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ON CONCIRCULARLY $\phi-$RECURRENT KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study concircularly $\phi-$recurrent Kenmotsu manifolds.

1. Introduction

A transformation of an n-dimensional Riemannian manifold M ,which transform every geodesic circle of M in to a geodesic circle,is called a concircular transformation . A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus ,the geometry of concircular transformations, that is, the concircular geometry,is a generalization of inversive geometry in the sence that the change of metric is more general than that induced by a circle preserving diffeomorphism . An interesting invariant of a concircular transformation is the concircular curavture tensor.

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. As a weaker version of local symmetry,in 1977, Takahashi [6] introduced the notion of locally $\phi-$symmetric Sasakian manifold and obtained their several interesting results .Later in 2009, De, Yildiz and Yaliniz [10] studied $\phi-$recurrent Kenmotsu manifold and obtained their some interesting results.In this paper we study a concircularly $\phi-$recurrent Kenmotsu manifold which generalizes the notion of locally concircular $\phi-$symmetric Kenmotsu manifold and obtained some interesting results.Again it is proved that a concircularly $\phi-$recurrent Kenmotsu manifold is an Einstein manifold and in a concircularly $\phi-$ recurrent Kenmotsu manifold, the characteristic vector field ξ and the vector field ρ associated to the $1-$form A are co-directional. Finally, we proved

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that a three dimensional locally concircularly $\phi-$recurrent Kenmotsu manifold is of constant curvature.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric.It is well known that the structure (ϕ, ξ, η, g) satisfy

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$(a)\ \ \eta(\xi) = 1, (b)\ \ g(X, \xi) = \eta(X),\ \ (c)\ \ \eta(\phi X) = 0, (d)\ \ \phi\xi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$(D_X \phi)(Y) = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

$$D_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(D_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.6}$$

for all vector fields X, Y, Z, where D denotes the operator of covariant differentiation with respect to g ,then $M^{2n+1}(\phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold [4].

Kenmotsu manifolds have been studied by many authors such as Binh, Tamassy, De and Tarafdar [7],De and Pathak [9],Jun, De and Pathak [3], Ozgur and De [1] and many others.

In a Kenmotsu manifold the following relations hold: [4]

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.9}$$

$$S(X, \xi) = -2n\eta(X), \tag{2.10}$$

for all vector fields X, Y, Z, where S is the Ricci tensor of type $(0, 2)$ and R is the Riemannian curvature tensor of the manifold.

Definition 2.1. A Kenmotsu manifold is said to be a locally ϕ-symmetric manifold if [6]

$$\phi^2((D_W R)(X, Y)Z) = 0, \tag{2.11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. A Kenmotsu manifold is said to be a locally concircularly $\phi-$symmetric manifold if

$$\phi^2((D_W C)(X, Y)Z) = 0, \tag{2.12}$$

for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 2.3. A Kenmotsu manifold is said to be concircularly $\phi-$recurrent Kenmotsu manifold if there exists a non-zero 1-form A such that

$$\phi^2((D_W C)(X, Y)Z) = A(W)C(X, Y)Z, \tag{2.13}$$

for arbitrary vector fields X, Y, Z, W, where C is a concircular curvature tensor given by [5]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y], \tag{2.14}$$

where R is the Riemann curvature tensor and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to a locally concircularly $\phi-$symmetric manifold.

3. Concircularly $\phi-$recurrent Kenmotsu manifold

Let us consider a concircularly $\phi-$recurrent Kenmotsu manifold. Then by virtue of (2.1) and (2.13), we get

$$-(D_W C)(X,Y)Z+\eta((D_W C)(X,Y)Z)\xi = A(W)C(X,Y)Z, \tag{3.1}$$

from which it follows that

$$-g((D_W C)(X,Y)Z,U)+\eta((D_W C)(X,Y)Z)\eta(U) = A(W)g(C(X,Y)Z,U). \tag{3.2}$$

Let $\{e_i\}$, $i = 1, 2, ...2n+1$ be an orthonormal basis of the tangent space at any point of the manifold.Then putting $X = U = e_i$ in (3.2) and taking summation over i , $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}(D_W S)(Y,Z) &= \frac{dr(W)}{(2n+1)}g(Y,Z) - \frac{dr(W)}{2n(2n+1)}[g(Y,Z)-\eta(Y)\eta(Z)] \\ &- A(W)[S(Y,Z) - \frac{r}{2n+1}g(Y,Z)]. \end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.5) and (2.10), we get

$$(D_W S)(Y,\xi) = \frac{dr(W)}{(2n+1)}\eta(Y) - A(W)[2n - \frac{r}{2n+1}]\eta(Y). \tag{3.4}$$

Now we have

$$(D_W S)(Y,\xi) = D_W S(Y,\xi) - S(D_W Y,\xi) - S(Y, D_W\xi)$$

Using (2.5) ,(2.6) and (2.10) in the above relation,it follows that

$$(D_W S)(Y,\xi) = -2ng(Y,W) - S(Y,W). \tag{3.5}$$

In view of (3.4) and (3.5),we get

$$S(Y,W) = -2ng(Y,W) - \frac{dr(W)}{2n+1}\eta(Y) + A(W)[2n - \frac{r}{2n+1}]\eta(Y). \tag{3.6}$$

. Replacing Y by ϕY in (3.6) ,we get

$$S(\phi Y,W) = -2ng(\phi Y,W). \tag{3.7}$$

Again replacing W by ϕW in (3.7) and using(2.3) and(2.9),we get

$$S(Y,W) = -2ng(Y,W),$$

for all Y,W.

