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STABILIZING OUTPUT FEEDBACK RECEDING HORIZON CONTROL OF SAMPLED-DATA NONLINEAR SYSTEMS VIA DISCRETE-TIME APPROXIMATIONS

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Abstract. This paper is devoted to the stabilization of sampled-data nonlinear systems by output feedback receding horizon control (RHC). The output feedback consists of a stabilizing state feedback RHC and an observer for estimating the unknown states. Since the exact discrete-time models of sampled-data nonlinear systems are often unavailable, both the RHC and the observer are designed via an approximate discrete-time model of the plant. We established a set of conditions which guarantee that the output feedback RHC designed via an approximate discrete-time model practically stabilizes the exact discrete-time model of the plant.

1. Introduction

The stabilization problem of nonlinear systems has recently received considerable attention due to a large number of technical applications. Among the solutions to this problem, receding horizon control (RHC) strategies, also known as model predictive control (MPC), have become quite popular. In receding horizon control, a state feedback control is obtained by solving a finite horizon optimal control problem at each time instant using the state of the system as the initial state for the optimization and applying the first part of the optimal control [11], [20].

Owing to the use of digital computers in the implementation of the controllers, the investigation of sampled-data control systems has become an important area of control engineering. An overview and analysis of existing approaches for the stabilization of sampled-data systems can be found in ([15], [16], [21] and [22], and the references of these papers therein). Two main categories of sampled-data control design can be distinguished. The first category is to design a continuous-time controller for the continuous-time plant and then to discretize the obtained controller for digital implementation. The second category is to discretize the continuous-time model and design a discrete-time controller on the basis of the discrete-time model. In connection with the RHC method, papers in the first category include among

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others in [11] and [17], while those in the second category include [5], [6], [7], [8], [15] and [16]. For nonlinear systems, we cannot compute the exact discrete-time model in general, therefore, an approximate discrete-time model has to be employed. In recent papers [15] and [16], sufficient conditions are presented to guarantee that the same family of state feedback RHCs that stabilize the approximate discrete-time model also practically stabilize the exact discrete-time model of the plant. The sampled-data RHC requires the availability of all the state of the system at every sampling instant. All of the works mentioned above are based on the assumption that the full state of the system is available for feedback. However, this assumption is often unrealistic in practice. In this situation the state feedback RHC cannot be realized. A possible solution to this problem is to use an observer to generate an estimate of the full state using the knowledge of the measured output and input of the system. In this case an output feedback RHC can be obtained by combining a state feedback RHC and an observer. The state feedback RHC is designed assuming that the full state is measured and then it is implemented using an estimate of the state that comes from the observer.

Several RHC schemes exist to guarantee stability when the full state information is available, see for example [9] and [20] for good recent reviews. However, even if the state feedback RHC and the observer used are both stable, there is no guarantee that the overall closed-loop is stable. Many researchers have addressed the question of output feedback RHC using observers for state recovery (see e.g. [9]). A great majority of works deal either with continuous-time systems with or without taking into account any sampling or with discrete-time systems considering the model given directly in discrete-time.

The aim of the present paper is to establish a set of conditions which guarantee that the output feedback RHC practically stabilizes the exact discrete-time model of the plant. The basic idea of handling this problem is similar to that of Findeisen, *et al.*, [10] but, in contrast to that work, the design of the controller and the observer are based on the approximate discrete-time model of the plant in the present paper. The importance of taking into account this fact is supported by a series of counter-examples (see e.g. [15], [21] and [22]), which show that even if the full state can be measured (i.e. zero observation error), one can design a state feedback controller that stabilizes the approximate discrete-time model, but the exact discrete-time model is destabilized by the same controller. In the presence of estimation errors, it has been shown by [12], where the plant model is given directly in discrete-time, that RHC may have zero robustness. This shows admonishes us that the problem of approximation has to be handled with care. In the present paper, the RHC is based on the solution of Bolza-type optimal control problems with implicit final state constraint. In this case, the Lipschitz continuity of the value function can be proven.

2. Preliminaries and problem statement

The sets of real and natural numbers (including zero) are denoted, respectively, by $\mathbb{R}$ and $\mathbb{N}$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if $\sigma(0) = 0$, $\sigma(s) > 0$ for all $s > 0$ and it is strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and $\sigma(s) \to \infty$ when $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(s, \tau)$ is of class-$\mathcal{K}$ in s for every $\tau \geq 0$, it is strictly decreasing in τ for every $s > 0$ and $\beta(s, \tau) \to 0$ when $\tau \to \infty$. The Euclidean norm of a vector x is denoted as $\|x\|$. Given $\rho > 0$ we define $\mathcal{B}_\rho$ to be a ball of radius ρ centered at the origin. We introduce the following notation $Y_\rho = Y \cap \mathcal{B}_\rho$ for any set Y.

We consider the output feedback stabilization of a nonlinear control system given by

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), u(t)\right), \quad x(0) = x_0 \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $\mathcal{X}$ is the state space, U is the control constraints set and $y(t) \in \mathbb{R}^p$ is the measured outputs, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $h : \mathbb{R}^n \times U \to \mathbb{R}^p$ with $f(0,0) = 0$ and $h(0,0) = 0$, U is closed and $0 \in U$, $0 \in \mathcal{X}$. We shall assume that f is continuous and for any pair of positive numbers (Δ', Δ'') there exists an $L_f = L_f(\Delta', \Delta'')$ such that

$$\|f(x,u) - f(y,u)\| \le L_f \|x - y\|, \tag{2.2}$$

for all $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$.

Let $\Gamma \subset \mathcal{X}$ be a given compact set containing the origin and consisting of all initial states to be taken into account. The control is taken to be a piecewise constant signal

$$u(t) = u(kT) =: u_k, \text{ for } t \in [kT, (k+1)T), \quad k \in \mathbb{N}, \tag{2.3}$$

where $T > 0$ is the sampling period, which is fixed and will be chosen later.

