ON SUMMABILITY METHODS $|A_f|_k$ AND $|C,0|_s$

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Abstract. In this paper we give necessary and sufficient conditions for $|C,0| \Rightarrow |A_f|_s$ and vice versa, and $|A_f| \Rightarrow |C,0|_s$ and vice versa for the case $1 \leq s < \infty$, where $|A_f|_k$ is absolute factorable summability. So we also complete some open problems in the paper of Sarıgöl [10].

1. Introduction

Let $\Sigma x_v$ be a given infinite series with partial sums $(s_n)$. By $\sigma^n_\alpha$ we denote n-th Cesàro mean of order $\alpha$, $\alpha > -1$, of the sequence $(s_n)$. The series $\Sigma x_v$ is said to be absolutely summable $(C,\alpha)$ with index $k$, or simply summable $|C,\alpha|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^\infty n^{k-1} |\sigma^n_\alpha - \sigma^{n-1}_\alpha|^k < \infty. \quad (1.1)$$

By $\sigma^n_0 = s_n$, the summability $|C,0|_k$ is equivalent to the condition

$$\sum_{n=1}^\infty n^{k-1} |x_n|^k < \infty. \quad (1.2)$$

Let $A_f = (a_{nv})$ be a factorable matrix which is the lower triangular with entries

$$a_{nv} = \begin{cases} \hat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v > n, \end{cases} \quad (1.3)$$

where $(\hat{a}_n)$ and $(a_n)$ are any sequences of real numbers. Then the series $\Sigma x_v$ is said to be summable $|A_f|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^\infty n^{k-1} \left| \hat{a}_n \sum_{v=1}^n a_v x_v \right|^k < \infty. \quad (1.4)$$

Note that if one takes $\hat{a}_n = p_n/P_n P_{n-1}$, $a_v = P_{v-1}$ and $\hat{a}_n = 1/n(n+1)$, $a_v = v$, then $|A_f|_k$ are reduced to the well known summabilities $|R,p|_k$ and $|C,1|_k$.

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respectively, where \((p_n)\) be a sequence of positive real constants with \(P_n = p_0 + p_1 + \ldots + p_n \to \infty\) as \(n \to \infty\). \([15]\).

If A and B are methods of summability, B is said to include A (written \(A \Rightarrow B\)) if every series summable by the method A is also summable by the method B. A and B said to be equivalent (written \(A \Leftrightarrow B\)) if each methods includes the other.

Problems on inclusion dealing absolute Cesàro and absolute weighted mean summabilities have been examined by many authors (see, \([1-5], [7-17]\)) . In this direction, Bor \([1]\) proved sufficient conditions for equivalence of the summabilities \(|R, p_n|_k\) and \(|C, 0|_k\). The more general result including Bor’s result has given by Sargöl \([12]\) under necessary and sufficient conditions. Quite recently, the main results of \([12]\) have been extended by a factorable matrix in \([10]\) as follows.

**Theorem 1.1.** Let \(1 < k \leq s < \infty\) and A be a factorable matrix given by \((1.3)\) such that \(\tilde{a}_n, a_n \neq 0\) for all \(n\). Then, \(|A_f|_k \Rightarrow |C, 0|_s\) if and only if

\[
\left( \sum_{v=m-1}^{m} \frac{1}{v} \left| a_v \right|^k \right)^{1/k^*} \left( \sum_{n=m}^{m+1} \frac{n^{s-1}}{|a_n|^s} \right)^{1/s} = O(1),
\]

where \(k^*\) denotes the conjugate index of \(k\), i.e., \(\frac{1}{k} + \frac{1}{k^*} = 1\).

**Theorem 1.2.** Let \(1 < k \leq s < \infty\) and A be a factorable matrix given by \((1.3)\) . Then, \(|C, 0|_k \Rightarrow |A_f|_s\) if and only if

\[
\left( \sum_{v=1}^{m} \frac{1}{v} \left| a_v \right|^k \right)^{1/k^*} \left( \sum_{n=m}^{\infty} n^{s-1} \left| \tilde{a}_n \right|^s \right)^{1/s} = O(1),
\]

where \(k^*\) denotes the conjugate index of \(k\).

