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FIXED POINT THEOREMS FOR $(\psi - \phi)$-CONTRACTIONS IN GENERALIZED NEUTROSOPHIC METRIC SPACES

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Abstract. In this article, we provide a fixed point theory for Generalized Neutrosophic Metric Spaces ($GNMS$) and present generalized $(\psi - \phi)$- contracting principle. Numerous modern fixed point concepts have been generalized as well as extended by what we have found. We support our argument with a specific example.

1. Introduction

In 1965, Zadeh [17] established the idea of fuzzy sets. In 1975, Kramosil and Michalek [8] introduced the idea of a fuzzy metric space, seen as an extension of the statistical metric space. Atanassov [1] researched intuitionistic fuzzy sets and noted their effectiveness in this context. The possibility of neutrosophic set was presented by Smarandache [13] as an augmentation of the intuitionistic fuzzy set. The paper authored by Ali Asghar et. al. explores Neutrosophic 2-Metric Spaces and their applications[2]. George and Veeramani delve into the exploration of neutrosophic metric spaces, contributing to the advancement of knowledge in fuzzy sets and systems.[3]. Different kinds of fuzzy contractive maps have been invented and generalized by numerous researchers, who also study various fixed point proofs in Intuitionistic and $GNMS$ [4,6,9,10]. Researchers additionally discovered distinct common fixed point results in generalized metric spaces with a V-fuzzy metric and a weakly non-archemedean intuitionistic metric[5,16]. These publications explore diverse aspects of neutrosophic metric spaces and related mathematical concepts. The researches by Uddin et al. [15] explores into Neutrosophic Double Controlled Metric Spaces and their applications, contributing to the understanding of this mathematical framework. Muhammad Saeed et al. [11] introduces new fixed point results in Neutrosophic b-Metric Spaces, demonstrating practical applications. Saleem et al.'s [14] work focuses on multivalued neutrosophic fractals and the Hutchinson-Barnsley operator in the context of neutrosophic metric spaces, providing insights into the broader field of mathematical chaos and fractals. Fixed point theories in $GNMS$ relies significantly upon the results of this research.

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2. Preliminaries

Now, we begin with some basic concepts.

Definition 2.1. [12] *Suppose $\mathfrak{A}$ is a nonempty set, and $\mathfrak{G} : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to (-\infty, \infty)$ is representing a function, then it must have the following conditions:*

(i) $\mathfrak{G}(\varpi, \vartheta, \xi) = 0$ *if* $\varpi = \vartheta = \xi$,
(ii) $0 < \mathfrak{G}(\varpi, \varpi, \vartheta)$ *for all* $\varpi, \vartheta \in \mathfrak{A}$ *with* $\varpi \neq \vartheta$,
(iii) $\mathfrak{G}(\varpi, \varpi, \vartheta) \leq \mathfrak{G}(\varpi, \vartheta, \xi)$ *for all* $\varpi, \vartheta, \xi \in \mathfrak{A}$ *with* $\vartheta \neq \xi$,
(iv) $\mathfrak{G}(\varpi, \vartheta, \xi) = \mathfrak{G}(\varpi, \xi, \vartheta) = \mathfrak{G}(\vartheta, \xi, \varpi) = \cdots$, *each of the three variables have symmetry,*
(v) $\mathfrak{G}(\varpi, \vartheta, \xi) \leq \mathfrak{G}(\varpi, \rho, \rho) + \mathfrak{G}(\rho, \vartheta, \xi)$ *for all* $\varpi, \vartheta, \xi, \rho \in \mathfrak{A}$.

The combination $(\mathfrak{A}, \mathfrak{G})$ is referred to as a $\mathfrak{G}$-metric space, whereas $\mathfrak{G}$ is a generalized metric or $\mathfrak{G}$-metric on X.

Definition 2.2. [12] *When each of the $\varpi, \vartheta \in \mathfrak{A}$, the $\mathfrak{G}$-metric space becomes symmetric, then $\mathfrak{G}(\varpi, \varpi, \vartheta) = \mathfrak{G}(\varpi, \vartheta, \vartheta)$.*

Definition 2.3. *A binary operation $\otimes : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (continuous φ-norm) if it satisfies the following conditions:*

(i) $\otimes$ *is commutative and associative,*
(ii) $\otimes$ *is continuous,*
(iii) $\otimes(\mathfrak{a}, 1) = \mathfrak{a}$ *for every* $\mathfrak{a} \in [0,1]$,
(iv) $\otimes(\mathfrak{a}, \mathfrak{b}) \leq \otimes(\mathfrak{c}, \mathfrak{d})$ *whenever* $\mathfrak{a} \leq \mathfrak{c}, \mathfrak{b} \leq \mathfrak{d}$ *and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1]$.

Definition 2.4. *A binary operation $\oplus : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular conorm (continuous φ-conorm) if it satisfies the following conditions:*

(i) $\oplus$ *is commutative and associative,*
(ii) $\oplus$ *is continuous,*
(iii) $\oplus(\mathfrak{a}, 0) = \mathfrak{a}$ *for every* $\mathfrak{a} \in [0,1]$,
(iv) $\oplus(\mathfrak{a}, \mathfrak{b}) \leq \oplus(\mathfrak{c}, \mathfrak{d})$ *whenever* $\mathfrak{a} \leq \mathfrak{c}, \mathfrak{b} \leq \mathfrak{d}$ *and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1]$.

Definition 2.5. [12] *A 6-tuple $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{I}, \otimes, \oplus)$ is referred to as a $GNMS$ if $\mathfrak{A}$ is a nonempty set, a continuous triangular φ -norm $\otimes$, continuous triangular φ -conorm $\oplus$ and neutrosophic sets $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{I}$ are defined from $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to (0, +\infty)$ satisfying the following requirements, for each $\varphi, \tau > 0$:*