Hence, we can state the following theorem:

Theorem 3.1. *A Concircularly $\phi-$recurrent Kenmotsu manifold (M^{2n+1}, g) is an Einstein manifold.*

Now from (3.1) ,we have

$$(D_W C)(X,Y)Z = \eta((D_W C)(X,Y)Z)\xi - A(W)C(X,Y)Z. \tag{3.8}$$

Using (2.14) in (3.8),we get

$$\begin{aligned}(D_W R)(X,Y)Z &= \eta((D_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z \\ &-\frac{dr(W)}{2n(2n+1)}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\ &+\frac{dr(W)}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \\ &+\frac{r}{2n(2n+1)}A(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \quad (3.9)$$

From (3.9) and the Bianchi identity,we get

$$\begin{aligned}&A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\ &= \frac{r}{2n(2n+1)}A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &+\frac{r}{2n(2n+1)}A(X)[g(Z,W)\eta(Y) - g(Y,Z)\eta(W)] \\ &+\frac{r}{2n(2n+1)}A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \end{aligned} \quad (3.10)$$

Putting $Y = Z = e_i$ in (3.10) and taking summation over i, $1 \le i \le 2n+1$, we get

$$A(W)\eta(X) = A(X)\eta(W), \quad (3.11)$$

for all vector fields X, W. Replacing X by ξ in (3.11) ,we get

$$A(W) = \eta(W)\eta(\rho), \quad (3.12)$$

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho), \rho$ being the vector field associated to the 1−form A i.e., $A(X) = g(X,\rho)$. From (3.11) and (3.12) ,we can state the following theorem:

Theorem 3.2. *In a Concircularly ϕ- recurrent Kenmotsu manifold (M^{2n+1}, g) $(n \ge 1)$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.12).*

4. On 3-dimensional locally concircularly ϕ-recurrent Kenmotsu manifolds

It is known that in a three dimensional Kenmotsu manifold the curvature tensor has the following form [9]

$$\begin{aligned}R(X,Y)Z &= (\frac{r+4}{2})[g(Y,Z)X - g(X,Z)Y] \\ &-\frac{(r+6)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \quad (4.1)$$

Taking covariant differentiation of (4.1), we get

$$\begin{aligned}(D_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &-\frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &-\frac{r+6}{2}[g(Y,Z)(D_W\eta)(X)\xi + g(Y,Z)\eta(X)(D_W\xi) \\ &-g(X,Z)(D_W\eta)(Y)\xi - g(X,Z)\eta(Y)(D_W\xi) \\ &+(D_W\eta)(Y)\eta(Z)X + (D_W\eta)(Z)\eta(Y)X \\ &-(D_W\eta)(X)\eta(Z)Y - (D_W\eta)(Z)\eta(X)Y]. \qquad (4.2)\end{aligned}$$

Taking X,Y,Z,W orthogonal to ξ and using (2.5) and (2.6),we get

$$\begin{aligned}(D_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &-\frac{r+6}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\xi. \qquad (4.3)\end{aligned}$$

From (4.3)it follows that

$$\phi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y]. \qquad (4.4)$$

Now taking X,Y,Z,W orthogonal to ξ and using (2.1) and (2.2) in(4.4),we get

$$\phi^2(D_W R)(X,Y)Z = -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \qquad (4.5)$$

Differentiating covariantly (2.14) with respect to W (for n=1),we get

$$(D_W C)(X,Y)Z = (D_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y]. \qquad (4.6)$$

Now applying ϕ^2 to the both sides of (4.6), we get

$$\phi^2(D_W C)(X,Y)Z = \phi^2(D_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y]. \qquad (4.7)$$

Now using (2.13), (4.5), (2.1) in (4.7),we obtain

$$\begin{aligned}A(W)C(X,Y)Z &= -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+\frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y \\ &+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \qquad (4.8)\end{aligned}$$

Taking X,Y,Z,W orthogonal to ξ, we get

$$C(X,Y)Z = -\frac{dr(W)}{3A(W)}[g(Y,Z)X - g(X,Z)Y]. \qquad (4.9)$$

Putting W=$\{e_i\}$ in (4.9), where $\{e_i\}$, i=1, 2, 3 is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$,we obtain

$$C(X,Y)Z = -\frac{dr(e_i)}{3A(e_i)}[g(Y,Z)X - g(X,Z)Y]. \tag{4.10}$$

Using (2.14) in (4.10), we get

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y], \tag{4.11}$$

where $\lambda = [\frac{r}{6} - \frac{dr(e_i)}{3A(e_i)}]$ is a scalar, Since A is a non-zero 1- form.Then by schur's theorem λ will be a constant on the manifold.Therefore, M^3 is of constant curvatare λ.Hence we can state the following theorem:

Theorem 4.1. *A 3- dimensional locally concircularly ϕ-recurrent kenmotsu manifold is of constant curvature*

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