The output y is measured at sampling instants kT that is

$$y(k) := y(kT).$$

Under the conditions on f, for any $\overline{x} \in \mathcal{X}_{\Delta'}$, $\overline{u} \in U_{\Delta''}$ equation (2.1) with $u(t) \equiv \overline{u}$, ($t \in [0,T]$) and initial condition $x(0) = \overline{x}$, has a unique solution on $[0,T]$ denoted by $\phi^E(.;\overline{x},\overline{u})$. Then the exact discrete-time model can be defined as

$$\begin{aligned} x^E_{k+1} &= F^E_T(x^E_k, u_k), \qquad\qquad x^E_0 = x_0, \\ y_k &= h(x^E_k, u_k), \end{aligned} \tag{2.4}$$

where $F^E_T(x,u) := \phi^E(T;x,u)$.

We emphasize that ϕ^E is not known in most cases since computing ϕ^E explicitly will require an analytic solution of a nonlinear initial value problem, therefore the controller design can be carried out by means of an approximate discrete-time model

$$x^A_{k+1} = F^A_{T,\delta}(x^A_k, u_k) \qquad\qquad x^A_0 = x_0 \tag{2.5}$$

where δ is the modeling parameter which will be used to refine the approximate model. It can be interpreted as the integration period in numerical schemes for solving differential equations: $F^A_{T,\delta}$ is derived by the multiple application of some numerical scheme (e.g. a Runge-Kutta formula) with step sizes $\delta^i_0, ..., \delta^i_{m_i}$, where $0 < \delta^i_k \le \delta$ and $\delta^i_0 + ... + \delta^i_{m_i} = T$. In what follows, we shall refer to such a subdivision by δ, for simplicity. To guarantee that (2.5) is a good approximation of (2.4), we shall assume that the approximate model is consistent with the exact model.

Assumption 2.1. *Let $T > 0$ be given. For any $\Delta' > 0$ and $\Delta'' > 0$ there exist $\gamma \in \mathcal{K}$ and $\delta^* > 0$ such that*

$$\left\|F^A_{T,\delta}(x,u) - F^E_T(x,u)\right\| \le T\gamma(\delta), \tag{2.6}$$

for all $(x,u) \in \mathcal{X}_{\Delta'} \times U_{\Delta''}$, and $\delta \in (0, \delta^*]$.

Remark. The consistency property described here is an adaptation of consistency property used in the numerical analysis literature (see e.g. [23]). We have to emphasize that, without explicit knowledge of the exact discrete-time model, the consistency property is checkable. Sufficient checkable conditions for consistency properties and further details can be found in [21] and [22].

For the approximate model (2.5), we design a family of discrete-time observers of the form (see [1])

$$\hat{x}_{k+1} = F^A_{T,\delta}(\hat{x}_k, u_k) + \theta_{T,\delta}(\hat{x}_k, y_k, u_k). \tag{2.7}$$

Since the observer is constructed based on the approximate models, there are two sources of errors, the observation error and the approximation error. Due to the mismatch of the exact and approximate models, the observer error system is now driven by the plant trajectories $x(t)$ and controls $u(t)$, which act as disturbance inputs. However, it is known that a deadbeat observer is always converging to the real value of the estimated states. Therefore, in this case the source of error is only coming from the approximation (see [19]).

The problem is to derive an output feedback controller that stabilizes the exact discrete-time model in an appropriate sense. The output feedback control scheme is given by a state feedback controller

$$\mathbf{v}_\delta : \widetilde{\Gamma} \to U,$$

and an observer for estimating the states, where $\widetilde{\Gamma}$ is a suitable set containing at least Γ.

In this paper the sampled-data RHC is based on the approximate discrete-time model with implicit final state constraint. In the next section we review how the stability for state feedback RHC can be achieved. Moreover, the Lipschitz continuity of the value function can be established.

3. State feedback RHC

In order to find a suitable feedback controller $\mathbf{v}_\delta$, we shall apply the receding horizon control method. To do so, we shall consider the following cost function. Let $1 \le N \in \mathbb{N}$ be given and for any event (x,k) (i.e. for state x at time kT), the cost function defined over the interval $[kT, (k+N)T]$ is given by

$$J_{T,\delta}(N, (x,k), \mathbf{u}) = \sum_{i=k}^{k+N-1} T l_\delta(x^A_i, u_i) + g(x^A_{k+N}), \tag{3.1}$$

where, $\mathbf{u} = \{u_k, u_{k+1}, ..., u_{k+N-1}\}$, $x^A_i = \phi^A(i; (x,k), \mathbf{u})$ is the solution of (2.5) resulting from an initial state x at time kT and a control sequence $\mathbf{u}$, l_δ and g are given functions, satisfying assumptions to be formulated below.

Consider the optimization problem

$$P^A_{T,\delta}(N, (x,k)) : \quad \min \{J_{T,\delta}(N, (x,k), \mathbf{u}) : u_i \in U\}. \tag{3.2}$$

