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ON A TYPE OF RIEMANNIAN MANIFOLD

(COMMUNICATED BY UDAY CHAND DE)

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Abstract. A new type of Riemannian manifold has been defined called AL-Watani manifold, and some of its geometric properties are derived. Also a non trivial example is obtained to prove the existence.

1. Introduction and Preliminaries

Consider a Riemannian manifold $(M^n, g) n \geq 2$,such that its curvature tensor R satisfies the relation,

$$\mathrm{R(X,Y,Z)} \ = \ \mathrm{k[S(Y,Z)X} \ + \ \mathrm{g(Y,Z)LX]}, \tag{1.1}$$

where L is the symmetric endomorphism of the Ricci tensor S such that,

$$S(X,Y) = g(LX,Y), \tag{1.2}$$

and k is a non-zero scalar.
Such a manifold shall be called AL-Watani manifold (after my father nick name). If in particular, $k = 0$, the manifold reduces to a flat manifold. This will justify the definition of the manifold defined by 1.1.

It is known [1] that on a Riemannian manifold the Ricci tensor is of Codazzi type if,

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = 0. \tag{1.3}$$

In section 2 it is shown that every AL-Watani manifold is an Einstein manifold, and obtained a necessary and sufficient condition for AL-Watani manifold to be a flat manifold, and conformally flat. Also a necessary and sufficient condition for AL-Watani manifold of constant scalar curvature to be a symmetric manifold is derived. Finally a non trivial example of AL-Watani manifold is given to prove the existence.

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2. Main results

AL-Watani manifold

Contracting 1.1 we get,

$$\mathrm{S(Y,Z)} \ = \frac{kr}{(1-kn)} \ \mathrm{g(Y,Z)}, \tag{2.1}$$

where r is the scalar curvature of the manifold.
Further contraction yields k = $\frac{1}{2n}$ .Therefore Einstein manifold [1] is a special case of AL-Watani manifold. Thus we can state,

Theorem 2.1. *Every AL-Watani manifold is an Einstein manifold.*

Let the manifold be flat, then we have,

$$\mathrm{S(Y,Z)X} \ + \ \mathrm{g(Y,Z)LX} \ = \ 0. \tag{2.2}$$

Contraction with respect to X gives,

$$\mathrm{nS(Y,Z)} \ + \ \mathrm{rg(Y,Z)} \ = \ 0.. \tag{2.3}$$

Further contraction yields,

$$\mathrm{r} \ = \ 0. \tag{2.4}$$

Therefore in consequence of (1.1), (2.3), and (2.4) we can state,

Theorem 2.2. *AL-Watani manifold is a flat manifold if and only if its scalar curvature is zero.*

Taking covariant derivative of (1.1) we get,

$$(\nabla_W R)(\mathrm{X,Y,Z}) \ = \ \mathrm{k}[(\nabla_W \mathrm{S})\,(\mathrm{Y,Z})\,\mathrm{X} \ + \ \mathrm{g}\,(\mathrm{Y,Z})\,(\nabla_W \mathrm{L})(\mathrm{X})]. \tag{2.5}$$

If the manifold is locally symmetric then we have, since k ?0,

$$(\nabla_W \mathrm{S})\,(\mathrm{Y,Z})\,\mathrm{X} \ + \ \mathrm{g}\,(\mathrm{Y,Z})\,(\nabla_W \mathrm{L})\,(\mathrm{X}) = 0. \tag{2.6}$$

Contracting this equation yields,

$$(\nabla_X \mathrm{S})\,(\mathrm{Y,Z}) \ + \mathrm{dr}\,(\mathrm{X})\,\mathrm{g}\,(\mathrm{Y,Z}) = 0. \tag{2.7}$$

Now if the manifold is of constant scalar curvature (2.7) imply the manifold is Ricci symmetric. Conversely, if the manifold of constant scalar curvature is Ricci symmetric, then it is obviously symmetric. Thus we can state,

Theorem 2.3. *AL-Watani manifold of constant scalar curvature is locally symmetric if and only if it is Ricci symmetric.*