**Corollary 1.3.** Let \(1 < k < \infty\) and A be a factorable matrix given by \((1.3)\) such that \(\tilde{a}_n, a_n \neq 0\) for all \(n\). Then, \(|C, 0|_k \Leftrightarrow |A_f|_s\) if and only if conditions \((1.5)\) and \((1.6)\) satisfied.

**2. Main Results**

Note that Theorem 1.1 and Theorem 1.2 do not include results \(|C, 0| \Rightarrow |A_f|_s\) vise versa, and \(|A_f| \Rightarrow |C, 0|_s\) vise versa for the case \(1 \leq s < \infty\). So, motivated by these theorems, a natural problem is that, what are the necessary and sufficient conditions in order that these results should be satisfied. The aim of this paper is to answer this open problem proving the following theorems.

**Theorem 2.1.** Let \(1 < s < \infty\) and A be a factorable matrix given by \((1.3)\) such that \(\tilde{a}_v, a_v \neq 0\) for all \(v\). Then, \(|A_f|_s \Rightarrow |C, 0|_s\) if and only if

\[
\sum_{v=1}^{\infty} \frac{1}{v} \left( \frac{1}{|\tilde{a}_v|} + \frac{1}{|a_{v+1}|} \right) < \infty,
\]

where \(s^*\) denotes the conjugate index of \(s\), i.e., \(\frac{1}{s} + \frac{1}{s^*} = 1\).

**Theorem 2.2.** Let \(1 < s < \infty\) and A be a factorable matrix given by \((1.3)\) . Then, \(|C, 0|_s \Rightarrow |A_f|_s\) if and only if

\[
\sum_{v=1}^{\infty} \frac{1}{v} \left( a_v \sum_{n=v}^{\infty} \tilde{a}_n \right)^{s^*} < \infty,
\]

where \(s^*\) denotes the conjugate index of \(s\).
Theorem 2.3 Let \( 1 \leq s < \infty \) and \( A \) be a factorable matrix given by (1.3). Then, \(|C,0| \Rightarrow |A_f|_s\) if and only if
\[
\sum_{n=\nu}^{\infty} n^{s-1} |\hat{a}_n a_\nu|^s = O(1) \text{ as } v \to \infty.
\] (2.3)

Theorem 2.4. Let \( 1 \leq s < \infty \) and \( A \) be a factorable matrix given by (1.3) such that \( \hat{a}_v, a_v \neq 0 \) for all \( v \). Then, \(|A_f| \Rightarrow |C,0|_s\) if and only if
\[
\sum_{\nu=1}^{\infty} \frac{1}{v^{s-1}} \left( \frac{1}{|a_v|^s} + \frac{1}{|a_{\nu+1}|^s} \right) = O(1) \text{ as } v \to \infty.
\] (2.4)

If one takes \( \hat{a}_n = p_n \backslash P_n P_{n-1} \) and \( a_\nu = P_{\nu-1} \) in Theorem 2.1 and Theorem 2.2, then the conditions (2.1) and (2.2) are reduced to
\[
\sum_{v=1}^{\infty} \frac{1}{v} \left( \frac{P_{\nu-1}}{p_\nu} + \frac{P_\nu}{p_\nu} \right)^{s^*} < \infty \text{ and } \sum_{v=1}^{\infty} \frac{1}{v} < \infty
\]
respectively, which is impossible. So we get the following results.

Corollary 2.5. If \( s > 1 \), then \(|R,p_n|_s \nRightarrow |C,0| \) and also \(|C,0|_s \nRightarrow |R,p_n| \).

Also, by taking \( \hat{a}_n = p_n \backslash P_n P_{n-1} \) and \( a_\nu = P_{\nu-1} \) in Theorem 2.3 and Theorem 2.4, we get the following results concerning the summability methods \(|C,0|\), \(|R,p_n|_s\), \(|R,p_n|\) and \(|C,0|_s\).