(i) $\mathfrak{G}(\varpi, \vartheta, \xi, \varphi) + \mathfrak{H}(\varpi, \vartheta, \xi, \varphi) + \mathfrak{I}(\varpi, \vartheta, \xi, \varphi) \leq 3$ *for all* $\varpi, \vartheta \in \mathfrak{A}$ *with* $\varpi \neq \vartheta$,
(ii) $\mathfrak{G}(\varpi, \varpi, \vartheta, \varphi) \geq \mathfrak{G}(\varpi, \vartheta, \xi, \varphi)$ *for all* $\varpi, \vartheta, \xi \in \mathfrak{A}$ *with* $\vartheta \neq \xi$,
(iii) $\mathfrak{G}(\varpi, \vartheta, \xi, \varphi) = 1$ *if and only if* $\varpi = \vartheta = \xi$,
(iv) $\mathfrak{G}(\varpi, \vartheta, \xi, \varphi) = \mathfrak{G}(p(\varpi, \vartheta, \xi), \varphi)$, *where* p *is a permutation function,*
(v) $\mathfrak{G}(\varpi, \rho, \rho, \varphi) \otimes \mathfrak{G}(\rho, \vartheta, \xi, \tau) \leq \mathfrak{G}(\varpi, \vartheta, \xi, \varphi + \tau)$ *(the triangle inequality),*
(vi) $\mathfrak{G}(\varpi, \vartheta, \xi, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*
(vii) $\mathfrak{H}(\varpi, \varpi, \vartheta, \varphi) \leq \mathfrak{H}(\varpi, \vartheta, \xi, \varphi)$ *for all* $\varpi, \vartheta, \xi \in \mathfrak{A}$ *with* $\vartheta \neq \xi$,
(viii) $\mathfrak{H}(\varpi, \vartheta, \xi, \varphi) = 0$ *if and only if* $\varpi = \vartheta = \xi$,
(ix) $\mathfrak{H}(\varpi, \vartheta, \xi, \varphi) = \mathfrak{H}(p(\varpi, \vartheta, \xi), \varphi)$, *where* p *is a permutation function,*
(x) $\mathfrak{H}(\varpi, \rho, \rho, \varphi) \oplus \mathfrak{H}(\rho, \vartheta, \xi, \tau) \leq \mathfrak{H}(\varpi, \vartheta, \xi, \varphi + \tau)$ *(the triangle inequality),*
(xi) $\mathfrak{H}(\varpi, \vartheta, \xi, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*
(xii) $\mathfrak{I}(\varpi, \varpi, \vartheta, \varphi) \leq \mathfrak{I}(\varpi, \vartheta, \xi, \varphi)$ *for all* $\varpi, \vartheta, \xi \in \mathfrak{A}$ *with* $\vartheta \neq \xi$,
(xiii) $\mathfrak{I}(\varpi, \vartheta, \xi, \varphi) = 0$ *if and only if* $\varpi = \vartheta = \xi$,
(xiv) $\mathfrak{I}(\varpi, \vartheta, \xi, \varphi) = \mathfrak{I}(p(\varpi, \vartheta, \xi), \varphi)$, *where* p *is a permutation function,*
(xv) $\mathfrak{I}(\varpi, \rho, \rho, \varphi) \oplus \mathfrak{I}(\rho, \vartheta, \xi, \tau) \leq \mathfrak{I}(\varpi, \vartheta, \xi, \varphi + \tau)$ *(the triangle inequality),*

(xvi) $\mathfrak{J}(\varpi, \vartheta, \xi, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

Definition 2.6. $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{J}, \otimes, \oplus)$ *a* $GNMS$ *, then*

(1) *a sequence* $\{\varpi_n\}$ *in* $\mathfrak{A}$ *is known to be convergent to* ϖ *if* $\lim_{n\to\infty} \mathfrak{G}(\varpi_n, \varpi_n, \varpi, \varphi) = 1$, $\lim_{n\to\infty} \mathfrak{H}(\varpi_n, \varpi_n, \varpi, \varphi) = 0$ *and* $\lim_{n\to\infty} \mathfrak{J}(\varpi_n, \varpi_n, \varpi, \varphi) = 0$ *for all* $\varphi > 0$.

(2) *a sequence* $\{\varpi_n\}$ *in* $\mathfrak{A}$ *is known to be a Cauchy sequence if* $\lim_{m\to\infty} \mathfrak{G}(\varpi_n, \varpi_n, \varpi_m, \varphi) = 1$, $\lim_{m\to\infty} \mathfrak{H}(\varpi_n, \varpi_n, \varpi_m, \varphi) = 0$ *and* $\lim_{m\to\infty} \mathfrak{J}(\varpi_n, \varpi_n, \varpi_m, \varphi) = 0$ *as* $n, m \to \infty$ *that is, for any* $\epsilon > 0$ *and for every* $\varphi > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $\mathfrak{G}(\varpi_n, \varpi_n, \varpi_m, \varphi) > 1 - \epsilon$, $\mathfrak{H}(\varpi_n, \varpi_n, \varpi_m, \varphi) < \epsilon$ *and* $\mathfrak{J}(\varpi_n, \varpi_n, \varpi_m, \varphi) < \epsilon$ *for* $n, m \geq n_0$.

(3) *Every Cauchy sequence in* $\mathfrak{A}$ *converges, so we say that* $GNMS$ $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{J}, \otimes, \oplus)$ *is complete.*

Lemma 2.7. *[7] Suppose* $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{J}, \otimes, \oplus)$ *is a* $GNMS$ *. At that instance,* $\mathfrak{G}(\varpi, \vartheta, \xi, \varphi)$ *is non-decreasing with reference to* φ *for all* $\varpi, \vartheta, \xi \in \mathfrak{A}$.

Lemma 2.8. *[7]* Ψ *indicate by the collection of non-decreasing continous functions,* $\phi, \psi : [0, \infty) \to [0, \infty)$ *so that* $\phi^n(\varphi) \to 0$ *as* $n \to \infty$ *and* $\psi^n(\varphi) \to 1$ *as* $n \to \infty$ *for every* $\varphi > 0$. *It is obvious that* $\phi(\varphi) > \varphi, \psi(\varphi) > \varphi$ *for all* $\varphi > 0$ *and* $\phi(0) = 0$ *and* $\psi(1) = 1$.

The objective of this work is to introduced generalized $(\psi - \phi)$- contractions and prove fixed point theorems in $GNMS$.