If this optimization problem has a solution denoted by

$$\mathbf{u}^*(x,k) = \left\{u^*_k(x,k), u^*_{k+1}(x,k), ..., u^*_{k+N-1}(x,k)\right\},$$

then the first element of $\mathbf{u}^*(x,k)$ is applied at the state x, i.e.,

$$\mathbf{v}_\delta(x,k) = u^*_k(x,k). \tag{3.3}$$

The value function for the optimal control problem is

$$V_N(x,k) = J_{T,\delta}(N,(x,k),\mathbf{u}^*(x,k)). \tag{3.4}$$

Since $F^A_{T,\delta}$ and l_δ are time invariant, the problem $P^A_{T,\delta}(N,(x,k))$ is time invariant in the sense that $V_N(x,k) = V_N(x,0)$ and $\mathbf{v}_\delta(x,k) = \mathbf{v}_\delta(x,0)$ for all k, so that it suffices, at each event (x,k) to solve $P^A_{T,\delta}(N,x) := P^A_{T,\delta}(N,(x,0))$, i.e. to regard current time as zero [20]. The solution of $P^A_{T,\delta}(N,x)$ will be denoted by $\mathbf{u}^*(x) = \mathbf{u}^*(x,0) = \left\{u^*_0(x), u^*_1(x), ..., u^*_{N-1}(x)\right\}$. In what follows we shall use the following notations

$$V_N(x) = V_N(x,0) \quad \text{and} \quad \mathbf{v}_\delta(x) = u^*_0(x).$$

To ensure the existence and the stabilizing property of the proposed controller, several assumptions are needed. Assumptions, under which $\mathbf{v}_\delta$ asymptotically stabilizes the origin for a fixed discrete-time system of type (2.5) are well-established in the literature (see e.g. [11] and [20]).

Assumption 3.1. *(i) g is continuous, positive definite, and for any $\Delta' > 0$ there exists an $L_g > 0$ such that*

$$|g(x) - g(y)| \le L_g \|x - y\|$$

for all $x, y \in \mathcal{B}_{\Delta'}$.
(ii) l_δ is continuous with respect to x and u, uniformly in small δ, and for any $\Delta' > 0$, $\Delta'' > 0$ there exist $\delta^* > 0$ and $L_l > 0$ such that

$$|l_\delta(x,u) - l_\delta(y,u)| \le L_l \|x - y\|$$

for all $\delta \in (0,\delta^*]$, $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$.
(iii) There exist a $\delta^* > 0$ and two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that the inequality

$$\varphi_1(\|x\|) + \varphi_1(\|u\|) \le l_\delta(x,u) \le \varphi_2(\|x\|) + \varphi_2(\|u\|), \tag{3.5}$$

holds for all $x \in \mathcal{X}$, $u \in U$ and $\delta \in (0,\delta^*]$.

Remark. The lower bound in (3.5) can be substituted by different conditions, e.g. $\varphi_1(\|u\|)$ may be omitted, if U is compact. The $\mathcal{K}_\infty$ lower estimation with respect to x is important in the consideration of the present paper.

Assumption 3.2. *$F^A_{T,\delta}(0,0) = 0$, $F^A_{T,\delta}$ is continuous in both variables uniformly in small δ, and it satisfies a local Lipschitz condition: there is a $\delta^* > 0$ such that for any pair of positive numbers (Δ', Δ'') there exists $L_{F^A} > 0$ such that*

$$\|F^A_{T,\delta}(x,u) - F^A_{T,\delta}(y,u)\| \le e^{L_{F^A}T}\|x - y\|,$$

holds for all $x, y \in \mathcal{B}_{\Delta'}$, $u \in \mathcal{B}_{\Delta''}$ and $\delta \in (0,\delta^*]$.

Remark. We note that, under the conditions on the function f, then Assumption 3.2 can be proven for many one step numerical methods.

Let $\mathcal{U}^\delta$ denote a family of control sequences parameterized by δ: $\mathbf{u}^\delta \in \mathcal{U}^\delta$ if $\mathbf{u}^\delta = \{u^\delta_0, u^\delta_1, ...\}$ and $u^\delta_i \in U$, $i = 0, 1, ...$. Since we want to find a state-feedback controller, it seems to be reasonable to investigate when it does exist. The next assumption formulates, roughly speaking, a necessary condition for the existence of a stabilizing feedback.

Definition 1 System (2.4) is *practically asymptotically controllable* (PAC) from a compact set Ω to the origin with the parameterized family $\mathcal{U}^\delta$, if there exist a $\beta(.,.) \in \mathcal{KL}$ and a continuous, positive and nondecreasing function $\sigma(.)$, both independent of δ, such that for any $r > 0$ there exists a $\delta^* > 0$ so that for all $x \in \Omega$ and for all $\delta \in (0, \delta^*]$ there exists a control sequence $\mathbf{u}^\delta(x) \in \mathcal{U}^\delta$, such that $\left\| u_i^\delta(x) \right\| \leq \sigma(\|x\|)$, and the corresponding solution ϕ^E of (2.4) satisfies the inequality

$$\left\| \phi^E(i; x, \mathbf{u}^\delta(x)) \right\| \leq \max\left\{ \beta(\|x\|, iT), r \right\}, \quad i \in \mathbb{N}. \tag{3.6}$$

Assumption 3.3. *There exists $\delta^* > 0$ such that the exact discrete-time model (2.4) is PAC from $\Omega \supset \Gamma$ to the origin with $\mathcal{U}^\delta$ for all $\delta \in (0, \delta^*]$.*

Remark. Observe that Assumption 3.3 implies that for any $x \in \Omega$ there exists a control function $\mathbf{u}^\delta(x) \in \mathcal{U}^\delta$ for which no finite escape time occurs.

We assume that there exists $\Delta_0 > 0$ such that $\Gamma \subset \Omega_{\Delta_0}$. Let β and σ are functions generated by Definition 1 and let Δ_1 and Δ_2 are such that $\Delta_1 = \beta(\Delta_0, 0)$ and $\Delta_2 = \sigma(\Delta_0)$.

The terminal cost function g and/or a terminal constraint set given explicitly or implicitly play a crucial role in establishing the desired stabilizing property. We shall assume that g has to be a local control Lyapunov function within the sampled data controllers.

