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ϕ -CONHARMONICALLY SYMMETRIC SASAKIAN MANIFOLDS

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Abstract. We consider some conditions on conharmonic curvature tensor $\tilde{C}$, which has many applications in physics and mathematics. We prove that every ϕ-conharmonically symmetric n-dimensional $(n > 3)$, Sasakain manifold is an Einstein manifold. Also we prove that a three-dimensional Sasakian manifold is locally ϕ-conharmonically symmetric if and only if it is locally ϕ- symmetric. Finally we give two examples of a three-dimensional ϕ-conharmonically symmetric Sasakian manifold.

1. Introduction

Let (M^n, g) be an n-dimensional, $n \geqslant 3$, Riemannian manifold of class C^∞.The conharmonic curvature tensor $\tilde{C}$ is considered as an invariant of the conharmonic transformation defined by Ishii [6]. It satisfies all the symmetric properties of the Riemannian curvature tensor R. There are many physical applications of the tensor $\tilde{C}$. For example, in [1], Abdussattar showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor $\tilde{C}$ vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or is filled with a distribution represented by energy momentum tensor T possesing the algebraic structure of an electromagnetic field and is conformal to flat space-time [1]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time.

On the other hand, the notion of locally symmetry of a Riemannian manifold has been weakened by many authors in several ways to different extent. As a weaker version of locally symmetry, T.Takashi [7] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Buecken and Vanhacke [5]. In [4], Boeckx proved that every non-Sasakian (κ, μ)- manifold is locally ϕ-symmetric in the strong sense.

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In the present work we study ϕ-conharmonically symmetry in a Sasakian manifold. The paper is organized as follows: In Section 2, we give a brief account of conharmonic curvature tensor, Weyl tensor and Sasakian manifold. In Section 3, we consider ϕ-conharmonically symmetric Sasakian manifold and prove that it is an Einstein manifold. Then using this result we concluded that a Sasakian manifold is ϕ-conharmonically symmetric if and only if it is ϕ-symmetric. In the next section we consider three-dimensional locally ϕ-conharmonically symmetric Sasakian manifold. Finally we give two examples of a three-dimensional ϕ-conharmonically symmetric Sasakian manifold.

2. **Preliminaries**

In this section, we collect some basic facts about contact metric manifolds. We refer to [3] for a more detailed treatment. An n-dimensional ($n = 2m + 1$) differentiable manifold M^n is called a *contact manifold* if there exists a globally defined 1-form η such that $(d\eta)^m \wedge \eta \neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

for any vector field X tangent to M.

Moreover, it is well-known that there exist a $(1,1)$-tensor field ϕ, a Riemannian metric g which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\xi, X) = \eta(X), \tag{2.3}$$

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.4}$$

for all X, Y tangent to M. As a consequence of the above relations we have

$$\phi\xi = 0, \ \ \eta o \phi = 0. \tag{2.5}$$

The structure (ϕ, ξ, η, g) is called a *contact metric structure* and the manifold M^n with a contact metric structure is said to be a *contact metric manifold*. Furthermore,if moreover the structure is normal, that is, $[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a *Sasakian structure* (normal contact metric structure) and M is called a *Sasakian manifold*.

We denote by ∇ the Levi-Civita connection on M. Then we have

$$(i)(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad (ii)\nabla_X \xi = -\varphi X, \tag{2.6}$$

$$(i)R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (ii)S(X, \xi) = 2n\eta(X) \tag{2.7}$$

for any vector fields X, Y tangent to M,where S denotes the Ricci tensor [3].

The *Weyl conformal curvature tensor* C and the *conharmonic curvature tensor* $\tilde{C}$ are defined by

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{n-2}\left[\begin{array}{c} g(Y, Z)QX - g(X, Z)QY \\ +S(Y, Z)X - S(X, Z)Y \end{array}\right] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y, Z)Z - g(X, Z)Y] \end{aligned} \tag{2.8}$$

and

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\left[\begin{array}{c} g(Y, Z)QX - g(X, Z)QY \\ +S(Y, Z)X - S(X, Z)Y \end{array}\right] \tag{2.9}$$

respectively, where Q denotes the Ricci operator, i.e. $S(X,Y) = g(QX,Y)$ and r is scalar curvature [6]. The curvature tensor R of a 3-dimensional Riemannian manifold can be written as

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right). \end{aligned} \tag{2.10}$$

3. ϕ-conharmonically symmetric Sasakian manifolds

Definition 3.1. *A Sasakian manifold M^n is said to be ϕ-symmetric if*

$$\phi^2(\nabla_X R)(Y,Z)W = 0,$$

for any vector fields X, Y, Z W of M. If the vector fields are orthogonal to ξ, then the manifold is called locally ϕ-symmetric .