3. Main Results

Definition 3.1. *Let* $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{J}, \otimes, \oplus)$ *be a* $GNMS$. *A mapping* $\Gamma : \mathfrak{A} \to \mathfrak{A}$ *is known to be a generalized* $(\psi - \phi)$- *contractions assuming there is* $\phi, \psi \in \Phi$ *so that, for any* $\varpi, \vartheta, \xi \in \mathfrak{A}$,

$$\begin{aligned} \mathfrak{G}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi) \neq 1 &\Rightarrow \phi(\mathfrak{G}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi)) \geq \psi[\phi(\mathfrak{K}(\varpi, \vartheta, \xi, \varphi))], \\ \mathfrak{H}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi) \neq 0 &\Rightarrow \phi(\mathfrak{H}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi)) \leq \psi[\phi(\mathfrak{L}(\varpi, \vartheta, \xi, \varphi))], \\ \mathfrak{J}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi) \neq 0 &\Rightarrow \phi(\mathfrak{J}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi, \varphi)) \leq \psi[\phi(\mathfrak{M}(\varpi, \vartheta, \xi, \varphi))], \end{aligned} \tag{3.1}$$

where

$$\mathfrak{K}(\varpi, \vartheta, \xi, \varphi) = \min \left\{ \begin{array}{c} \mathfrak{G}(\varpi, \vartheta, \xi, \varphi), \mathfrak{G}(\varpi, \Gamma\varpi, \Gamma\varpi, \varphi), \mathfrak{G}(\vartheta, \Gamma\vartheta, \Gamma\vartheta, \varphi), \\ \mathfrak{G}(\xi, \Gamma\xi, \Gamma\xi, \varphi), \frac{1}{2}[\mathfrak{G}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{G}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi)], \\ \frac{1}{3}[\mathfrak{G}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{G}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi) + \mathfrak{G}(\xi, \Gamma\varpi, \Gamma\varpi, \varphi)] \end{array} \right\}$$

$$\mathfrak{L}(\varpi, \vartheta, \xi, \varphi) = \max \left\{ \begin{array}{c} \mathfrak{H}(\varpi, \vartheta, \xi, \varphi), \mathfrak{H}(\varpi, \Gamma\varpi, \Gamma\varpi, \varphi), \mathfrak{H}(\vartheta, \Gamma\vartheta, \Gamma\vartheta, \varphi), \\ \mathfrak{H}(\xi, \Gamma\xi, \Gamma\xi, \varphi), \frac{1}{2}[\mathfrak{H}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{H}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi)], \\ \frac{1}{3}[\mathfrak{H}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{H}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi) + \mathfrak{H}(\xi, \Gamma\varpi, \Gamma\varpi, \varphi)] \end{array} \right\}$$

$$\mathfrak{M}(\varpi, \vartheta, \xi, \varphi) = \max \left\{ \begin{array}{c} \mathfrak{J}(\varpi, \vartheta, \xi, \varphi), \mathfrak{J}(\varpi, \Gamma\varpi, \Gamma\varpi, \varphi), \mathfrak{J}(\vartheta, \Gamma\vartheta, \Gamma\vartheta, \varphi), \\ \mathfrak{J}(\xi, \Gamma\xi, \Gamma\xi, \varphi), \frac{1}{2}[\mathfrak{J}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{J}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi)], \\ \frac{1}{3}[\mathfrak{J}(\varpi, \Gamma\vartheta, \Gamma\vartheta, \varphi) + \mathfrak{J}(\vartheta, \Gamma\xi, \Gamma\xi, \varphi) + \mathfrak{J}(\xi, \Gamma\varpi, \Gamma\varpi, \varphi)] \end{array} \right\} \tag{3.2}$$

Theorem 3.2. *Let us consider* $(\mathfrak{A}, \mathfrak{G}, \mathfrak{H}, \mathfrak{J}, \otimes, \oplus)$ *as a complete* $GNMS$ *and* $\Gamma : \mathfrak{A} \to \mathfrak{A}$ *as a generalized* $(\psi - \phi)$- *contraction. Then* Γ *has unique fixed point* $\varpi^* \in \mathfrak{A}$.

Proof. Choose $\varpi_0 \in \mathfrak{A}$ be any point in $\mathfrak{A}$. There is sequence $\{\varpi_n\}$ in $\mathfrak{A}$ in such way that $\Gamma\varpi_n = \varpi_{n+1}$ for all $n \in \mathbb{N}$. In case that, $\varpi_{n+1} = \varpi_n$ for certain $n \in \mathbb{N}$, therefore, $\varpi^* = \varpi_n$ is a fixed point for Γ. The following assumption, $\varpi_{n+1} \neq \varpi_n$ for every $n \in \mathbb{N}$. Obviously, $\mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) > 0$ for all $n \in \mathbb{N}$. Execution inequality (3.1) along $\varpi = \varpi_n, \vartheta = \varpi_{n+1}, \xi = \varpi_{n+1}$, we obtain

$$\phi(\mathfrak{G}(\Gamma\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)) \geq \psi[\phi(\mathfrak{K}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi))]$$
$$\phi(\mathfrak{H}(\Gamma\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)) \leq \psi[\phi(\mathfrak{L}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi))]$$
$$\phi(\mathfrak{J}(\Gamma\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)) \leq \psi[\phi(\mathfrak{M}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi))]$$

where,

$$\begin{aligned}
&\mathfrak{K}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) \\
&= \min\left\{\begin{array}{c} \mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{G}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{G}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \mathfrak{G}(\varpi_{n+1}, \\ \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \frac{1}{2}[\mathfrak{G}(\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)], \\ \frac{1}{3}[\mathfrak{G}(\varpi_n, \Gamma\varpi_{n+1}\Gamma\varpi_{n+1}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array}\right\} \\
&= \min\left\{\begin{array}{c} \mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \\ \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \frac{1}{2}[\mathfrak{G}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)] \\ \frac{1}{3}[\mathfrak{G}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{G}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+1}, \varphi)] \end{array}\right\} \\
&= \min\left\{\mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)\right\}.
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{L}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) \\
&= \max\left\{\begin{array}{c} \mathfrak{H}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{H}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{H}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \mathfrak{H}(\varpi_{n+1}, \\ \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \frac{1}{2}[\mathfrak{H}(\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)], \\ \frac{1}{3}[\mathfrak{H}(\varpi_n, \Gamma\varpi_{n+1}\Gamma\varpi_{n+1}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \mathfrak{H}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{H}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{H}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \\ \mathfrak{H}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \frac{1}{2}[\mathfrak{H}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)] \\ \frac{1}{3}[\mathfrak{H}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{H}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+1}, \varphi)] \end{array}\right\} \\
&= \max\left\{\mathfrak{H}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{H}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)\right\}.
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{M}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) \\
&= \max\left\{\begin{array}{c} \mathfrak{J}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{J}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{J}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \mathfrak{J}(\varpi_{n+1}, \\ \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi), \frac{1}{2}[\mathfrak{J}(\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)], \\ \frac{1}{3}[\mathfrak{J}(\varpi_n, \Gamma\varpi_{n+1}\Gamma\varpi_{n+1}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \mathfrak{J}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{J}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{J}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \\ \mathfrak{J}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi), \frac{1}{2}[\mathfrak{J}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)] \\ \frac{1}{3}[\mathfrak{J}(\varpi_n, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi) + \mathfrak{J}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+1}, \varphi)] \end{array}\right\} \\
&= \max\left\{\mathfrak{J}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi), \mathfrak{J}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)\right\}.
\end{aligned}$$

