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ON CERTAIN SUBCLASSES OF P-VALENT ANALYTIC FUNCTIONS INVOLVING THE CHO-KWON-SRIVASTAVA OPERATOR

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, we introduce new classes $B_k^\lambda(a, c, p, \alpha, \rho)$ and $T_k^\lambda(a, c, p, \alpha, \rho)$ of multivalent analytic functions defined by using the Cho-Kwon-Srivastava operator. We use a strong convolution technique. Inclusion results, a radius problem and some other interesting properties of these classes are discussed.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_n z^{n+p}, \ (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$. Also let the Hadamard product (or convolution) of two functions

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \, (j = 1, 2) \, ,$$

be given by

$$(f_1 \star f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} = (f_2 \star f_1)(z) \quad (z \in E) \, .$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \ z = re^{i\theta}, \tag{1.2}$$

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where $k \geqslant 2$ and $0 \leq \rho < 1$. This class has been introduced in [8]. We note that $P_k(0) = P_k$, see [11], $P_2(\ \rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (1.2) we can easily deduce that $p(z) \in P_k(\rho)$ if, and only if, there exists $p_1(z)$, $p_2(z) \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{1.3}$$

In [13] Saitoh introduced a linear operator $\mathcal{L}_p(a, c) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ defined by

$$\mathcal{L}_p(a, c) = \phi_p(a, c; z) * f(z),\ (z \in E;\ f(z) \in \mathcal{A}(p), \tag{1.4}$$

where

$$\phi_p(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p},\ \left(a \in \mathbb{R};\ c \in \mathbb{R} \backslash \mathbb{Z}_0^- = \{0, -1, \ldots\},\ z \in E\right), \tag{1.5}$$

and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)\ldots(x+n-1),\ n \in \mathbb{N} \\ 1,\ n = 0. \end{cases}$$

The operator $\mathcal{L}_p(a, c)$ is an extension of the Carlson-Shaffer operator, see [1]. Very recently, Cho, Kwon and Srivastava [2] introduced the following linear operator $\mathcal{I}_p^{\lambda}(a, c)$ analogous to $\mathcal{L}_p(a, c)$:

$$\mathcal{I}_p^{\lambda}(a, c) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$$

$$\mathcal{I}_p^{\lambda}(a, c) f(z) = \phi_p^{(\dagger)}(a, c; z) * f(z),\ z \in E, \tag{1.6}$$

where $a \in \mathbb{R}$; $c \in \mathbb{R} \backslash \mathbb{Z}_0^-$, $\lambda > -p$ and $\phi_p^{(\dagger)}(a, c; z)$ is the function defined in terms of the Hadamard product (or convolution) by the following ways.

$$\phi_p(a, c; z) \star \phi_p^{\dagger}(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}},\ z \in E, \tag{1.7}$$

where $\phi_p(a, c; z)$ is given by (1.5). It is well known that for $\lambda > -p$

$$\frac{z^p}{(1-z)^{\lambda+p}} = \sum_{n=0}^{\infty} \frac{(\lambda+p)_n (c)_n}{n!} z^{n+p},\ z \in E, \tag{1.8}$$

Cho, Kwon and Srivastava, see [2] have obtained the following properties of the operator $\mathcal{I}_p^{\lambda}(a, c)$

$$\mathcal{I}_p^{0}(p, 1) f(z) = \mathcal{I}_p^{1}(p+1, 1) f(z) = f(z),\ \mathcal{I}_p^{1}(p, 1) f(z) = \frac{z f'(z)}{p}, \tag{1.9}$$

$$z\left(\mathcal{I}_p^{\lambda}(a+1, c) f(z)\right)' = a\mathcal{I}_p^{\lambda}(a, c) f(z) - (a-p)\mathcal{I}_p^{\lambda}(a+1, c) f(z), \tag{1.10}$$

$$z\left(\mathcal{I}_p^{\lambda}(a, c) f(z)\right)' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a, c) f(z) - \lambda\mathcal{I}_p^{\lambda}(a, c) f(z), \tag{1.11}$$