Definition 3.2. *A Sasakian manifold $M^n\,(\phi,\xi,\eta,g)$ is said to be ϕ-conharmonically symmetric if*

$$\phi^2(\nabla_X \tilde{C})(Y,Z)W = 0, \tag{3.1}$$

for any vector fields X, Y, Z, W of M. If the vector fields are orthogonal to ξ, then the manifold is called locally ϕ-conharmonically symmetric .

From the definition it follows that a ϕ- symmetric Sasakian manifold is ϕ-conharmonically symmetric. But the converse is not true in general.

Firstly, differentiating (2.9) covariantly with respect to X, we obtain

$$(\nabla_X \tilde{C})(Y,Z)W = (\nabla_X R)(Y,Z)W \tag{3.2}$$

$$-\frac{1}{n-2}[(\nabla_X S)(Z,W)Y - (\nabla_X S)(Y,W)Z + g(Z,W)(\nabla_X Q)Y - g(Y,W)(\nabla_X Q)Z].$$

Using (3.1) and (2.2) , we get

$$\begin{aligned} &-g((\nabla_X R)(Y,Z)W,U) + \frac{1}{n-2}[(\nabla_X S)(Z,W)g(Y,U) - (\nabla_X S)(Y,W)g(Z,U) \\ &+g(Z,W)g((\nabla_X Q)Y,U) - g(Y,W)g((\nabla_X Q)Z,U)] + g((\nabla_X R)(Y,Z)W,\xi)\xi \\ &+\frac{1}{n-2}[g((\nabla_X S)(Z,W)Y - (\nabla_X S)(Y,W)Z;\xi)\eta(U) \\ &+g(Z,W)g((\nabla_X Q)Y,\xi)\eta(U) - g(Y,W)g((\nabla_X Q)Z,\xi)\eta(U)] = 0. \end{aligned} \tag{3.3}$$

Applying contraction to the equation (3.3) with respect to Y and U, we have

$$\begin{aligned} &-(\nabla_X S)(Z,W) + \frac{1}{n-2}[(n-2)(\nabla_X S)(Z,W) + g(Z,W)X(r)] \\ &+g((\nabla_X R)(\xi,Z)W,\xi) - \frac{1}{n-2}[(\nabla_X S)(Z,W) - (\nabla_X S)(\xi,W)\eta(Z) \\ &+g(Z,W)g((\nabla_X Q)\xi,\xi) - \eta(W)g((\nabla_X Q)Z,\xi)] = 0. \end{aligned} \tag{3.4}$$

Taking $W = \xi$ in (3.4) it follows that

$$-\,(\nabla_X S)(Z,\xi) + \frac{1}{n-2}\eta(Z)X(r) = 0. \tag{3.5}$$

Then putting $Z = \xi$ in (3.5), we obtain $X(r) = 0$,that is, r is constant. Thus we can state the following:

Theorem 1. *Let M be a Sasakian manifold. If M is ϕ-conharmonically symmetric then the scalar curvature r is constant.*

From the equation (3.5) and Theorem 1 we obtain

$$(\nabla_X S)(Z,\xi) = 0,$$

that is,

$$\nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X \xi) = 0.$$

Now using 6(ii) and 7(ii) yields

$$2n(\nabla_X \eta)(Z) + S(Z,\phi X) = 0. \tag{3.6}$$

Also in a Sasakian manifold it is known that $(\nabla_X \eta)(Z) = g(X,\phi Z)$. Therefore putting $X = \phi X$ in (3.6) we get

$$S(X,Z) = 2ng(X,Z).$$

Hence we are in a position to state the following:

Theorem 2. *Let M be a Sasakian manifold. If M is ϕ-conharmonically symmetric then M is an Einstein manifold.*

Then using the above theorem in the equation (3.2) , we get easily $(\nabla_X \tilde{C})(Y,Z)W = (\nabla_X R)(Y,Z)W$. So, we state the following:

Corollary 1. *Let M^n be a Sasakian manifold. M^n is ϕ-conharmonically symmetric if and only if it is ϕ-symmetric.*

4. Three-dimensional locally ϕ-conharmonically symmetric Sasakian manifolds

Now, we suppose that M is a three-dimensional locally ϕ-conharmonically symmetric Sasakian manifold. Using the equation (2.9), we get

$$\phi^2(\nabla_X \tilde{C})(Y,Z)W = -\frac{X(r)}{2}[g(Y,W)Z - g(Z,W)Y]$$

for any vector fields X, Y, Z W are orthogonal to ξ. Thus we can easily get the following:

Theorem 3. *A three-dimensional Sasakian manifold is locally ϕ-conharmonically symmetric if and only if the scalar curvature r is constant.*

It is known from Watanabe's result [9] that a three-dimensional Sasakian manifold is locally ϕ- symmetric if and only if the scalar curvature r is constant. Using Watanabe's result we state the following:

Theorem 4. *A three-dimensional Sasakian manifold is locally ϕ-conharmonically symmetric if and only if it is locally ϕ- symmetric.*

5. Example

In this section we give two examples to prove the existence of a three-dimensional ϕ-conharmonically symmetric Sasakian manifold.