If $\mathfrak{K}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) = \mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)$, then it follows from (3.1) that

$$\begin{aligned}
\phi(\mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi)) &= \phi(\mathfrak{G}(\Gamma\varpi_n, \Gamma\varpi_{n+1}, \Gamma\varpi_{n+1}, \varphi)) \\
&\geq \psi[\phi(\mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi))] \\
&= \psi[\phi(\mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi))] \\
&> \phi(\mathfrak{G}(\varpi_{n+1}, \varpi_{n+2}, \varpi_{n+2}, \varphi),
\end{aligned}$$

utilizing Lemma (2.8), that itself is a contradiction. Thereupon, for all $n \in \mathbb{N}$,

$$\mathfrak{K}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi) = \mathfrak{G}(\varpi_n, \varpi_{n+1}, \varpi_{n+1}, \varphi).$$

If $\mathfrak{L}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)=\mathfrak{H}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi)$, then it follows from (3.1) that

$$\begin{aligned}\phi(\mathfrak{H}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi))&=\phi(\mathfrak{H}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))\\&\leq\psi[\phi(\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi))]\\&=\psi[\phi(\mathfrak{H}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi))]\\&<\phi(\mathfrak{H}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi),\end{aligned}$$

by Lemma (2.8), which is a contradiction. Hence for all $n\in\mathbb{N}$,

$$\mathfrak{L}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)=\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi).$$

If $\mathfrak{M}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)=\mathfrak{J}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi)$, this means from (3.1) that

$$\begin{aligned}\phi(\mathfrak{J}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi))&=\phi(\mathfrak{J}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))\\&\leq\psi[\phi(\mathfrak{J}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi))]\\&=\psi[\phi(\mathfrak{J}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi))]\\&<\phi(\mathfrak{J}(\varpi_{n+1},\varpi_{n+2},\varpi_{n+2},\varphi),\end{aligned}$$

according to Lemma (2.8), this itself is a contradiction. Thus, for all $n\in\mathbb{N}$,

$$\mathfrak{M}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)=\mathfrak{J}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi). \tag{3.3}$$

Thus, (3.1) becomes

$$\begin{aligned}\phi(\mathfrak{G}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))&\geq\psi[\phi(\mathfrak{G}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))]\\\phi(\mathfrak{H}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))&\leq\psi[\phi(\mathfrak{H}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))]\\\phi(\mathfrak{J}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))&\leq\psi[\phi(\mathfrak{J}(\Gamma\varpi_n,\Gamma\varpi_{n+1},\Gamma\varpi_{n+1},\varphi))]\end{aligned}$$

Repeating this process, we get

$$\begin{aligned}\phi(\mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi))&=\phi(\mathfrak{G}(\Gamma\varpi_{n-1},\Gamma\varpi_n,\Gamma\varpi_n,\varphi))\\&\geq\psi[\phi(\mathfrak{G}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))]\\&\geq\psi^2[\phi(\mathfrak{G}(\varpi_{n-2},\varpi_{n-1},\varpi_{n-1},\varphi))]\\&\geq\psi^3[\phi(\mathfrak{G}(\varpi_{n-3},\varpi_{n-2},\varpi_{n-2},\varphi))]\\&\geq\cdots\geq\psi^n[\phi(\mathfrak{G}(\varpi_0,\varpi_1,\varpi_1,\varphi))].\\\phi(\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi))&=\phi(\mathfrak{H}(\Gamma\varpi_{n-1},\Gamma\varpi_n,\Gamma\varpi_n,\varphi))\\&\leq\psi[\phi(\mathfrak{H}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))]\\&\leq\psi^2[\phi(\mathfrak{H}(\varpi_{n-2},\varpi_{n-1},\varpi_{n-1},\varphi))]\\&\leq\psi^3[\phi(\mathfrak{H}(\varpi_{n-3},\varpi_{n-2},\varpi_{n-2},\varphi))]\\&\leq\cdots\leq\psi^n[\phi(\mathfrak{H}(\varpi_0,\varpi_1,\varpi_1,\varphi))]\\\phi(\mathfrak{J}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi))&=\phi(\mathfrak{J}(\Gamma\varpi_{n-1},\Gamma\varpi_n,\Gamma\varpi_n,\varphi))\\&\leq\psi[\phi(\mathfrak{J}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))]\\&\leq\psi^2[\phi(\mathfrak{J}(\varpi_{n-2},\varpi_{n-1},\varpi_{n-1},\varphi))]\\&\leq\psi^3[\phi(\mathfrak{J}(\varpi_{n-3},\varpi_{n-2},\varpi_{n-2},\varphi))]\\&\leq\cdots\leq\psi^n[\phi(\mathfrak{J}(\varpi_0,\varpi_1,\varpi_1,\varphi))].\end{aligned}$$

We have

$$\left.\begin{array}{ll}\phi(\mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)) & \geq \psi[\phi(\mathfrak{G}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))] \geq \cdots \geq \psi^n[\phi(\mathfrak{G}(\varpi_0,\varpi_1,\varpi_1,\varphi))] \\ \phi(\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)) & \leq \psi[\phi(\mathfrak{H}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))] \leq \cdots \leq \psi^n[\phi(\mathfrak{H}(\varpi_0,\varpi_1,\varpi_1,\varphi))] \\ \phi(\mathfrak{I}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)) & \leq \psi[\phi(\mathfrak{I}(\varpi_{n-1},\varpi_n,\varpi_n,\varphi))] \leq \cdots \leq \psi^n[\phi(\mathfrak{I}(\varpi_0,\varpi_1,\varpi_1,\varphi))]\end{array}\right\} \tag{3.4}$$

