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THREE DIMENSIONAL LORENTZIAN PARA α-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to introduce the notion of Lorentzian Para (LP) α -Sasakian manifolds and study its basic results. Further, these results are used to establish some of the properties of three dimensional semisymmetric and locally φ-symmetric LP α-Sasakian manifolds. An Example of three dimensional Lorentzian Para α -Sasakian manifold is given which verifies all the Theorems.

1. Introduction

The notion of Lorentzian manifold was first introduced by K. Matsumoto [10] in 1989. The same was independently studied by I. Mihai and R. Rosca [13]. Lorentzian Para (LP) Sasakian manifolds are extensively studied by U. C. De and Anupkumar Sengupta [3], U. C. De and A.A. Shaikh [4], [5], U. C. De , K. Matsumoto and A. A. Shaikh [6], U. C. De , Adnan Al-Aqeel and A. A. Shaikh [7], U. C. De , Ion Mihai and A. A. Shaikh [8]. Some of the other authors have also studied LP-Sasakian spaces such as Matsumoto and I. Mihai [11], Abolfazl Taleshian and Nader Asghar [1], Lovjoy Das [9], Mobin Ahmad and Janardhan Ojha [12], S. M. Bhati [2].

In this paper in Section 2, we have introduced the notion of Lorentzian Para α-Sasakian manifold which is the generalised form of the LP-Sasakian manifolds. In 2009, A. Yildiz, M. Turan and B. E. Acet [14] have studied the notion of three dimensional Lorentzian α-Sasakian manifolds and established series of Theorems. Though the concepts of Lorentzian Para α-Sasakian manifolds and Lorentzian α-Sasakian manifolds are different and the basic definitions are also disagreed each other. However,in Section 3, 4 and 5, it is shown that most of the basic results and Theorems of both the manifolds are agreed one another.

In Section 3, basic results of LP α-Sasakian manifolds have been established. Further in Section 4, it has been shown that three dimensional Ricci semisymmetric

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semisymmetric LP α-Sasakian manifold is locally isometric to a sphere. The Section 5 is devoted to the study of locally φ- symmetric three dimensional Lorentzian Para α-Sasakian manifolds. We constructed two Examples of Lorentzian Para α-Sasakian manifolds of which Example 2.1 verifies all Theorems. In fact, it is shown that there exists a Lorentzian Para α-Sasakian manifold which is not a Lorentzian α-Sasakian manifold.

2. **Lorentzian Para α-Sasakian Manifolds**

For a almost Lorentzian contact manifold M (see [4],[10]) of dimension 2n+1, we have

$$\varphi^2X = X + \eta(X)\xi, \eta(\xi) = -1, \eta(X) = g(X, \xi) \tag{2.1}$$

$$g(\varphi(X), \varphi(Y)) = g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

for a C^∞ vector field X on M and φ is a tensor field of type (1,1), ξ is a characteristic vector field and η is 1-form. From these conditions, one can deduce that

$$\varphi(\xi) = 0 \textit{ and } \eta(\xi) = 0$$

Definition 2.1. A Manifold M with Lorentzian almost contact metric structure (φ,ξ,η,g) is said to be the Lorentzian α- Sasakian manifold if

$$(\nabla_X\varphi)Y = \alpha\{g(X, Y)\xi + \eta(Y)X\},$$

where α is a constant function on M.

An almost contact metric structure (for details see [1], [7], [9]) is called a Lorentzian Para Sasakian manifold (or simply LP-Sasakian manifold) if

$$(\nabla_X\varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

where ∇ is the Levi-Civita connection with respect to g. Using above formula, one can deduce

$$\nabla_X\xi = \varphi(X), (\nabla_X\eta)(Y) = g(X, \varphi(Y))$$

More generally in this paper, we introduce the notion of Lorentzian Para α - Sasakian manifold as follows and study its basic properties.

Definition 2.2. A Manifold M with Lorentzian almost contact metric structure (φ,ξ,η,g) is said to be the Lorentzian Para (LP) α- Sasakian manifold if

$$(\nabla_X\varphi)Y = \alpha\{g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\}, \tag{2.3}$$

where α is a smooth function on M.

Note that if $\alpha = 1$, then LP- Sasakian manifold is the special case of Lorentzian Para α Sasakian manifold

Lemma 2.1. *With usual notations, for a Lorentzian Para α- Sasakian manifold M, we have*

$$\nabla_X\xi = \alpha\varphi(X) \tag{2.4}$$

for any vector field X on M.