Assumption 3.4. *There exist $\delta^* > 0$ and $\eta > 0$ such that for all $x \in \mathcal{G}_\eta = \{x \in \mathcal{X} : g(x) \leq \eta\}$ there exists a $\kappa(x) \in U_{\Delta_2}$ (which may depend on parameter δ) such that inequality*

$$g\left(F^A_{T,\delta}(x, \kappa(x))\right) - g(x) \leq -T l_\delta(x, \kappa(x)) \tag{3.7}$$

holds true for all $\delta \in (0, \delta^*]$.

Remark. In the RHC literature, it has been shown in several works that an appropriate choice of the terminal cost g can enforce stability: in fact, if g is a strict control Lyapunov function within one of its level sets, then the RHC makes the origin to be asymptotically stable with respect to the closed-loop system with a domain of attraction containing the above mentioned level set of g (the terminal constraint set is implicit e.g. in [15], [16], [17]). This domain of attraction can be enlarged up to an arbitrary compact set, which is asymptotically controllable to the origin, by a suitable - finite - choice of the horizon length. To find a suitable g, several approaches have been proposed in the literature: in the case when the system has a stabilizable linearization and a quadratic cost function is applied, one can find g in a quadratic form by solving an algebraic Riccati equation (though the corresponding level set may be unacceptably small). More sophisticated methods are, e.g. the quasi-infinite horizon method of [3], the method of infinite horizon costing of [4]. Sometimes it may be difficult to derive an appropriate terminal cost. Recently, it has been proven by [13] and [18] that the required stability can be enforced merely by a sufficiently large time horizon, having obvious advantages, but at the cost of a possibly substantial enlargement of the computational burden.

Let δ_0^* denote the minimum of the values δ^* generated by Assumptions 2.1 and 3.1-3.4 with $\Delta' = \Delta_1$ and $\Delta'' = \Delta_2$.

Lemma 3.5. [15] *If Assumptions 2.1 and 3.1-3.4 hold true, then there exist a δ_1^* with $0 < \delta_1^* \leq \delta_0^*$, and a constant $V^A_{\max} > 0$ such that $V_N(x) \leq V^A_{\max}$ for all $x \in \Omega_{\Delta_0}$, $\delta \in (0, \delta_1^*]$ and $N \in \mathbb{N}$.*

Let $\Gamma_{\max} = \left\{x \in \mathcal{X} : V_N(x) \leq V^A_{\max}\right\}$ and $\sigma_1(.) = T\varphi_1(.)$.

The stability property for the approximate discrete-time model can be characterized by using the well known criterion of Lyapunov for asymptotic stability which will be given in the following theorem.

Theorem 3.6. [15] *If Assumptions 2.1 and 3.1-3.4 hold true, then there exist constants N^*, M, L_V and c and functions $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that for all $x \in \Gamma_{\max}$, $N > N^*$ and $\delta \in (0, \delta_1^*]$*

$$\sigma_1(\|x\|) \leq V_N(x) \leq \sigma_2(\|x\|), \tag{3.8}$$

$$\|u_i^*(x)\| \leq M,\ i = 0, 1, ..., N-1, \tag{3.9}$$

$$V_N\left(F_T^A(x, u_0^*(x))\right) - V_N(x) \leq -T\varphi_1(\|x\|), \tag{3.10}$$

and for all $x, y \in \Gamma_{\max}$ with $\|x - y\| \leq c$

$$|V_N(x) - V_N(y)| \leq L_V \|x - y\|. \tag{3.11}$$

It can be seen that, $\Omega_{\Delta_0} \subset \Gamma_{\max} \subset \mathcal{B}_{\widetilde{\Delta}}$, where $\widetilde{\Delta} = \sigma_1^{-1}(V^A_{\max})$.

Theorem 3.6 shows that under suitable conditions the state feedback RHC renders the origin to be asymptotically stable for the approximate discrete-time model. These conditions concern directly with the data of the problem and the design parameters (the horizon length N, the stage cost l_δ, the terminal cost g, and the terminal constraint set $\mathcal{G}_\eta$) of the method, but not the results of the design procedure.

In [15] and [16], it has been proven that the same family of state feedback RHCs that stabilize the approximate discrete-time model also practically stabilize the exact discrete-time model for sufficiently small integration and/or sampling parameters. The results are based on the assumption that the full state can be measured. In some applications, not all state are directly measurable. Therefore, an output feedback can be obtained by combining a stabilizing state feedback RHC and an observer for estimating the unknown states. The results given in Lemma 3.5 and Theorem 3.6 help us to show that an output feedback designed via an approximate discrete-time model will practically stabilize the exact discrete-time model.

4. Output feedback RHC

In this section, we propose an output feedback RHC for sampled-data nonlinear systems. The state feedback RHC is implemented directly by substituting the required, unknown, state variables values $x(kT)$ by their estimates $\hat{x}(kT)$. Thus, the problem (3.2) will be solved with respect to $\hat{x}$, i.e. $P^A_{T,\delta}(N, \hat{x})$ and the following "disturbed" feedback is applied:

$$\mathbf{v}_\delta(\hat{x}) = u_0^*(\hat{x}). \tag{4.1}$$

We want to show that the output feedback controller (4.1) will practically stabilize the exact discrete-time system for small approximation and observation errors. For our analysis we note from (2.4) and (2.7) that the observer error $e : \hat{x} - x$ satisfies

$$\begin{aligned} e_{k+1} &= F^A_{T,\delta}(\hat{x}_k, u_k) + \theta_{T,\delta}(\hat{x}_k, y_k, u_k) - F^E_T(x^E_k, u_k) \\ &= E_{T,\delta}(e_k, x^E_k, u_k) + F^A_{T,\delta}(x^E_k, u_k) - F^E_T(x^E_k, u_k) \end{aligned}$$

where

$$E_{T,\delta}(e, x, u) = F^A_{T,\delta}(\hat{x}, u) + \theta_{T,\delta}(\hat{x}, y, u) - F^A_{T,\delta}(x, u)$$

represents the nominal observer error dynamics for the approximate design (see [1]). In the present paper we do not consider the observer design, instead we state conditions on the approximation and observer errors that must be satisfied to achieve practical stability of the closed-loop system, allowing to consider different types of observers such as Newton observer (see [2]).