Based to the definition ψ and ϕ, there is,

$$\left.\begin{array}{ll}\lim_{n\to\infty}\psi^n\left[\phi\left(\mathfrak{G}\left(\varpi_0,\varpi_1,\varpi_1,\varphi\right)\right)\right] & = 1, \lim_{n\to\infty}\phi\left(\mathfrak{G}\left(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi\right)\right)] = 0 \\ \lim_{n\to\infty}\psi^n\left[\phi\left(\mathfrak{H}\left(\varpi_0,\varpi_1,\varpi_1,\varphi\right)\right)\right] & = 0, \lim_{n\to\infty}\phi\left(\mathfrak{H}\left(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi\right)\right)] = 1 \\ \lim_{n\to\infty}\psi^n\left[\phi\left(\mathfrak{I}\left(\varpi_0,\varpi_1,\varpi_1,\varphi\right)\right)\right] & = 0, \lim_{n\to\infty}\phi\left(\mathfrak{I}\left(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi\right)\right)] = 1\end{array}\right\}$$

To demonstrate that $\{\varpi_n\}$ is $\mathfrak{G}$-Cauchy sequence in $\mathfrak{A}$, currently, let's assume for $m > n$, we have

$$\begin{aligned}&\mathfrak{K}(\varpi_n,\varpi_n,\varpi_m,\varphi)\\ &= \min\left\{\begin{array}{c}\mathfrak{G}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{G}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi), \mathfrak{G}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi),\\ \mathfrak{G}(\varpi_m,\Gamma\varpi_m,\Gamma\varpi_m,\varphi), \frac{1}{2}[\mathfrak{G}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{G}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)],\\ \frac{1}{3}[\mathfrak{G}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{G}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)+\mathfrak{G}(\varpi_m,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)]\end{array}\right\}\\ &= \min\left\{\begin{array}{c}\mathfrak{G}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi), \mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi),\\ \mathfrak{G}(\varpi_m,\varpi_{m+1},\varpi_{m+1},\varphi), \frac{1}{2}[\mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{G}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)],\\ \frac{1}{3}[(\mathfrak{G}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{G}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)+\mathfrak{G}(\varpi_m,\varpi_{n+1},\varpi_{n+1},\varphi)]\end{array}\right\}\\ &\geq \min\{\mathfrak{G}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{G}(\varpi_{m+1},\varpi_{m+1},\varpi_n,\varphi)\}\\ &> 1-\epsilon,\end{aligned}$$

for $n, m \geq n_0$.

$$\begin{aligned}&\mathfrak{L}(\varpi_n,\varpi_n,\varpi_m,\varphi)\\ &= \max\left\{\begin{array}{c}\mathfrak{H}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{H}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi), \mathfrak{H}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi),\\ \mathfrak{H}(\varpi_m,\Gamma\varpi_m,\Gamma\varpi_m,\varphi), \frac{1}{2}[\mathfrak{H}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{H}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)],\\ \frac{1}{3}[\mathfrak{H}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{H}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)+\mathfrak{H}(\varpi_m,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)]\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\mathfrak{H}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi), \mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi),\\ \mathfrak{H}(\varpi_m,\varpi_{m+1},\varpi_{m+1},\varphi), \frac{1}{2}[\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{H}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)],\\ \frac{1}{3}[(\mathfrak{H}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{H}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)+\mathfrak{H}(\varpi_m,\varpi_{n+1},\varpi_{n+1},\varphi)]\end{array}\right\}\\ &\leq \max\{\mathfrak{H}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{H}(\varpi_{m+1},\varpi_{m+1},\varpi_n,\varphi)\}\\ &< \epsilon,\end{aligned}$$

for $n, m \leq n_0$.

$$\begin{aligned}&\mathfrak{M}(\varpi_n,\varpi_n,\varpi_m,\varphi)\\ &= \max\left\{\begin{array}{c}\mathfrak{I}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{I}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi), \mathfrak{I}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi),\\ \mathfrak{I}(\varpi_m,\Gamma\varpi_m,\Gamma\varpi_m,\varphi), \frac{1}{2}[\mathfrak{I}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{I}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)],\\ \frac{1}{3}[\mathfrak{I}(\varpi_n,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)+\mathfrak{I}(\varpi_n,\Gamma\varpi_m,\Gamma\varpi_m,\varphi)+\mathfrak{I}(\varpi_m,\Gamma\varpi_n,\Gamma\varpi_n,\varphi)]\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\mathfrak{I}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{I}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi), \mathfrak{I}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi),\\ \mathfrak{I}(\varpi_m,\varpi_{m+1},\varpi_{m+1},\varphi), \frac{1}{2}[\mathfrak{I}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{I}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)],\\ \frac{1}{3}[(\mathfrak{I}(\varpi_n,\varpi_{n+1},\varpi_{n+1},\varphi)+\mathfrak{I}(\varpi_n,\varpi_{m+1},\varpi_{m+1},\varphi)+\mathfrak{I}(\varpi_m,\varpi_{n+1},\varpi_{n+1},\varphi)]\end{array}\right\}\\ &\leq \max\{\mathfrak{I}(\varpi_n,\varpi_n,\varpi_m,\varphi), \mathfrak{I}(\varpi_{m+1},\varpi_{m+1},\varpi_n,\varphi)\}\\ &< \epsilon,\end{aligned}$$

for $n, m \leq n_0$.

Now,

$$\begin{aligned}
\mathfrak{G}(\varpi_n, \varpi_n, \varpi_m, \varphi) &\geq \psi[\mathfrak{G}(\varpi_n, \varpi_n, \varpi_{n+1}, \varphi)] + \psi^2[\phi(\mathfrak{G}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+2}, \varphi))] + \cdots \\
&+ \psi^n[\phi(\mathfrak{G}(\varpi_{m-1}, \varpi_{m-1}, \varpi_m, \varphi))] \\
&\geq \psi^n[\phi(\mathfrak{G}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \psi^{n+1}[\phi(\mathfrak{G}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \cdots \\
&+ \psi^{m-1}[\phi(\mathfrak{G}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] \\
&\to 1 \,\mathrm{as}\, n, m \to \infty. \\
\mathfrak{H}(\varpi_n, \varpi_n, \varpi_m, \varphi) &\leq \psi[\mathfrak{H}(\varpi_n, \varpi_n, \varpi_{n+1}, \varphi)] + \psi^2[\phi(\mathfrak{H}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+2}, \varphi))] + \cdots \\
&+ \psi^n[\phi(\mathfrak{H}(\varpi_{m-1}, \varpi_{m-1}, \varpi_m, \varphi))] \\
&\leq \psi^n[\phi(\mathfrak{H}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \psi^{n+1}[\phi(\mathfrak{H}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \cdots \\
&+ \psi^{m-1}[\phi(\mathfrak{H}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] \\
&\to 0 \,\mathrm{as}\, n, m \to \infty. \\
\mathfrak{J}(\varpi_n, \varpi_n, \varpi_m, \varphi) &\leq \psi[\mathfrak{J}(\varpi_n, \varpi_n, \varpi_{n+1}, \varphi)] + \psi^2[\phi(\mathfrak{J}(\varpi_{n+1}, \varpi_{n+1}, \varpi_{n+2}, \varphi))] + \cdots \\
&+ \psi^n[\phi(\mathfrak{J}(\varpi_{m-1}, \varpi_{m-1}, \varpi_m, \varphi))] \\
&\leq \psi^n[\phi(\mathfrak{J}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \psi^{n+1}[\phi(\mathfrak{J}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] + \cdots \\
&+ \psi^{m-1}[\phi(\mathfrak{J}(\vartheta_0, \vartheta_1, \vartheta_1, \varphi)] \\
&\to 0 \,\mathrm{as}\, n, m \to \infty.
\end{aligned}$$