Proof. For a Lorentzian Para α- Sasakian manifold M, from (2.3), we have

$$\nabla_X(\varphi(Y)) - \varphi(\nabla_XY) = \alpha\{(g(X, Y)\xi + \eta(Y)X + 2\eta(Y)\eta(X)\xi\}$$

Now taking Y=ξ in the above equation using (2.1), we get

$$-\varphi(\nabla_X\xi) = \alpha\{\eta(X)\xi - X - 2\eta(X)\xi\}$$

Applying φ on both sides of the above equation and using the fact that $(\nabla_X g)(\xi,\xi) = 0$ implies $g((\nabla_X \xi), \xi) = 0$ so that $\eta((\nabla_X \xi)) = g(\nabla_X \xi, \xi) = 0$ and simplifying, we get (2.4). □

Example 2.1.We consider the 3-dimensional manifold $M = \{(x, y, z)\epsilon R^3 : z \neq 0\}$, where x, y, z are the standard co-ordinates in R. Let $\{e_1, e_2, e_3\}$ be the linearly independent global frame on M given by

$$e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), e_3 = \alpha \frac{\partial}{\partial z}$$

where α is a nonzero constant on M. Let g be the Lorentzian metric on M defined by
$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$ and $g(e_1, e_1) = 1, g(e_2, e_2) = 1, g(e_3, e_3) = -1$
Let $e_3 = \xi$. Then Lorentzian metric on M is given by

$$g = (e^{-z})^2 \{2(dx)^2 + (dy)^2 - 2dxdy\} - \alpha^{-2}(dz)^2$$

Clearly g is a Lorentzian metric on M. Let η be the 1-form defined by

$$\eta(U) = g(U, e_3)$$

for any vector field U on M. Let φ be the 1-1 tensor field defined by

$$\varphi(e_1) = -e_1, \varphi(e_2) = -e_2, \varphi(e_3) = 0$$

Then using the linearity property, one obtains

$$\eta(e_3) = -1 \ , \varphi^2 U = U + \eta(U)e_3$$

$$g(\varphi(U), \varphi(W)) = g(U, W) + \eta(U)\eta(W)$$

Also for $\xi = e_3$, it is easy to see that

$$\eta(e_1) = 0 \, , \eta(e_2) = 0 \, , \eta(e_3) = -1$$

Hence for $e_3 = \xi$, (φ, ξ, η, g) defines a Lorentzian almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. The the following results hold.

$$[e_1, e_2] = 0 \, , [e_1, e_3] = -\alpha e_1 \, , [e_2, e_3] = -\alpha e_2$$

Using Koszul's formula for Levi-Civita connection ∇ with respect to g, that is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ - g(X, [Y, Z]) + g[Y, [Z, X]) + g(Z, [X, Y]).$$

One can easily calculate

$$\nabla_{e_1} e_3 = -\alpha e_1 \, , \nabla_{e_3} e_3 = 0 \, , \nabla_{e_2} e_3 = -\alpha e_2$$

$$\nabla_{e_2} e_2 = -\alpha e_3 \, , \nabla_{e_1} e_2 = 0 \, , \nabla e_2 e_1 = 0$$

$$\nabla_{e_1} e_1 = -\alpha e_3 \, , \nabla_{e_3} e_2 = 0 \, , \nabla_{e_3} e_1 = 0$$

Also one can verify the condition (2.3) of the Definition 2.2. Hence $M(\varphi, \xi, \eta, g)$ defines a 3 - dimensional Lorentzian Para α-Sasakian manifold and satisfies (2.4).

Theorem 2.2. *There exists a Lorentzian Para α-Sasakian manifold which is not a Lorentzian α-Sasakian manifold*

Corollary 2.3. *There exists a LP Sasakian manifold which is not a Lorentzian Sasakian manifold*

Example 2.2. We consider the 3-dimensional manifold $M = \{(x, y, z) \epsilon R^3 : z \neq 0\}$, where x, y, z are the standard co-ordinates in R. Let $\{e_1, e_2, e_3\}$ be the linearly independent global frame on M given by

$$e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), e_3 = e^z \frac{\partial}{\partial z}$$