Assumption 4.1. *For any $E_{\max} > 0$ there exist observer parameters, $1 \leq k_0 \in \mathbb{N}$ and $T^* > 0$, and for each fixed $T \in (0, T^*]$ there exists $\delta_2^* > 0$ such that for all $x \in \Gamma$ and $\delta \in (0, \delta_2^*]$ there exists a control sequence $\overline{\mathbf{u}} = \{\overline{u}_0, ...\overline{u}_{k_0-1}\}$ with $\overline{u}_k \in U_{\Delta_2}$ such that*

(i) for all $k \geq k_0$

$$\left\| E_{T,\delta}(e_{k-1}, x_{k-1}^E, \overline{u}_{k-1}) \right\| \leq E_{\max}, \tag{4.2}$$

(ii) for $i = 0, 1, ..., k_0$

$$\phi_i^E(x, \overline{\mathbf{u}}) \in \Omega_{\Delta_0} \quad \text{and} \quad \hat{x}_{k_0} \in \Omega_{\Delta_0},$$

(iii) for all $x_{j-1}^E, \hat{x}_{j-1} \in \Gamma_{\max}$, $j \geq k_0 + 1$ and all $\mathbf{v}_\delta(\hat{x}_{j-1}) \in U_M$

$$\left\| E_{T,\delta}(e_{j-1}, x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1})) \right\| \leq E_{\max}.$$

Assumption 4.1 means that after an initial phase the observer error at the sampling instants can be achieved the desired maximum nominal observer error $E_{\max}$ and the trajectory of the exact discrete-time system does not leave the set of PAC of the exact system during the initial phase. Therefore we need to utilize observers for which the speed of convergence of the observer error can be made sufficiently fast and the absolute achieved observer error can be made sufficiently small. Assume that we have such kind of observers then a precomputed control sequence $\overline{\mathbf{u}}$ will be used during the time interval $[0, k_0T)$. Here k_0 is a freely chosen, but fixed number of sampling instants after which the observer error has to satisfy (4.2). To guarantee that the trajectory of the exact discrete-time system does not leave the set of PAC of the exact system during the initial phase, we may choose a small sampling period T. At the time instants kT, $k = k_0, k_0 + 1, ...$, the RHC strategy will be applied. Since we solve the optimal control problem with initial $\hat{x}_{k_0}$ i.e. $P_{T,\delta}^A(N, \hat{x}_{k_0})$ at the time instant k_0T, we have to ensure that the initial $\hat{x}_{k_0}$ belongs to the set of PAC of the exact system so that Theorem 3.6 is applicable. Moreover, we have to ensure that the observer error can be made sufficiently small after applying the RHC strategy.

In what follows we shall fix $T \in (0, T^*]$.

Lemma 4.2. *Let $d > 0$, suppose that Assumptions 2.1, 3.1-3.4 and 4.1 are valid, N is chosen so that $N > N^*$, and the following condition is satisfied:*

(C) if for any $j \geq k_0+1$, $x_{j-1}^E \in \Gamma_{\max}$, $\hat{x}_{j-1} \in \Gamma_{\max}$, and there exists a $\varepsilon_1(\delta) \in \mathcal{K}$ such that

$$\left\| x_{j-1}^E - \hat{x}_{j-1} \right\| \leq \varepsilon_1(\delta) + E_{\max}$$

for all $\delta \in (0, \delta_2^*]$,

then there exist $\overline{\delta}^* > 0$ and $\overline{E}_{\max} > 0$ such that for any $\delta \in (0, \overline{\delta}^*]$, and $E_{\max} \leq \overline{E}_{\max}$ inequality

$$\max\left\{ V_N\left(F_T^E\left(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}) \right) \right), V_N(x_{j-1}^E) \right\} \geq d \tag{4.3}$$

implies that

$$V_N\left(F_T^E\left(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1})\right)\right)-V_N(x_{j-1}^E)\le -\frac{T}{2}\varphi_1(\frac{1}{2}\left\|x_{j-1}^E\right\|), \tag{4.4}$$

where $\mathbf{v}_\delta(\hat{x}_{j-1})$ *is the first element of the optimal solution of problem* $P_{T,\delta}^A(N,\hat{x}_{j-1})$.

Proof. From condition (C) and Theorem 3.6, for $\hat{x}_{j-1}\in\Gamma_{\max}$ we have

$$V_N\left(F_{T,\delta}^A\left(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1})\right)\right)-V_N(\hat{x}_{j-1})\le -T\varphi_1(\|\hat{x}_{j-1}\|). \tag{4.5}$$

Using (3.8) we obtain

$$\sigma_1\left(\left\|F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\right)\le V_N\left(F_{T,\delta}^A\left(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1})\right)\right)\le V_N(\hat{x}_{j-1})\le V_{\max}^A,$$

it follows that

$$\left\|F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\le\sigma_1^{-1}\left(V_{\max}^A\right)=\widetilde{\Delta}. \tag{4.6}$$

Since $\hat{x}_{j-1},x_{j-1}^E\in\Gamma_{\max}\subset\mathcal{B}_{\widetilde{\Delta}}$ and $\|\mathbf{v}_\delta(\hat{x}_{j-1})\|\le M$, then by integrating equation (2.2) over $[0,T]$ and using Gronwall's Lemma, it can be easily shown that (see [6])

$$\left\|F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1}))-F_T^E(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\le e^{L_fT}\left\|x_{j-1}^E-\hat{x}_{j-1}\right\|. \tag{4.7}$$