$$\begin{aligned}
\mathcal{I}_p^0\,(a+1,1)\,f(z) &= p\int_0^z \frac{f(t)}{t}dt,\ \ \mathcal{I}_p^0\,(p,1)\,f(z)=\mathcal{I}_p^1\,(p+1,1)\,f(z)=f(z),\\
\mathcal{I}_p^1\,(p,1)\,f(z) &= \frac{zf'(z)}{p},\ \ \mathcal{I}_p^2\,(p,1)\,f(z)=\frac{2zf'(z)+z^2f''(z)}{p\,(p+1)},\\
\mathcal{I}_p^n\,(a,a)\,f(z) &= D^{n+p-1}f(z),\ n\in\mathbb{N},\ n>-p,
\end{aligned}$$

where $D^{n+p-1}f(z)$ is the Ruscheweyh derivative of $(n+p-1)$th order, see [3]. Many interesting result of multivalent analytic functions associated with the linear operator $\mathcal{I}_p^\lambda\,(a,c)$ have been studied in [2]. Also the authors [2] presented a long list of papers connected with the operator (1.4) and (1.6) and classes of functions defined by means of those operators. The interested reader are refered to the work done by authors [4, 7, 9, 15]. Using the Cho-Kown-Srivastava operator $\mathcal{I}_p^\lambda\,(a,c)$, we now define a subclasses of $\mathcal{A}(p)$ as follows:

Definition 1.1. *Let $f\in\mathcal{A}(p)$. Then $f\in\mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$, if and only if*

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\frac{\alpha}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\right\}\in P_k(\rho), \tag{1.12}$$

where α is a complex number, $k\geq 2$, $z\in E$, $0\leq\rho<p$, $\lambda>-p$, a, $c\in\mathbb{R}\backslash\mathbb{Z}_0^-$.

Definition 1.2. *Let $f\in\mathcal{A}(p)$. Then $f\in\mathcal{T}_k^\lambda(a,c,p,\alpha,\rho)$, if and only if*

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\alpha\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right\}\in P_k(\rho), \tag{1.13}$$

where $\alpha>0$, $k\geq 2$, $z\in E$, $0\leq\rho<p$, $\lambda>-p$, and a, $c\in\mathbb{R}\backslash\mathbb{Z}_0^-$.

2. PRELIMINARY RESULTS

Lemma 2.1. [12]. *If $p(z)$ is analytic in E with $p(0)=1$, and if λ_1 is a complex number satisfying $Re(\lambda_1)\geq 0$ $(\lambda_1\neq 0)$, then*

$$Re\{p(z)+\lambda_1 zp'(z)\}>\beta,\ (0\leq\beta<1).$$

implies

$$Rep(z)>\beta+(1-\beta)(2\gamma-1),$$

where γ is given by

$$\gamma=\int_0^1(1+t^{Re\lambda_1})dt,$$

which is an increasing function of $Re\lambda_1$ and $\frac{1}{2}\leq\gamma<1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2. [14]. *If $p(z)$ is analytic in E, $p(0)=1$ and $Rep(z)>\frac{1}{2}$, $z\in E$, then for any function F analytic in E, the function $p*F$ takes values in the convex hull of the image of E under F.*

Lemma 2.3. [10]. *Let $p(z)=1+b_1z+b_2z^2+...\in P(\rho)$. Then*

$$\text{Re}p(z)\geq 2\rho-1+\tfrac{2(1-\rho)}{1+|z|}.$$

3. MAIN RESULTS

Theorem 3.1. *Let $Re\alpha > 0$. Then*

$$\mathcal{B}_k^\lambda(a,c,p,\alpha,\rho) \subset \mathcal{B}_k^\lambda(a,c,p,0,\rho_1).$$

where ρ_1 is given by

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{3.1}$$

and

$$\gamma = \int_0^1 \left(1 + t^{Re\frac{\alpha}{p}}\right)^{-1} dt.$$

Proof. Let $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$ and set

$$\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{3.2}$$

Then $p(z)$ is analytic in E with $p(0) = 1$. By a simple computation we have

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} + \frac{\alpha}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\right\} = \left\{p(z) + \frac{\alpha}{p}zp'(z)\right\}.$$

Since $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$, so $\left\{p(z) + \frac{\alpha}{p}zp'(z)\right\} \in P_k(\rho)$ for $z \in E$.
This implies that

$$\left\{p_i(z) + \frac{\alpha}{p}zp_i'(z)\right\} > \rho, \quad i = 1,2.$$

Using Lemma 2.1, we see that $\mathrm{Re}\{p_i(z)\} > \rho_1$, where ρ_1 is given by (3.1).
Consequently $p \in P_k(\rho_1)$ for $z \in E$, and the proof is complete. □

Now we take the converse case of Theorem 3.1.