Using the condition (3.1)

$$\begin{aligned}
\phi(\mathfrak{G}(\varpi_{n+1}, \varpi_{m+1}, \varpi_{m+1}, \varphi)) &= \phi(\mathfrak{G}(\Gamma\varpi_n, \Gamma\varpi_m, \Gamma\varpi_m, \varphi)) \geq \psi[\phi(\mathfrak{K}(\varpi_n, \varpi_m, \varpi_m, \varphi))] \\
\phi(\mathfrak{H}(\varpi_{n+1}, \varpi_{m+1}, \varpi_{m+1}, \varphi)) &= \phi(\mathfrak{H}(\Gamma\varpi_n, \Gamma\varpi_m, \Gamma\varpi_m, \varphi)) \leq \psi[\phi(\mathfrak{L}(\varpi_n, \varpi_m, \varpi_m, \varphi))] \\
\phi(\mathfrak{J}(\varpi_{n+1}, \varpi_{m+1}, \varpi_{m+1}, \varphi)) &= \phi(\mathfrak{J}(\Gamma\varpi_n, \Gamma\varpi_m, \Gamma\varpi_m, \varphi)) \leq \psi[\phi(\mathfrak{M}(\varpi_n, \varpi_m, \varpi_m, \varphi))]
\end{aligned}$$

Over the limit $n, m \to \infty$, next there is,$\phi(1 - \epsilon) \geq \psi[\phi(1 - \epsilon)]$, $\phi(\epsilon) \leq \psi[\phi(\epsilon)]$ applying Lemma (2.8), $\psi[\phi(1 - \epsilon)] > \phi(1 - \epsilon)$, $\psi[\phi(\epsilon)] < \phi(\epsilon)$ then $\phi(1 - \epsilon) \geq \psi[\phi(1 - \epsilon)] > \phi(1 - \epsilon)$, $\phi(\epsilon) \leq \psi[\phi(\epsilon)] < \phi(\epsilon)$ which are the contradictions.
Hence $\{\varpi_n\}$ is $\mathfrak{G}$-Cauchy. Since $\Gamma(\mathfrak{A})$ is $\mathfrak{G}$-complete. After it,there is $\varpi^* \in \mathfrak{A}$ so that $\{\varpi_n\}$ convergence to ϖ^*. In particular,

$$\lim_{n\to\infty} \mathfrak{G}(\varpi_n, \varpi^*, \varpi^*) = 1, \lim_{n\to\infty} \mathfrak{H}(\varpi_n, \varpi^*, \varpi^*) = 0, \lim_{n\to\infty} \mathfrak{J}(\varpi_n, \varpi^*, \varpi^*) = 0. \quad (3.5)$$

We take use of the fact that $\mathfrak{G}$ is continuous on each variable,

$$\begin{aligned}
\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*) &= \lim_{n\to\infty} \mathfrak{G}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*) \\
\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*) &= \lim_{n\to\infty} \mathfrak{H}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*) \\
\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*) &= \lim_{n\to\infty} \mathfrak{J}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*). \quad (3.6)
\end{aligned}$$

We assert that ϖ^* is a fixed point of Γ. Let's say, on the other hand, if $\varpi^* \neq \Gamma\varpi^*$, then by (3.5) and (3.6)

$$\mathfrak{K}(\varpi_n, \varpi^*, \varpi^*, \varphi) = \min \left\{ \begin{array}{c} \mathfrak{G}(\varpi_n, \varpi^*, \varpi^*, \varphi), \mathfrak{G}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \\ \mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \frac{1}{2}[\mathfrak{G}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)], \\ \frac{1}{3}[\mathfrak{G}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{G}(\varpi^*, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array} \right\}$$
$$\to \mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi),$$

$$\mathfrak{L}(\varpi_n, \varpi^*, \varpi^*, \varphi) = \max \left\{ \begin{array}{c} \mathfrak{H}(\varpi_n, \varpi^*, \varpi^*, \varphi), \mathfrak{H}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \\ \mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \frac{1}{2}[\mathfrak{H}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)], \\ \frac{1}{3}[\mathfrak{H}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{H}(\varpi^*, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array} \right\}$$
$$\to \mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi),$$

$$\mathfrak{M}(\varpi_n, \varpi^*, \varpi^*, \varphi) = \max \left\{ \begin{array}{c} \mathfrak{J}(\varpi_n, \varpi^*, \varpi^*, \varphi), \mathfrak{J}(\varpi_n, \Gamma\varpi_n, \Gamma\varpi_n, \varphi), \mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \\ \mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi), \frac{1}{2}[\mathfrak{J}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)], \\ \frac{1}{3}[\mathfrak{J}(\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi) + \mathfrak{J}(\varpi^*, \Gamma\varpi_n, \Gamma\varpi_n, \varphi)] \end{array} \right\}$$
$$\to \mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi),$$

as $n \to \infty$, using the condition (3.1),

$$\phi(\mathfrak{G}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) = \phi(\mathfrak{G}(\Gamma\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \geq \psi[\phi(\mathfrak{K}(\varpi_n, \varpi^*, \varpi^*, \varphi))]$$
$$\phi(\mathfrak{H}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) = \phi(\mathfrak{H}(\Gamma\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{L}(\varpi_n, \varpi^*, \varpi^*, \varphi))]$$
$$\phi(\mathfrak{J}(\varpi_{n+1}, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) = \phi(\mathfrak{J}(\Gamma\varpi_n, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{M}(\varpi_n, \varpi^*, \varpi^*, \varphi))].$$