Conformal curvature tensor of a Riemannian manifold [1] is given by the relation,

$$C\,(X,Y,Z) = R\,(X,Y,Z) - \frac{1}{n-2}\,[S\,(Y,Z)\,X - S(X,Z)Y] + g\,(Y,Z)\,LX - g\,(Z,X)\,LY] \tag{2.8}$$

$$- \frac{r}{(n-1)\,(n-2)}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y]\,. \tag{2.9}$$

If the manifold is conformally flat then we have,

$$R\,(X,Y,Z) = \frac{1}{n-2}\,[S\,(Y,Z)\,X - S(X,Z)Y] + \\ g\,(Y,Z)\,LX - g\,(Z,X)\,LY] + \frac{r}{(n-1)\,(n-2)}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y]\,.$$

Contraction with respect to X gives,

$$n^2 S(Y,Z) + (n-2) r g(Y,Z) = 0. \tag{2.10}$$

Further contraction yields r = 0. Therefore in consequence of (1.1), (2.8), and (2.10) we can state,

Theorem 2.4. *AL-Watani manifold is a conformally flat manifold if and only if it is of constant scalar curvature.*

It is known [1] that in a conformally flat $(M^n, g)(n \geq 3)$,

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}[dr(X) g(Y,Z) - dr(Z) g(X,Y)]. \tag{2.11}$$

Now contracting (2.5) we get,

$$(\nabla_W S)(Y,Z) = \frac{1}{n} dr(W) g(Y,Z). \tag{2.12}$$

Substituting (2.12) in (2.11) we get,

$$dr(X) g(Y,Z) = dr(Z) g(X,Y). \tag{2.13}$$

Therefore in consequence of (2.13), and (2.12) we can state,

Theorem 2.5. *In a conformally flat AL-Watani manifold the Ricci tensor is of Codazzi type.*

3. Example of AL-Watani manifold

Let us consider R^4endowed with the Riemannian metric [2],

$$d^2 = \mathrm{g_{ij} dx^i dx^j} = (\mathrm{x}^4)^{\frac{4}{3}} [(\mathrm{dx}^1)^2 + (\mathrm{dx}^2)^2 + \left(\mathrm{dx}^3)^2\right] + (\mathrm{dx}^4)^2, \tag{3.1}$$

where i, j = 1,2,3,4. Then it is known [2] that the only non vanishing Ricci tensors and the curvature tensors are,

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}; \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3(x^4)^{\frac{1}{3}}},$$

$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}}, \tag{3.2}$$

$$S_{11} = S_{22} = S_{33} = \frac{-2}{9(x^4)^{\frac{2}{3}}}; S_{44} = \frac{-2}{3(\mathrm{x}^4)^2}, \tag{3.3}$$

And the scalar curvature r = $\frac{-4}{3(\mathrm{x}^4)^2}$.

To verify the definition by (1.1) we have to verify the following relations:

$$R_{1441} = k[S_{44} g_{11} + S_{11} g_{44}], \tag{3.4}$$

$$R_{2442} = k[S_{44} g_{22} + S_{22} g_{44}], \tag{3.5}$$

$$R_{3443} = k[S_{44} g_{33} + S_{33} g_{44}]. \tag{3.6}$$

Let k = $\frac{1}{4}$ and using (3.2), (3.3) on (3.4) we get,

$$\text{R.H.S.} = k[S_{44} g_{11} + S_{11} g_{44}] = \frac{1}{4}[\frac{-2}{3(\mathrm{x}^4)^2}(\mathrm{x}^4)^{\frac{4}{3}} + \frac{-2}{9(x^4)^{\frac{2}{3}}}(1)]$$

$$= \frac{-2}{9(x^4)^{\frac{2}{3}}} = \text{ L.H.S.}$$

Similarly we can show (3.5) and (3.6) are true, whereas the other cases are trivially true. Hence R^4 along with the metric g defined by (3.1) is AL-Watani manifold. Thus we can state,

Theorem 3.1. *A Riemannian manifold (M^4,g) endowed with the metric* (3.1) *is AL-Watani manifold with non constant scalar curvature.*

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