Let g be the Lorentzian metric on M defined by
$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$ and $g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1$. Let $e_3 = \xi$. Then Lorentzian metric on M is given by

$$g = (e^{-z})\{2(dx)^2 + (dy)^2 - 2dxdy\} - e^{-2z}(dz)^2$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_3)$$

for any vector field U on M. Let φ be the 1-1 tensor field defined by

$$\varphi(e_1) = -e_1,\ \varphi(e_2) = -e_2,\ \varphi(e_3) = 0$$

Then using the linearity property, one obtains

$$\eta(e_3) = -1\ , \varphi^2 U = U + \eta(U) e_3$$

$$g(\varphi(U), \varphi(W)) = g(U, W) + \eta(U)\eta(W) \tag{2.5}$$

It is easy to see that

$$\eta(e_1) = 0\,, \eta(e_2) = 0\,, \eta(e_3) = -1$$

. Replacing W by $\varphi(W)$ in (2.5) we have $g(\varphi(U), W) = g(U, \varphi(W)$, that is, φ is symmetric. Hence for $e_3 = \xi$, (φ, ξ, η, g) defines a Lorentzian almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then the following results hold.

$$[e_1, e_2] = 0, [e_1, e_3] = -e^z e_1, [e_2, e_3] = -e^z e_2$$

Using Koszul's formula for Levi-Civita connection ∇ with respect to g, one can easily calculate

$$\nabla_{e_1} e_3 = -e^z e_1\,, \nabla_{e_3} e_3 = 0\,, \nabla_{e_2} e_3 = -e^z e_2$$
$$\nabla_{e_2} e_2 = -e^z e_3\,, \nabla_{e_1} e_2 = 0\,, \nabla e_2 e_1 = 0$$
$$\nabla_{e_1} e_1 = -e^z e_3\,, \nabla_{e_3} e_2 = 0\,, \nabla_{e_3} e_1 = 0$$

Also one can verify the condition (2.3) of the Definition 2.2.

Hence $M(\varphi, \xi, \eta, g)$ defines a 3- dimensional Lorentzian Para α-Sasakian manifold with $\alpha = e^z$ and satisfies (2.4).

Lemma 2.4. *For a Lorentzian Para α- Sasakian manifold M, we have*

$$(\nabla_X \eta)(Y) = \alpha g(\varphi(X), Y) \tag{2.6}$$

for all X, Y on M.

Proof. Consider,

$$\begin{aligned}(\nabla_X \eta) Y &= \nabla_X(\eta(Y)) - \eta(\nabla_X Y)\\ &= \nabla_X(g(Y, \xi)) - g(\nabla_X Y, \xi)\\ &= g(Y, \nabla_X \xi)\end{aligned}$$

By virtue of (2.2)and (2.4), we get (2.6). □

Lemma 2.5. *With usual notations, for a Lorentzian Para α- Sasakian manifold M, we have*

$$R(X,Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\} + \{(X\alpha)\varphi(Y) - (Y\alpha)\varphi(X)\}, \tag{2.7}$$

$$R(\xi,Y)\xi = \alpha^2\{Y + \eta(Y)\xi\} + (\xi\alpha)\varphi(Y)\},\ R(\xi,\xi)\xi = 0 \tag{2.8}$$

for all vector fields X, Y on M and R is the curvature tensor of M.

Proof. From (2.1), (2.3) and (2.4), further using the fact that $[X,Y] = \nabla_X Y - \nabla_Y X$ we have

$$\begin{aligned} R(X,Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ &= \nabla_X\{\alpha\varphi(Y)\} - \nabla_Y\{\alpha\varphi(X)\} - \{\alpha\varphi(\nabla_X Y - \nabla_Y X)\} \\ &= [(X\alpha)\varphi(Y) - (Y\alpha)\varphi(X)] + \alpha[\alpha\{(g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\}] \\ &\quad - \alpha[\alpha\{(g(X,Y)\xi + \eta(X)Y + 2\eta(X)\eta(Y)\xi\}] + \alpha\varphi(\nabla_X Y - \alpha\varphi(\nabla_Y X) \\ &\quad - \{\alpha\varphi(\nabla_X Y - \nabla_Y X)\} \end{aligned}$$

Finally, after simplification, we get (2.7). (2.8) follows from (2.7). □

Lemma 2.6. *With usual notations, for a Lorentzian Para α- Sasakian manifold M, we have*

$$R(\xi,Y)X = \alpha^2\{g(X,Y)\xi - \eta(X)Y\} - (X\alpha)\varphi(Y) + g(\varphi(X),Y)(grad\alpha) \tag{2.9}$$