From Assumption 2.1 with $\Delta'=\widetilde{\Delta}+2$ and $\Delta''=M$ and using (4.7) and condition (C) we obtain

$$\begin{aligned}
&\left\|F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1}))-F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\le\\
&\left\|F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1}))-F_T^E(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\\
&+\left\|F_T^E(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))-F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\\
&\le e^{L_fT}\left\|x_{j-1}^E-\hat{x}_{j-1}\right\|+T\gamma(\delta)\\
&\le e^{L_fT}E_{\max}+\varepsilon_2(\delta)
\end{aligned} \tag{4.8}$$

where, $\varepsilon_2(\delta)=e^{L_fT}\varepsilon_1(\delta)+T\gamma(\delta)$. Let $E_{\max}^{(0)}>0$ and $\delta_3^*>0$ be such that $E_{\max}^{(0)}\le e^{-L_fT}$ and $\varepsilon_2(\delta_3^*)\le 1$. Using (4.6) and (4.8), for any $\delta\in(0,\delta_3^*]$ and $E_{\max}\le E_{\max}^{(0)}$ we obtain

$$\begin{aligned}
\left\|F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|&\le\left\|F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|+e^{L_fT}E_{\max}+\varepsilon_2(\delta)\\
&\le\widetilde{\Delta}+2.
\end{aligned}$$

First suppose that $V_N(F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1})))\ge d/2$. Let $E_{\max}^{(1)}>0$ and $\delta_4^*>0$ be such that $E_{\max}^{(1)}\le\frac{3d}{16L_Ve^{L_fT}}$ and $\delta_4^*\le\varepsilon_2^{-1}\left(\frac{d}{16L_V}\right)$. Applying the Lipschitz continuity of V_N (3.11) and using (4.8), we obtain for any $\delta\in(0,\delta_4^*]$ and $E_{\max}\le E_{\max}^{(1)}$

$$\begin{aligned}
V_N(F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1})))&\ge V_N(F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1})))\\
&\quad-L_V\left\|F_T^E(x_{j-1}^E,\mathbf{v}_\delta(\hat{x}_{j-1}))-F_T^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1}))\right\|\\
&\ge\frac{d}{2}-L_V\left[e^{L_fT}E_{\max}+\varepsilon_2(\delta)\right]\ge\frac{d}{4}.
\end{aligned}$$

From (4.5), we obtain

$$V_N(\hat{x}_{j-1})\ge V_N(F_{T,\delta}^A(\hat{x}_{j-1},\mathbf{v}_\delta(\hat{x}_{j-1})))\ge\frac{d}{4},$$

and using (3.8) we get

$$\|\hat{x}_{j-1}\| \geq \sigma_2^{-1}\left(\frac{d}{4}\right) = \mu_1. \tag{4.9}$$

Choose δ_5^* and $E_{\max}^{(2)}$ as $\delta_5^* \leq \varepsilon_1^{-1}\left(\frac{\mu_1}{4}\right)$ and $E_{\max}^{(2)} \leq \frac{3\mu_1}{4}$, then for any $\delta \in (0, \delta_5^*]$ and $E_{\max} \leq E_{\max}^{(2)}$ and using (4.9) we have

$$\begin{aligned}\left\|x_{j-1}^E\right\| &\leq \|\hat{x}_{j-1}\| + \left\|x_{j-1}^E - \hat{x}_{j-1}\right\| \\ &\leq \|\hat{x}_{j-1}\| + E_{\max} + \varepsilon_1(\delta) \\ &\leq \|\hat{x}_{j-1}\| + \mu_1 \leq 2\|\hat{x}_{j-1}\|. \end{aligned} \tag{4.10}$$

Therefore, if $V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) \geq d/2$, then using (3.11), (4.5) as well as condition (C) and (4.8) we obtain

$$\begin{aligned} V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) - V_N(x_{j-1}^E) &= V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) \\ &- V_N(F_{T,\delta}^A(\hat{x}_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1}))) + V_N(F_{T,\delta}^A(\hat{x}_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1}))) \\ &- V_N(\hat{x}_{j-1}) + V_N(\hat{x}_{j-1}) - V_N(x_{j-1}^E) \\ &\leq L_V\left\|F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1})) - F_{T,\delta}^A(\hat{x}_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1}))\right\| \\ &- T\varphi_1(\|\hat{x}_{j-1}\|) + L_V\left\|\hat{x}_{j-1} - x_{j-1}^E\right\| \\ &\leq L_V E_{\max}\left[e^{L_f T} + 1\right] + L_V\left[\varepsilon_1(\delta) + \varepsilon_2(\delta)\right] - T\varphi_1(\|\hat{x}_{j-1}\|). \end{aligned} \tag{4.11}$$

Let $\delta_6^* > 0$ and $E_{\max}^{(3)} > 0$ be such that

$$\varepsilon_1(\delta_6^*) + \varepsilon_2(\delta_6^*) \leq \min\left\{\frac{T\varphi_1(\mu_1)}{8L_V}, \frac{c}{2}\right\},$$

$$E_{\max}^{(3)} \leq \min\left\{\frac{3T\varphi_1(\mu_1)}{8L_V\left[e^{L_f T} + 1\right]}, \frac{3c}{2}\right\},$$

then for any $\delta \in (0, \delta_6^*]$ and $E_{\max} \leq E_{\max}^{(3)}$, inequality (4.11) becomes

$$V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) - V_N(x_{j-1}^E) \leq \frac{T}{2}\varphi_1(\mu_1) - T\varphi_1(\|\hat{x}_{j-1}\|) \tag{4.12}$$

and from (4.9) and (4.10), inequality (4.12) becomes

$$\begin{aligned} V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) - V_N(x_{j-1}^E) &\leq \frac{T}{2}\varphi_1(\|\hat{x}_{j-1}\|) - T\varphi_1(\|\hat{x}_{j-1}\|) \\ &= -\frac{T}{2}\varphi_1(\|\hat{x}_{j-1}\|) \\ &\leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\left\|x_{j-1}^E\right\|\right). \end{aligned} \tag{4.13}$$

Then we take

$$\overline{\delta}^* = \min\{\delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*, \delta_5^*, \delta_6^*\}, \qquad \overline{E}_{\max} = \min\{E_{\max}^{(0)}, E_{\max}^{(1)}, E_{\max}^{(2)}, E_{\max}^{(3)}\}.$$

Suppose now that $V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) \leq d/2$ and $V_N(x_{j-1}^E) \geq d$.