Corollary 2.6. Let \( s \geq 1 \). Then, \(|C,0| \Rightarrow |R,p_n|_s\) if and only if
\[
\sum_{n=\nu}^{\infty} n^{s-1} \left( \frac{p_n}{P_n P_{n-1}} \right)^{s} = O \left( \frac{1}{P_{\nu-1}^{s}} \right).
\]

Corollary 2.7. Let \( s \geq 1 \). Then, \(|R,p_n| \Rightarrow |C,0|_s\) if and only if
\[
v^{1/s^*} P_\nu = O(p_\nu) \text{ as } v \to \infty.
\]

3. Needed Lemmas

In this subtitle we give the following lemmas which are needed in proving our Theorems.

Lemma 3.1 ([11]). Let \( 1 < s < \infty \). Then, \( A : \ell_s \to \ell \) if and only if
\[
\sum_{v=0}^{\infty} \left( \sum_{n=0}^{\infty} |a_{nv}| \right)^{s^*} < \infty.
\] (3.1)

Lemma 3.2 ([7]). Let \( 1 \leq s < \infty \). Then, \( A : \ell \to \ell_s \) if and only if
\[
\sum_{n=0}^{\infty} |a_{nv}|^s = O(1) \text{ as } v \to \infty.
\] (3.2)
4. Proof of Theorems

Since the proofs of Theorem 2.2 and Theorem 2.4 are similar to those of Theorem 2.1 and Theorem 2.3, respectively, we only give proofs of Theorem 2.1 and Theorem 2.3.

Proof of Theorem 2.1. Let \( A_n^* (x) = n^{1/s} A_n (x) \) for \( n \geq 1 \), where

\[
A_n (x) = \hat{a}_n \sum_{\nu=1}^{n} a_{\nu} x_{\nu}.
\]

(4.1)

Then \( \Sigma x_n \) is summable \( |A|_s \) and \( |C,0| \) iff \( A^* (x) \in l_s \) and \( x \in l \), respectively. On the other hand, it can be written from (4.1) that

\[
x_n = \frac{1}{a_n} \left( A_n^* (x) \right) - \frac{A_{n-1}^* (x)}{(n-1)^{1/s} \hat{a}_{n-1}}
\]

which gives us

\[
x_n = \sum_{\nu=1}^{\infty} b_{n\nu} A_n^* (x),
\]

where

\[
b_{n\nu} = \begin{cases} \frac{1}{a_n} \left( \frac{1}{(n-1)^{1/s} \hat{a}_{n-1}} \right), & v = n-1 \\ \frac{1}{a_n} \left( \frac{1}{n^{1/s} \hat{a}_n} \right), & v = n \\ 0, & v \neq n-1, n \end{cases}
\]

(4.2)

Then \( |A|_s \Rightarrow |C,0| \) if and only if \( H : l_s \rightarrow l \), where \( H \) is the matrix whose entries are defined by (4.2). Therefore applying (3.1) to the matrix \( B \), by Lemma 2.1, we have that \( |A|_s \Rightarrow |C,0| \) iff the condition (2.1) holds, which completes the proof.

Proof of Theorem 2.3. Let, for \( n \geq 1 \),

\[
A_n^* (x) = n^{1/s} \hat{a}_n \sum_{\nu=1}^{n} a_{\nu} x_{\nu}.
\]

Then \( \Sigma x_n \) is summable \( |A|_s \) whenever \( \Sigma x_n \) is summable \( |C,0| \) if and only if \( A(x) \in l_s \) whenever \( x \in l \). Also, it follows that

\[
A_n^* (x) = n^{1/s} \hat{a}_n \sum_{\nu=1}^{n} a_{\nu} x_{\nu} = \sum_{\nu=1}^{\infty} h_{n\nu} x_{\nu}
\]

where

\[
h_{n\nu} = \begin{cases} n^{1/s} \hat{a}_n a_{\nu}, & 1 \leq \nu \leq n \\ 0, & \nu > n \end{cases}
\]

Hence \( |C,0|_s \Rightarrow |A|_s \) if and only if \( H : l \rightarrow l_s \). So, applying the matrix \( H \) to (3.2) gives
\[ \sum_{v=1}^{\infty} \left| \frac{1}{v} \sum_{n=v}^{\infty} a_n \right| < \infty, \]

which completes the proof by Lemma 3.2.

References

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