Proof. We have the identity,

$$g(R(\xi,Y)X,Z) = g(R(X,Z)\xi,Y)$$

$$\begin{aligned} g(R(\xi,Y)X,Z) &= g(R(X,Z)\xi,Y) \\ &= \alpha^2\{g(Z,\xi)g(X,Y) - \eta(X)g(Z,Y)\} - \{(Z\alpha)g(\varphi(X),Y)\} \\ &\quad + \{(X\alpha)g(Z,\varphi(Y)\} \end{aligned}$$

After simplification, we get (2.9). □

Lemma 2.7. *With usual notations, for a Lorentzian Para α- Sasakian manifold M, we have*

$$S(Y,\xi) = 2n\alpha^2\eta(Y) - \{(Y\alpha)\omega + (\varphi(Y)\alpha\} \tag{2.10}$$

$$S(\xi,\xi) = -2n\alpha^2 - (\xi\alpha)\omega \tag{2.11}$$

for any vector field Y on M , $\omega = g(\varphi(e_i), e_i)$ and S is the Ricci curvature on M. Note that repeated indices imply the summation.

Proof. From (2.7), we have

$$\begin{aligned} g(R(X,Y)\xi,Z) &= \alpha^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\} \\ &\quad + \{-(Y\alpha)g(\varphi(X),Z) + (X\alpha)g(\varphi(Y),Z)\} \end{aligned} \tag{2.12}$$

Let$\{e_i\}$, for i=1,2,.........,$2n+1$ be the orthonormal basis at each point of the tangent space of M. Then in the equation (2.12), taking $X = Z = e_i$, we have

$$\begin{aligned} g(R(e_i,Y)\xi,e_i) &= \alpha^2\{\eta(Y)g(e_i,e_i) - \eta(e_i)g(Y,e_i)\} \\ &\quad + \{-(Y\alpha)g(\varphi(e_i),e_i) + (e_i\alpha)g(\varphi(Y),e_i)\} \end{aligned}$$

which after simplification gives (2.10).

Put $Y = \xi$ in (2.10) to get (2.11). □

Lemma 2.8. *With usual notations, for a Lorentzian Para α- Sasakian manifold M, we have*

$$\eta(R(X,Y)Z) = \alpha^2\{\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \\ - (X\alpha)g(\varphi(Y),Z) - (Y\alpha)g(\varphi(X),Z) \tag{2.13}$$

Proof. From (2.7, we have

$$\begin{aligned}\eta(R(X,Y)Z) &= g(R(X,Y)Z,\xi)\\ &= g(R(X,Y)\xi,Z)\\ &= -\alpha^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\\ &\quad - \{(X\alpha)g(\varphi(Y),Z) - (Y\alpha)g(\varphi(X),Z)\}\end{aligned}$$

which proves (2.13). □

3. Three Dimensional Lorentzian Para α-Sasakian Manifolds

In this Section and in the rest of the Sections, we assume that α is constant on M. Following definition is needed to prove some Theorems.

Definition A Lorentzian Para α-Sasakian manifold is said to be η-Einstein if its Ricci curvature tensor S of type (0,2) satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{3.1}$$

where X and Y are any vector fields on M and a, b are smooth functions on M.

In three dimensional Lorentzian Para α-Sasakian manifold, the curvature tensor satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{3.2}$$

where r is the scalar curvature of M and Q is the Ricci operator such that $S(X,Y) = g(QX,Y)$

Now putting $Z = \xi$ in (3.2), we have

$$R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + S(Y,\xi)X - S(X,\xi)Y \\ - \frac{r}{2}[\eta(Y)X - \eta(X)Y], \tag{3.3}$$

Further using (3.2)and (2.10)(2.10) in (3.3) and simplifying , we get

$$\eta(Y)QX - \eta(X)QY = [\frac{r}{2} - \alpha^2][\eta(Y)X - \eta(X)Y], \tag{3.4}$$

where r is the scalar curvature of M. The above equation (3.4) may be written as

$$\eta(Y)S(X,Z) - \eta(X)S(Y,Z) = [\frac{r}{2} - \alpha^2][\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{3.5}$$

Now put $Y = \xi$ in (3.5)and simplifying using (2.10)(2.10). Finally we get

$$S(X,Z) = [\frac{r}{2} - \alpha^2]g(X,Z) + [\frac{r}{2} - 3\alpha^2]\eta(X)\eta(Z) \tag{3.6}$$

which by (3.1) of definition shows that M is η-Einstein. Hence we state

Theorem 3.1. *A three dimensional Lorentzian Para α -Sasakian manifold is η -Einstein.*

If $r = 6\alpha^2$, then from (3.6), M is η -Einstein By virtue of (3.6) and (3.2), we find

$$R(X,Y)Z = [\frac{r}{2} - 2\alpha^2][g(Y,Z)X - g(X,Z)Y] + [\frac{r}{2} - 3\alpha^2][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \quad (3.7)$$

From (3.7), one can state the following Theorem.