Over to limit as $n \to \infty$, at that instant, we have

$$\phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \geq \psi[\phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))]$$
$$\phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))]$$
$$\phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))]$$

By Lemma (2.8),

$$\psi[\phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] > \phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)),$$
$$\psi[\phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] < \phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \text{ and}$$
$$\psi[\phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] < \phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)).$$

Then

$$\phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \geq \psi[\phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] > \phi(\mathfrak{G}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)),$$
$$\phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] < \phi(\mathfrak{H}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)),$$
$$\phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)) \leq \psi[\phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi))] < \phi(\mathfrak{J}(\varpi^*, \Gamma\varpi^*, \Gamma\varpi^*, \varphi)),$$

which are the contradictions. This leads us to the conclusion with $\Gamma\varpi^* = \varpi^*$. Let's show that there's no more than a single fixed point in Γ. Instead, let us assume that, additional unique fixed point ϑ^* of Γ so that $\Gamma\varpi^* = \varpi^* \neq \Gamma\vartheta^* = \vartheta^*$. Then
$\mathfrak{G}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi) = \mathfrak{G}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) > 0$ and $\mathfrak{K}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) = \mathfrak{G}(\varpi^*, \vartheta^*, \vartheta^*, \varphi)$,
$\mathfrak{G}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi) = \mathfrak{H}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) < 1$ and $\mathfrak{L}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) = \mathfrak{H}(\varpi^*, \vartheta^*, \vartheta^*, \varphi)$,
$\mathfrak{H}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi) = \mathfrak{J}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) < 1$ and $\mathfrak{M}(\varpi^*, \vartheta^*, \vartheta^*, \varphi) = \mathfrak{J}(\varpi^*, \vartheta^*, \vartheta^*, \varphi)$,
and then by (3.1)

$$\phi(\mathfrak{G}(\Gamma\varpi, \Gamma\vartheta, \Gamma\vartheta)) = \psi(\mathfrak{G}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi)) \geq \psi[\phi(\mathfrak{K}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))] = \psi[\phi(\mathfrak{K}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))],$$
$$\phi(\mathfrak{H}(\Gamma\varpi, \Gamma\vartheta, \Gamma\vartheta)) = \psi(\mathfrak{H}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi)) \leq \psi[\phi(\mathfrak{L}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))] = \psi[\phi(\mathfrak{L}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))],$$
$$\phi(\mathfrak{J}(\Gamma\varpi, \Gamma\vartheta, \Gamma\vartheta)) = \psi(\mathfrak{J}(\Gamma\varpi^*, \Gamma\vartheta^*, \Gamma\vartheta^*, \varphi)) \leq \psi[\phi(\mathfrak{M}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))] = \psi[\phi(\mathfrak{M}(\varpi^*, \vartheta^*, \vartheta^*, \varphi))],$$

and by Lemma (2.8),

$$\phi(\mathfrak{G}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))\geq\psi[\phi(\mathfrak{K}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))]>\phi(\mathfrak{K}(\varpi^*,\vartheta^*,\varpi^*,\varphi)),$$
$$\phi(\mathfrak{H}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))\leq\psi[\phi(\mathfrak{L}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))]<\phi(\mathfrak{L}(\varpi^*,\vartheta^*,\varpi^*,\varphi)),$$
$$\phi(\mathfrak{J}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))\leq\psi[\phi(\mathfrak{M}(\varpi^*,\vartheta^*,\vartheta^*,\varphi))]<\phi(\mathfrak{M}(\varpi^*,\vartheta^*,\varpi^*,\varphi)),$$

which are the contradictions. As a result, there can be only one fixed point of Γ. ☐

Corollary 3.3. *Let $(\mathfrak{A},\mathfrak{G},\mathfrak{H},\mathfrak{J},\otimes,\oplus)$ be a complete GNMS and $\Gamma:\mathfrak{A}\to\mathfrak{A}$ be a self-mapping which satisfies the following condition, for all $\varpi,\vartheta\in\mathfrak{A}$,*

$$\mathfrak{G}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)\geq\min\left\{\begin{array}{c}\rho\mathfrak{G}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{G}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{G}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}$$
$$\mathfrak{H}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)\leq\max\left\{\begin{array}{c}\rho\mathfrak{H}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{H}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{H}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}$$
$$\mathfrak{J}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)\leq\max\left\{\begin{array}{c}\rho\mathfrak{J}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{J}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{J}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}$$

where $0\leq\rho<\frac{1}{2}$ and $0\leq\upsilon<\frac{1}{3}$. In that case, Γ has a unique fixed point $\varpi^*\in\mathfrak{A}$.

Proof. Take $\eta=\min\{2\rho,3\upsilon\}$, $\mu=\max\{2\rho,3\upsilon\}$, then $0\leq\eta,\mu<1$. And let $\phi(\varphi)=\varphi\eta,\psi(\varphi)=\varphi$, $\phi(\varphi)=\frac{\varphi}{\eta},\psi(\varphi)=\varphi$ then $\phi,\psi\in\Phi$. Since