Theorem 3.2. *Let $f \in \mathcal{B}_k^\lambda(a,c,p,0,\rho)$ for $z \in E$. Then $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$ for $|z| < R(\alpha,p)$, where*

$$Re(\alpha,p) = \frac{p}{|\alpha| + \sqrt{|\alpha|^2 + p}}. \tag{3.3}$$

Proof. Set

$$\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} = (p-\rho)h(z) + \rho, \quad h \in P_k.$$

Now proceeding as in Theorem 3.1, we have

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} + \frac{\alpha}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}} - \rho\right\} = (p-\rho)\left\{h(z) + \frac{\alpha}{p}zh'(z)\right\}.$$

$$= (p-\rho)\left[(\frac{k}{4} + \frac{1}{2})\left\{h_1(z) + \frac{\alpha z h_1(z)}{p}\right\}\right] - (\frac{k}{4} - \frac{1}{2})\left\{h_2(z) + \frac{\alpha z h_2(z)}{p}\right\}, \tag{3.4}$$

where we have used (1.3) and $h_1, h_2 \in P$, $z \in E$. Using the following well known estimates, see [5]

$$|zh_i'(z)| \le \frac{2r}{1-r^2}\mathrm{Re}\{h_i(z)\}, \qquad (|z| = r < 1), \quad i = 1,2,$$

we have

$$\begin{aligned}\mathrm{Re}\left\{h_i(z)+\frac{\alpha}{p}zh_i'(z)\right\} &\geq \mathrm{Re}\left\{h_i(z)-\frac{|\alpha|}{p}\left|zh_i'(z)\right|\right\}\\ &\geq \mathrm{Re}h_i(z)\left\{1-\frac{2\left|\alpha\right|r}{p(1-r^2)}\right\}.\end{aligned}$$

The right hand side of this inequality is positive if $r < R(\alpha,p)$, where $R(\alpha,p)$ is given by (3.3). Consequently it follows from (3.4) that $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$ for $|z| < R(\alpha,p)$. Sharpness of this result follows by taking $h_i(z) = \frac{1+z}{1-z}$ in (3.4), $i = 1,2$. □

Theorem 3.3.

$$\mathcal{B}_k^\lambda(a,c,p,\alpha_1,\rho) \subset \mathcal{B}_k^\lambda(a,c,p,\alpha_2,\rho) \textit{ for } 0 \leq \alpha_2 < \alpha_1.$$

Proof. For $\alpha_2 = 0$ the proof is immediate. Let $\alpha_2 > 0$ and let $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha_1,\rho)$. Then there exist two functions $H_1, H_2 \in P_k(\rho)$ such that , from Definition 1.1 and Theorem 3.1

$$\left\{(1-\alpha_1)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\frac{\alpha_1}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\right\} = H_1(z),$$

and

$$\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} = H_2(z).$$

Hence

$$\left\{(1-\alpha_2)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\frac{\alpha_2}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\right\} = \frac{\alpha_2}{\alpha_1}H_1(z)+(1-\frac{\alpha_2}{\alpha_1})H_2(z). \quad (3.5)$$

Since the class $P_k(\rho)$ is a convex set, see [6], it follows that the right hand side of (3.5) belong to $P_k(\rho)$ and this proves the result. □

Theorem 3.4. *Let $f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$, and let $\phi \in C(p)$, where $C(p)$ is the class of p-valent convex functions. Then $\phi * f \in \mathcal{B}_k^\lambda(a,c,p,\alpha,\rho)$.*