Example 5.1. *In* [8]*(p.275) , K.Yano and M.Kon gave an example of a Sasakian manifolds which is three-dimensional sphere. Three-dimensional sphere is an Einstein manifold and hence a manifold of constant scalar curvature. Hence by Theorem 3 the three-dimensional sphere is locally ϕ-conharmonically symmetric.*

Example 5.2. *We consider the three-dimensional manifold $M = \{(x,y,z)\varepsilon\mathbb{R}^3, (x,y,z) \neq 0\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$.*

The vector fields

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{1}{2}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g *be the Riemannian metric defined by*

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η *be the 1-form defined by* $\eta(Z) = g(Z, e_3)$ *for any* $Z\varepsilon\chi(M)$.

Further, let ϕ *be the* $(1,1)$ *tensor field defined by*

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

So, using the linearity of ϕ *and* g, *we have*

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, *the structure* (ϕ, ξ, η, g) *defines an almost contact metric structure on* M.

Let ∇ *be the Levi-Civita connection with respect to metric* g. *Then we have*

$$\begin{aligned}
[e_1, e_2] &= e_1 e_2 - e_2 e_1 \\
&= (\frac{\partial}{\partial x} - y\frac{\partial}{\partial z})\frac{\partial}{\partial y} - \frac{\partial}{\partial y}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}) \\
&= \frac{\partial^2}{\partial x \partial y} - y\frac{\partial^2}{\partial z \partial y} - \frac{\partial^2}{\partial y \partial x} + \frac{\partial}{\partial z} + y\frac{\partial^2}{\partial y \partial z} \\
&= \frac{\partial}{\partial z} = 2e_3.
\end{aligned}$$

Similarly

$$[e_2, e_3] = 0 \quad and \quad [e_1, e_3] = 0.$$

The Riemannian connection ∇ *of the metric* g *is given by*

$$\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
&+ g(Z, [X, Y]) - g(X, [Y, Z]) + g(Y, [Z, X]),
\end{aligned} \tag{5.1}$$

which is known as Koszul's formula. Using (5.1) we have

$$2g(\nabla_{e_1} e_3, e_1) = 0 = 2g(-e_2, e_1). \tag{5.2}$$

Again by (5.1)

$$\begin{aligned}
2g(\nabla_{e_1} e_3, e_2) &= g(-2e_3, e_3) \\
&= 2g(-e_2, e_2)
\end{aligned} \tag{5.3}$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_2, e_3). \tag{5.4}$$

From (5.2), (5.3) and (5.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_2, X)$$

for all $X \in \chi(M)$.

Thus

$$\nabla_{e_1} e_3 = -e_2.$$

Therefore, (5.1) further yields

$$\nabla_{e_1} e_3 = -e_2, \quad \nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_1 = 0$$

$$\nabla_{e_2} e_3 = e_1, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = -e_3$$

$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = e_1, \quad \nabla_{e_3} e_1 = -e_2 \tag{5.5}$$

(5.5) tells us that the (ϕ, ξ, η, g) *structure satisfies the formula* $\nabla_X \xi = -\phi X$ *for* $\xi = e_3$. *Hence* $M(\phi, \xi, \eta, g)$ *is a three-dimensional Sasakian manifold.*

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5.6}$$

With the help of the above results and using (5.6) it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = e_2, \quad R(e_1, e_3)e_3 = e_1$$

$$R(e_1, e_2)e_2 = -3e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0$$

$$R(e_1, e_2)e_1 = 3e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the above expression of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2.$$

Similarly we have

$$S(e_2, e_2) = -2, \quad S(e_3, e_3) = 2 \quad \textit{and} \quad S(e_i, e_j) = 0 \quad \textit{for} \quad i \neq j.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -2.$$

Hence we obtain that the scalar curvature is constant. Therefore from Theorem 4, it follows that M *is a three-dimensional locally* ϕ*-conharmonically symmetric Sasakian manifold.*

6. **Conclusions**

As a generalization of ϕ-symmetric Sasakian manifolds, ϕ-conharmonically symmetric Sasakian manifolds have been introduced in this paper. Conharmonic curvature tensor has some physical applications. Examples of three-dimensional locally ϕ-conharmonically symmetric Sasakian manifolds are given and prove that a ϕ-conharmonically symmetric Sasakian manifold is an Einstein manifold.

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