From (3.8) we have

$$V_N(x_{j-1}^E) \geq \sigma_1(\left\|x_{j-1}^E\right\|) = T\varphi_1(\left\|x_{j-1}^E\right\|),$$

then

$$V_N(F_T^E(x_{j-1}^E, \mathbf{v}_\delta(\hat{x}_{j-1}))) - V_N(x_{j-1}^E) \le \frac{1}{2}\left[d - V_N(x_{j-1}^E) - V_N(x_{j-1}^E)\right]$$
$$\le \frac{-T}{2}\varphi_1(\|x_{j-1}^E\|) \le \frac{-T}{2}\varphi_1(\frac{1}{2}\|x_{j-1}^E\|).$$

□

Theorem 4.3. *Suppose that Assumptions 2.1, 3.1-3.4 and 4.1 hold true. Then there exists a function $\beta \in \mathcal{KL}$, and for any $\overline{r} > 0$ there exist $\delta^* > 0$ and $E_{\max}^* > 0$ such that for any fixed $N \ge N^*$, $\delta \in (0, \delta^*]$, $E_{\max} \le E_{\max}^*$ and $x \in \Gamma$, the trajectory of the exact discrete-time system*

$$x_{k+1}^E = F_T^E(x_k^E, \mathbf{v}_\delta(\hat{x}_k)), \qquad k = k_0, k_0+1, ... \tag{4.14}$$

with the output feedback RHC $\mathbf{v}_\delta(\hat{x}_k)$, satisfies that $x_k^E \in \Gamma_{\max}$ and

$$\|x_k^E\| \le \max\left\{\beta\left(\|x_{k_0}^E\|, (k-k_0)T\right), \overline{r}\right\}, \qquad k = k_0, k_0+1, ...$$

Moreover, $\hat{x}_k \in \Gamma_{\max}$, as well, and

$$\|\hat{x}_k\| \le \max\left\{\beta\left(\|\hat{x}_{k_0}\|, (k-k_0)T\right) + r_0, \overline{r}\right\}, \qquad k = k_0, k_0+1, ...,$$

where r_0 depends on $E_{\max}$ and δ.

Proof. To prove the theorem, we have to show that for any $j = k_0+1, k_0+2, ...$, Lemma 4.2 is applicable. As in the proof of Lemma 4.2, we take $\Delta' = \widetilde{\Delta} + 2$ and $\Delta'' = M$ in Assumptions 2.1 and 3.1-3.4. For $j = k_0+1$ we have

$$\begin{aligned}\|x_{k_0}^E - \hat{x}_{k_0}\| &\le \|E_{T,\delta}(e_{k_0-1}, x_{k_0-1}^E, \overline{u}_{k_0-1}\| \\ &+ \|F_{T,\delta}^A(x_{k_0-1}^E, \overline{u}_{k_0-1}) - F_T^E(x_{k_0-1}^E, \overline{u}_{k_0-1})\| \\ &\le E_{\max} + T\gamma(\delta).\end{aligned} \tag{4.15}$$

Let us define $\varepsilon_1(\delta) = T\gamma(\delta)$. Then, from (4.15) it follows that $x_{k_0}^E$ and $\hat{x}_{k_0}$ satisfy condition (C) of Lemma 4.2. Let $\overline{r} > 0$ be arbitrary, let $d = \sigma_1\left(\frac{1}{2}\sigma_2^{-1}\left(\sigma_1\left(\overline{r}\right)\right)\right)$ and let $\mu_2 = \sigma_2^{-1}(d)$. Suppose that for some $j \ge k_0+1$ condition (C) is satisfied. Let $\overline{\delta}^*$ and $\overline{E}_{\max}$ be defined as in Lemma 4.2 and consider $\delta \in (0, \overline{\delta}^*]$, $E_{\max} \le \overline{E}_{\max}$.

If $V_N\left(x_{j-1}^E\right) \ge d$, then from (3.8) we obtain

$$\begin{aligned}\sigma_2\left(\|x_{j-1}^E\|\right) &\ge V_N\left(x_{j-1}^E\right) \ge d, \\ \|x_{j-1}^E\| &\ge \sigma_2^{-1}(d) = \mu_2,\end{aligned} \tag{4.16}$$

and from (4.4) and (4.16), the following inequality

$$\begin{aligned}V_N\left(x_j^E\right) - V_N\left(x_{j-1}^E\right) &\le -\frac{T}{2}\varphi_1\left(\frac{1}{2}\|x_{j-1}^E\|\right) \\ &\le -\frac{T}{2}\varphi_1\left(\frac{\mu_2}{2}\right)\end{aligned} \tag{4.17}$$

holds true. Since $x_{j-1}^E \in \Gamma_{\max}$ we have

$$V_N\left(x_j^E\right) \le V_N\left(x_{j-1}^E\right) \le V_{\max}^A,$$

thus $x_j^E \in \Gamma_{\max}$.