Proof. Let $F = \phi * F$.Then we have

$$\begin{aligned}&\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)F(z)}{z^p}+\frac{\alpha}{p}\frac{(\mathcal{I}_p^\lambda(a,c)F(z))'}{z^{p-1}}\right\}\\ =\ & (1-\alpha)(\frac{\phi(z)}{z^p})*\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\frac{\alpha}{p}(\frac{\phi(z)}{z^p})*\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\\ =\ & (\frac{\phi(z)}{z^p})*G(z),\end{aligned}$$

where

$$G(z) = \left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}+\frac{\alpha}{p}\frac{(\mathcal{I}_p^\lambda(a,c)f(z))'}{z^{p-1}}\right\} \in P_k(\rho).$$

Therefore we have

$$(\frac{\phi(z)}{z^p})*G(z) = (p-\rho)\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\phi(z)}{z^p}*g_1(z)\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\phi(z)}{z^p}*g_2(z)\right)\right\}+\rho,$$

with $g_1, g_2 \in P$. Since $\phi \in C(p)$, $\operatorname{Re}\left\{\frac{\phi(z)}{z^p}\right\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi * F \in \mathcal{B}_k^{\lambda}(a, c, p, \alpha, \rho)$. □

Now we study the interesting properties of the class $\mathcal{T}_k^{\lambda}(a, c, p, \alpha, \rho)$.

Theorem 3.5. *Let $f \in \mathcal{T}_k^{\lambda}(a, c, p, \alpha, \rho_2)$ and $g \in \mathcal{T}_k^{\lambda}(a, c, p, \alpha, \rho_3)$, and let $F = f*g$. Then $F \in \mathcal{T}_k^{\lambda}(a, c, p, \alpha, \rho_4)$ where*

$$\rho_4 = 1 - 4(1-\rho_2)(1-\rho_3)\left[1 - \frac{\lambda+p}{\alpha}\int_0^1 \frac{u^{\frac{\lambda+p}{\alpha}-1}}{1+u}du\right]. \tag{3.6}$$

Proof. Since $f \in \mathcal{T}_k^{\lambda}(a, c, p, \alpha, \rho_2)$, it follows that

$$H(z) = \left\{(1-\alpha)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^p} + \alpha\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right\} \in P_k(\rho_2),$$

and so using identity (1.11) in the above equation, we have

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = \frac{\lambda+p}{\alpha}z^{p-\frac{\lambda+p}{\alpha}}\int_0^z t^{\frac{\lambda+p}{\alpha}-1}H(t)dt. \tag{3.7}$$

Similarly

$$\mathcal{I}_p^{\lambda}(a,c)g(z) = \frac{\lambda+p}{\alpha}z^{p-\frac{\lambda+p}{\alpha}}\int_0^z t^{\frac{\lambda+p}{\alpha}-1}H^*(t)dt, \tag{3.8}$$

where $H^* \in P_k(\rho_3)$. Using (3.7) and (3.8), we have

$$\mathcal{I}_p^{\lambda}(a,c)F(z) = \frac{\lambda+p}{\alpha}z^{p-\frac{\lambda+p}{\alpha}}\int_0^z t^{\frac{\lambda+p}{\alpha}-1}Q(t)dt, \tag{3.9}$$

where

$$Q(z) = \left(\frac{k}{4}+\frac{1}{2}\right)q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)q_2(z), \tag{3.10}$$

$$= \frac{\lambda+p}{\alpha}z^{-\frac{\lambda+p}{\alpha}}\int_0^z t^{\frac{\lambda+p}{\alpha}-1}(H*H^*)dt.$$

