frac{1}{k} + \frac{1}{k'} = 1 \), as in \( I_{n,1} \), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) |\lambda_v| |t_v| \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \left( \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right) \times \left( \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \left( \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v|^k \right)
\]

\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v| |t_v|^k m \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta_v(\hat{a}_{nv})|
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k - 1} |\lambda_v| |t_v|^k
\]

\[
= O(1) \text{ as } m \to \infty,
\]
Finally, since $\Delta$ by (2.1), (2.3), (2.6), (3.1), (3.2), (3.5), (3.6), (3.7) and (3.8). Now, using Hölder’s inequality, we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{n=1}^{n-1} |\hat{a}_{n+1}||\Delta \lambda_v| |t_v| \right)^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{n=1}^{n-1} v|\Delta_v(\hat{a}_{n+1})| |\beta_v| |t_v| \right)^k \times \left( \sum_{n=1}^{n-1} v|\Delta_v(\hat{a}_{n+1})| |\beta_v| \right)^{k-1} \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \left( \sum_{v=1}^{m+1} v|\Delta_v(\hat{a}_{n+1})| |\beta_v| |t_v| \right)^k \]

\[ = O(1) \sum_{v=1}^{m+1} v|\beta_v| |t_v|^k \sum_{n=1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} \sum_{n=1}^{n-1} |\Delta_v(\hat{a}_{n+1})| \left| \beta_v \right| \left| t_v \right|^k \]

\[ = O(1) \sum_{v=1}^{m+1} (v + 1)|\Delta \beta_v| X_v + O(1)m \beta_m X_m \]

\[ = O(1) \sum_{v=1}^{m+1} (v + 1)|\Delta \beta_v| X_v + O(1)m \beta_m X_m \]

\[ = O(1) \text{ as } m \to \infty, \]

by (2.1), (2.3), (2.6), (3.1), (3.2), (3.5), (3.6), (3.7) and (3.8).

Finally, since $\Delta \left( \frac{P_v}{p_v} \right) = O \left( \frac{1}{v^\theta} \right)$ by Lemma 3.3, as in $I_{n,1}$, we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,4}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{n=1}^{n-1} |\hat{a}_{n+1}||\lambda_{v+1}| |t_v| \right)^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{n=1}^{n-1} |\Delta \lambda_{v+1}||\lambda_{v+1}| |t_v| \right)^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{n=1}^{n-1} |\Delta \lambda_{v+1}||\lambda_{v+1}| |t_v| \right)^k \times \left( \sum_{v=1}^{n-1} |\Delta \lambda_{v+1}| \right)^{k-1} \]

\[ = O(1) \left( \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta_v(\hat{a}_{n+1})| \right) \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_v|^k = O(1) \text{ as } m \to \infty, \]
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by (2.1), (2.4), (3.1), (3.2), (3.5), (3.6) and (3.11).

Therefore we get

\[ \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} | I_{n,r} |^{k} = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \]

This completes the proof of Theorem 3.1.

If we take \( a_{nv} = \frac{p_n}{P_n} \) and \( \delta = 0 \), then we get a result of Bor [4] for \( |\tilde{N}, p_n|_k \) summability factors. Also, if we take \( \delta = 0 \), then we get a result of Özarslan [8] for \( |A, p_n|_k \) summability factors. Furthermore, if we take \( (X_n) \) as a positive non-decreasing sequence, then we get a new result dealing with \( |A, p_n; \delta|_k \) summability factors.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References


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