Now we show that $\hat{x}_j \in \Gamma_{\max}$. Since $x^E_{j-1}, \hat{x}_{j-1} \in \Gamma_{\max} \subset \mathcal{B}_{\widehat{\Delta}}$ then from Assumptions 2.1 and 4.1 we obtain

$$\begin{aligned}\left\|x^E_j - \hat{x}_j\right\| &\le \left\|E_{T,\delta}(e_{j-1}, x^E_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1})\right\| \\ &\quad + \left\|F^A_{T,\delta}(x^E_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1})) - F^E_T(x^E_{j-1}, \mathbf{v}_\delta(\hat{x}_{j-1}))\right\| \\ &\le E_{\max} + T\gamma(\delta) = E_{\max} + \varepsilon_1(\delta). \end{aligned} \tag{4.18}$$

Let $\delta' \le \overline{\delta}^*$, $E'_{\max} \le \overline{E}_{\max}$ be such that for any $\delta \in (0, \delta']$ and $E_{\max} \le E'_{\max}$ inequalities

$$\varepsilon_1(\delta) \le \min\left\{\frac{T\varphi_1\left(\mu_2/2\right)}{8L_V}, \frac{c}{4}\right\}, \qquad E_{\max} \le \min\left\{\frac{3T\varphi_1\left(\mu_2/2\right)}{8L_V}, \frac{3c}{4}\right\}$$

are satisfied. Then, using (3.11) and (4.17) we obtain

$$\begin{aligned} V_N\left(\hat{x}_j\right) &= V_N\left(\hat{x}_j\right) - V_N\left(x^E_j\right) + V_N\left(x^E_j\right) \\ &\le L_V\left\|\hat{x}_j - x^E_j\right\| + V_N\left(x^E_{j-1}\right) - \frac{T}{2}\varphi_1\left(\frac{\mu_2}{2}\right) \\ &\le V_N\left(x^E_{j-1}\right) + L_V[\varepsilon_1(\delta) + E_{\max}] - \frac{T}{2}\varphi_1\left(\frac{\mu_2}{2}\right) \\ &\le V_N\left(x^E_{j-1}\right) \le V^A_{\max}, \end{aligned}$$

therefore $\hat{x}_j \in \Gamma_{\max}$. Thus condition (C) is valid for $j+1$ as long as $V_N\left(x^E_{j-1}\right) \ge d$ holds true. From (4.17) it follows that after a finite number of steps $V_N\left(x^E_{j-1}\right) < d$ will occur. Then, by Lemma 4.2, we know that $V_N\left(x^E_j\right) < d$ must also be valid. Define the level set $\mathcal{V}_q = \{x : V_N(x) \le q\}$, and let $d_1 = \sigma_1(\overline{r})$, $r_1 = \sigma_2^{-1}\left(\sigma_1(\overline{r})\right)$. Obviously $\mathcal{V}_d \subset \mathcal{B}_{r_1/2} \subset \mathcal{B}_{r_1} \subset \mathcal{V}_{d_1} \subset \mathcal{B}_{\overline{r}}$ and

$$\left\|x^E_{j-1}\right\| \le \sigma_1^{-1}(d) = \frac{r_1}{2} \text{ and } \left\|x^E_j\right\| \le \frac{r_1}{2}. \tag{4.19}$$

On the other hand, choose $\delta'' \le \delta'$, $E''_{\max} \le E'_{\max}$ such that $\varepsilon_1(\delta'') = \sigma_2^{-1}\left(\sigma_1(\overline{r})\right)/8$, $E''_{\max} = 3\sigma_2^{-1}\left(\sigma_1(\overline{r})\right)/8$ then

$$\left\|\hat{x}_{j-1} - x^E_{j-1}\right\| \le E_{\max} + \varepsilon_1(\delta) \le \frac{1}{2}\sigma_2^{-1}\left(\sigma_1(\overline{r})\right) = \frac{r_1}{2}$$

and from (4.19) we obtain

$$\|\hat{x}_{j-1}\| \le r_1.$$

Since $\left\|x^E_j\right\| \le \frac{r_1}{2}$ and from (4.18) and the choice of $E''_{\max}$ and δ'' it follows that

$$\left\|\hat{x}_j - x^E_j\right\| \le \frac{r_1}{2} \quad \text{and} \quad \|\hat{x}_j\| \le r_1 < \overline{r}.$$

We take $E^*_{\max} = E''_{\max}$ and $\delta^* = \delta''$. Thus the ball $\mathcal{B}_{\overline{r}}$ is positively invariant with respect to the exact and the estimation states. The existence of the function $\beta \in \mathcal{KL}$ can be constructed in the standard way (see [22]). □

Remark. We note that the statement of Theorem 4.3 is similar to the practical asymptotic stability of the closed-loop system (4.14) and the observer state with respect to the initial state $x^E_{k_0}$, $\hat{x}_{k_0}$. This is not true for the original initial state x_0, $\hat{x}_0$ because - due to the initial phase - the ball $\mathcal{B}_{\overline{r}}$ is not invariant over the time interval $[0, k_0T)$.

Theorem 4.3 lays the foundation for the design of an output feedback RHC via an approximate discrete-time model to achieve practical stability of the exact discrete-time model. Achieving practical stability of the exact model requires that both the approximation error and the observer error can be made sufficiently small. By application of some one step numerical approximation formula with possibly variable step sizes (e.g. a Runge-Kutta formula), the approximation error can be made sufficiently small. Since the maximum observer error bound depends on many parameters as given in Theorem 4.3, using fixed RHC parameters requires that the observer has some sort of tuning parameters to sufficiently decrease the observer error. To design such kind of observers via an approximate model, further research is needed and this will be considered in the future work.

5. Conclusion

The stabilization problem of sampled data nonlinear system by output feedback RHC was investigated. Both the state feedback RHC and the observer was designed via an approximate discrete-time model. It was shown that under a set of conditions the output feedback RHC practically stabilizes the exact discrete-time model of the plant.

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