Now

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \tag{3.11}$$

$$H^*(z = \left(\frac{k}{4}+\frac{1}{2}\right)h_1^*(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2^*(z),$$

where $h_i \in P(\rho_2)$ and $h_i^* \in P(\rho_3), i = 1, 2$. Since

$$P_i^* = \frac{h_i^*(z)-\rho_3}{2(1-\rho_3)} + \frac{1}{2} \in P(\frac{1}{2}), \quad i = 1, 2,$$

we obtain that $(h_i * p_i^*) \in P(\rho_3)$, by using Herglotz formula. Thus

$$(h_i * h_i^*) \in P(\rho_4),$$

with

$$\rho_4 = 1 - 2(1-\rho_2)(1-\rho_3). \tag{3.12}$$

Using (3.9), (3.10), (3.11), (3.12) and Lemma 2.3, we have

$$\begin{aligned}
\mathrm{Re} q_i(z) &= \frac{(\lambda+p)}{\alpha}\int_0^1 u^{\frac{(\lambda+p)}{\alpha}-1}\mathrm{Re}\{(h_i * h_i^*)(uz)\}du \\
&\geq \frac{(\lambda+p)}{\alpha}\int_0^1 u^{\frac{(\lambda+p)}{\alpha}-1}\left(2\rho_4 - 1 + \frac{2(1-\rho_4)}{1+u\,|z|}\right)du \\
&\geq \frac{(\lambda+p)}{\alpha}\int_0^1 u^{\frac{(\lambda+p)}{\alpha}-1}\left(2\rho_4 - 1 + \frac{2(1-\rho_4)}{1+u}\right)du \\
&= 1-4(1-\rho_2)(1-\rho_3)\left[1-\frac{(\lambda+p)}{\alpha}\int_0^1 \frac{u^{\frac{(\lambda+p)}{\alpha}-1}}{1+u}du\right].
\end{aligned}$$

From this we conclude that $F \in \mathcal{T}_k^{\lambda}(a,c,p,\alpha,\rho_4)$ where ρ_4 is given by (3.6).
We discuss the sharpness as follows:
We take

$$\begin{aligned}
H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_2)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_2)z}{1+z}, \\
H^*(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_3)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_3)z}{1+z}.
\end{aligned}$$

Since

$$\left(\frac{1+(1-2\rho_2)z}{1-z}\right) * \left(\frac{1+(1-2\rho_3)z}{1-z}\right) = 1-4(1-\rho_2)(1-\rho_3) + \frac{4(1-\rho_2)(1-\rho_3)}{1-z}.$$

It follows from (3.10) that

$$\begin{aligned}
q_i(z) &= \frac{(\lambda+p)}{\alpha}\int_0^1 u^{\frac{(\lambda+p)}{\alpha}-1}\left\{1-4(1-\rho_2)(1-\rho_3)+\frac{4(1-\rho_2)(1-\rho_3)}{1-z}\right\}du \\
&\longrightarrow 1-4(1-\rho_2)(1-\rho_3)\left[1-\frac{(\lambda+p)}{\alpha}\int_0^1 \frac{u^{\frac{(\lambda+p)}{\alpha}-1}}{1+u}du\right] \quad \text{as } z \longrightarrow -1.
\end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $f \in \mathcal{A}(p)$ and define the one parameter integral operator J_c by*

$$J_c f(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt, \quad (f \in \mathcal{A}(p);\ c > -p). \tag{3.13}$$

If

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^{\lambda}(a,c)J_c f(z)}{z^p} + \alpha\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^p}\right\} \in P_k(\rho), \tag{3.14}$$

then

$$\frac{\mathcal{I}_p^\lambda(a,c)J_cf(z)}{z^p} \in P_k(\beta), \quad z \in E,$$

where

$$\beta = \rho + (1-\rho)(2\gamma_1 - 1), \tag{3.15}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{Re(\frac{\alpha}{c+p})}\right) dt.$$

Proof. First of all it follows from (3.13) that

$$z(\mathcal{I}_p^\lambda(a,c)J_cf(z))' = (c+p)\mathcal{I}_p^\lambda(a,c)f(z) - c\mathcal{I}_p^\lambda(a,c)J_cf(z). \tag{3.16}$$

Let

$$\frac{\mathcal{I}_p^\lambda(a,c)J_cf(z)}{z^p} = h(z) = (\frac{k}{4} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z). \tag{3.17}$$

Then the hypothesis (3.14) in connection with (3.16) would yield

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)J_cf(z)}{z^p} + \alpha\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right\} = \left\{h(z) + \frac{\alpha z h'(z)}{c+p}\right\} \in P_k(\rho) \text{ for } z \in E.$$