Theorem 3.2. *A three dimensional Lorentzian Para α -Sasakian manifold is of constant curvature $\frac{r}{2} - 2\alpha^2$ if and only if the scalar curvature is $6\alpha^2$*

Remark. *If the scalar curvature is $6\alpha^2$, then (3.7) gives*

$$R(X,Y)Z = \alpha^2[g(Y,Z)X - g(X,Z)Y]$$

which shows that M is locally isometric to a sphere $S^{2n+1)}(c)$, where $c = \alpha^2$.

4. Three Dimensional Ricci Semisymmetric Lorentzian Para α-Sasakian Manifolds

Definition 4.1: A Lorentzian Para α-Sasakian manifold M is said to be Ricci symmetric if the Ricci tensor of M satisfies

$$R(X,Y).S = 0, \quad (4.1)$$

where $R(X,Y)$ is the derivation of the tensor algebra at each point of the manifold.

Theorem 4.1. *A three dimensional Ricci semisymmetric Lorentzian Para α -Sasakian manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

Proof. Suppose (4.1) holds for Lorentzian Para α -Sasakian manifold. Then

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \quad (4.2)$$

Setting $X = \xi$ in (4.2), further using (2.9), we have

$$2\alpha^2 g(Y,U)\eta(V) - S(Y,V)\eta(U) + 2\alpha^2 g(Y,V)\eta(U) - S(Y,U)\eta(V) = 0 \quad (4.3)$$

where $\alpha \neq 0$.

Let $\{e_1, e_2, \xi\}$ be an orthonormal basis of the tangent space at each point of M. Putting $Y = U = \xi$ in (4.3)and further using (3.6), we obtain

$$\eta(V)[2\xi^2 g(e_i,e_i) - S(e_i,e_i)] = 0$$

From which we have $r = 6\alpha^2$ so that Theorem follows from (3.7). □

5. Locally φ -Symmertic Three dimensional Lorentzian Para α-Sasakian Manifolds

Definition. A Lorentzian Para α -Sasakian manifold is said to be locally φ symmetric if

$$\varphi^2(\nabla_W R)(X,Y)Z = 0 \quad (5.1)$$

for all vector fields X, Y, Z orthogonal to ξ.

Let us prove the following Theorem for three dimensional Lorentzian Para α-Sasakian Manifolds

Theorem 5.1. *A three dimensional Lorentzian Para α-Sasakian manifold is locally φ-symmetric if and only if the scalar curvature r is constant on M.*

Proof. Differentiating (3.2) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &+ [\frac{r}{2} - 3\alpha^2][g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi \\ &+ g(Y,Z)\eta(X)\nabla_W\xi - g(X,Z)\eta(Y)\nabla_W\xi + (\nabla_W\eta)(Y)\eta(Z)X \\ &+ \eta(Y)(\nabla_W\eta)(Z)X - (\nabla_W\eta)X\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y].\end{aligned} \tag{5.2}$$

Now taking X, Y, Z, W vector fields orthogonal to ξ in (5.2), we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+ [\frac{r}{2} - 3\alpha^2][g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi]\end{aligned} \tag{5.3}$$

Using (2.2) in (5.3), after simplification, we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+ \alpha[\frac{r}{2} - 3\alpha^2][g(Y,Z)g(W,\varphi(X))\xi - g(X,Z)g(W,\varphi(Y))\xi]\end{aligned} \tag{5.4}$$

Now applying φ^2 on both sides of (5.4), finally we have

$$\varphi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y], \tag{5.5}$$

from which proof of the Theorem follows from (5.1)(5.1) of the definition stated above. □

Theorem 5.2. *A three dimensional Ricci semisymmetric Lorentzian Para α-Sasakian manifold is locally φ- symmetric.*

Proof. For a semisymmetric Lorentzian Para α-Sasakian manifold M , it is seen in the proof of Theorem 4.1 that the scalar curvature $r = 6\alpha^2$ i.e. r is constant on M so that from Theorem 5.1, the proof follows. □

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