Consequently

$$\left\{h(z) + \frac{\alpha z h'(z)}{c+p}\right\} \in P(\rho), \ i = 1, 2, \ 0 \le \rho \le p, \text{ and } z \in E.$$

Using Lemma 2.1 with $\lambda_1 = \frac{a}{(c+p)}$, we have $\mathrm{Re}\{h_i(z)\} > \beta$, where β is given by (3.15), and the proof is complete. □

Theorem 3.7. *Let $f \in \mathcal{T}_k^\lambda(a,c,p,a,\rho)$, and let $\phi \in C(p)$, where $C(p)$ is the class of p-valent convex functions. Then $\phi * f \in \mathcal{T}_k^\lambda(a,c,p,a,\rho)$.*

Proof. Let $F = \phi * f$. Then, we have

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)F(z)}{z^p} + \alpha\frac{\mathcal{I}_p^{\lambda+1}(a,c)F(z)}{z^p}\right\} = \frac{\phi\,(z))}{z^p} * G(z),$$

where

$$G(z) = \left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} + \alpha\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right\} \in P_k(\rho).$$

Therefore, we have

$$\frac{\phi(z)}{z^p} * G(z) = (\frac{k}{4} + \frac{1}{2})\left\{(p-\rho)\left(\frac{\phi\,(z))}{z^p} * g_1(z)\right) + \rho - (\frac{k}{4} - \frac{1}{2})(p-\rho)\left(\frac{\phi\,(z))}{z^p} * g_2(z)\right) + \rho\right\},$$

Since $\phi \in C(p)$, $\mathrm{Re}\left\{\frac{\phi(z)}{z^p}\right\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi * f \in \mathcal{T}_k^\lambda(a,c,p,a,\rho)$. □

Theorem 3.8. *For $0 \le \alpha_2 < \alpha_1$,*

$$\mathcal{T}_k^\lambda(a,c,p,a_1,\rho) \subset \mathcal{T}_k^\lambda(a,c,p,a_2,\rho).$$

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and $f \in \mathcal{T}_k^\lambda(a, c, p, a_1, \rho)$. Then

$$\begin{aligned}
&\left\{(1-\alpha_2)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} + \alpha_2\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right\} \\
= \; & \frac{\alpha_2}{a_1}\left[\left(\frac{\alpha_1}{a_2}-1\right)\left(\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right) + (1-a_1)\left(\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right) + a_1\left(\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right)\right] \\
= \; & \left(1-\frac{\alpha_2}{a_1}\right)H_1(z) + \frac{\alpha_2}{a_1}H_2(z), \quad H_1,\ H_2 \in P_k(\rho).
\end{aligned}$$

Since $P_k(\rho)$ is a convex set, see [6], we conclude that $f \in \mathcal{T}_k^\lambda(a, c, p, a_2, \rho)$, for $z \in E$. □

Theorem 3.9. *Let $f \in \mathcal{T}_k^\lambda(a, c, p, 0, \rho)$. Then $f \in \mathcal{T}_k^\lambda(a, c, p, a, \rho)$ for*

$$|z| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}, \quad \alpha \neq \frac{1}{2}, \quad 0 < \alpha < 1.$$

Proof. Let

$$\begin{aligned}
\Psi_\alpha(z) &= (1-\alpha)\frac{z^p}{(1-z)} + \alpha\frac{z^p}{(1-z)^2} \\
&= z^p + \sum_{n=2}^{\infty}(1+(n-1)\alpha)z^{n+p-1}.
\end{aligned}$$

$\Psi_\alpha \in C(p)$ for

$$|z| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}, \quad \alpha \neq \frac{1}{2}, \quad 0 < \alpha < 1.$$

We can write

$$\left\{(1-\alpha)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p} + \alpha\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^p}\right\} = \frac{\Psi_\alpha(z)}{z^p} * \frac{I_p^\lambda(a,c)f(z)}{z^p}.$$

Applying Theorem 3.8, we see that $f \in \mathcal{T}_k^\lambda(a, c, p, a, \rho)$ for $|z| < r_\alpha$. □

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