$$\begin{aligned}
&\min\left\{\begin{array}{c}\rho\mathfrak{G}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{G}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{G}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\geq\min\left\{\begin{array}{c}\mathfrak{G}(\varpi,\vartheta,\vartheta,\varphi),\frac{1}{3}[\mathfrak{G}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \frac{1}{3}[\mathfrak{G}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\geq\min\left\{\begin{array}{c}\mathfrak{G}(\varpi,\vartheta,\vartheta,\varphi),\mathfrak{G}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi),\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi),\\ \frac{1}{3}[\mathfrak{G}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{G}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\geq\eta\mathfrak{G}(\varpi,\vartheta,\vartheta)\\
&\max\left\{\begin{array}{c}\rho\mathfrak{H}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{H}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{H}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\max\left\{\begin{array}{c}\mathfrak{H}(\varpi,\vartheta,\vartheta,\varphi),\frac{1}{3}[\mathfrak{H}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \frac{1}{3}[\mathfrak{H}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\max\left\{\begin{array}{c}\mathfrak{H}(\varpi,\vartheta,\vartheta,\varphi),\mathfrak{H}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi),\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi),\\ \frac{1}{3}[\mathfrak{H}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{H}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\mu\mathfrak{H}(\varpi,\vartheta,\vartheta)\\
&\max\left\{\begin{array}{c}\rho\mathfrak{J}(\varpi,\vartheta,\vartheta,\varphi),\upsilon[\mathfrak{J}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \upsilon[\mathfrak{J}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\max\left\{\begin{array}{c}\mathfrak{J}(\varpi,\vartheta,\vartheta,\varphi),\frac{1}{3}[\mathfrak{J}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi)+2\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)],\\ \frac{1}{3}[\mathfrak{J}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\max\left\{\begin{array}{c}\mathfrak{J}(\varpi,\vartheta,\vartheta,\varphi),\mathfrak{J}(\varpi,\Gamma\varpi,\Gamma\varpi,\varphi),\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi),\\ \frac{1}{3}[\mathfrak{J}(\varpi,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\vartheta,\Gamma\vartheta,\varphi)+\mathfrak{J}(\vartheta,\Gamma\varpi,\Gamma\varpi,\varphi)]\end{array}\right\}\\
&\leq\mu\mathfrak{J}(\varpi,\vartheta,\vartheta).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(\mathfrak{G}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta)) &= (\mathfrak{G}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta))^{\eta} \geq (\eta\mathfrak{K}(\varpi,\vartheta,\vartheta))^{\eta} = \phi(\eta\mathfrak{K}(\varpi,\vartheta,\vartheta)) = \psi(\phi(\eta\mathfrak{K}(\varpi,\vartheta,\vartheta))),\\
\phi(\mathfrak{H}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta)) &= (\mathfrak{H}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta))^{\mu} \leq (\mu\mathfrak{L}(\varpi,\vartheta,\vartheta))^{\mu} = \phi(\mu\mathfrak{L}(\varpi,\vartheta,\vartheta)) = \psi(\phi(\mu\mathfrak{L}(\varpi,\vartheta,\vartheta))),\\
\phi(\mathfrak{J}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta)) &= (\mathfrak{J}(\Gamma\varpi,\Gamma\vartheta,\Gamma\vartheta))^{\mu} \leq (\mu\mathfrak{M}(\varpi,\vartheta,\vartheta))^{\mu} = \phi(\mu\mathfrak{M}(\varpi,\vartheta,\vartheta)) = \psi(\phi(\mu\mathfrak{M}(\varpi,\vartheta,\vartheta))).
\end{aligned}$$

Therefore, Γ has only one fixed point $\varpi^* \in \mathfrak{A}$. □

Example 3.4. *Let* $\mathfrak{A} = \left\{\frac{1}{\mathfrak{n}} : \mathfrak{n} \in \mathbb{N}\right\} \cup \{0\}$ *be endowed with the GNMS*

$$\begin{aligned}
\mathfrak{G}(\varpi,\vartheta,\xi,\varphi) &= e^{-(|\varpi-\vartheta|+|\vartheta-\xi|+|\xi-\varpi|)}\\
\mathfrak{H}(\varpi,\vartheta,\xi,\varphi) &= 1-e^{-(|\varpi-\vartheta|+|\vartheta-\xi|+|\xi-\varpi|)}\\
\mathfrak{J}(\varpi,\vartheta,\xi,\varphi) &= e^{-(|\varpi-\vartheta|+|\vartheta-\xi|+|\xi-\varpi|)}-1
\end{aligned}$$

for all $\varpi,\vartheta,\xi \in \mathfrak{A}$. *Then* $(\mathfrak{A},\mathfrak{G},\mathfrak{H},\mathfrak{J},\otimes,\oplus)$ *is a complete GNMS.*
Define the mapping $\Gamma : \mathfrak{A} \to \mathfrak{A}$ *by* $\Gamma(\varpi) = \begin{cases} \frac{1}{\mathfrak{n}^4} & if \quad \varpi = \frac{1}{\mathfrak{n}}, \mathfrak{n} \geq 2 \\ 0 & Otherwise \end{cases}$
If so, there is only one fixed point $\varpi \in \mathfrak{A}$.

Solution: The three cases below are taken into account.
Case-I: Take $\varpi = 0$ (or $\varpi = 1$) $\vartheta = \frac{1}{\mathfrak{n}}$ and $\xi = \frac{1}{\mathfrak{n}}$
Since $\Gamma_\varpi = 0$ (or $\Gamma_\varpi = 1$), $\Gamma_\vartheta = \frac{1}{\mathfrak{n}^4}$ and $\Gamma_\xi = \frac{1}{\mathfrak{n}^4}$ for all $\mathfrak{n} \in \mathbb{N}$, then

$$\mathfrak{K}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right) = \min\left\{\begin{array}{c} \mathfrak{G}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right), \mathfrak{G}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi), \mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{G}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right), \frac{1}{2}\left[\mathfrak{G}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{G}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\} = 1,$$

$$\mathfrak{L}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right) = \max\left\{\begin{array}{c} \mathfrak{H}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right), \mathfrak{H}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi), \mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{H}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right), \frac{1}{2}\left[\mathfrak{H}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{H}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\} = 0,$$

$$\mathfrak{M}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right) = \max\left\{\begin{array}{c} \mathfrak{J}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right), \mathfrak{J}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi), \mathfrak{J}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{J}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right), \frac{1}{2}\left[\mathfrak{J}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{J}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{J}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{J}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{J}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\} = 0.$$

Hence the LHS of (3.1)
$\phi(\mathfrak{G}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = e^{\frac{2}{\mathfrak{n}^4}}, \phi(\mathfrak{H}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = 1-e^{\frac{2}{\mathfrak{n}^4}}, \phi(\mathfrak{J}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = e^{\frac{2}{\mathfrak{n}^4}}-1$,
and the RHS of (3.1)
$\psi[\phi(\mathfrak{K}(\varpi,\vartheta,\xi))] = 1, \psi[\phi(\mathfrak{L}(\varpi,\vartheta,\xi))] = 0, \psi[\phi(\mathfrak{M}(\varpi,\vartheta,\xi))] = 0.$
Therefore,
$\phi(\mathfrak{G}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = \psi[\phi(\mathfrak{K}(\varpi,\vartheta,\xi))], \phi(\mathfrak{H}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = \psi[\phi(\mathfrak{L}(\varpi,\vartheta,\xi))]$
and $\phi(\mathfrak{J}((\Gamma_\varpi,,\Gamma_\vartheta,\Gamma_\xi,\varphi)) = \psi[\phi(\mathfrak{M}(\varpi,\vartheta,\xi))]$.
Case-II: Let $\varpi = \frac{1}{\mathfrak{n}}, y = \frac{1}{\mathfrak{m}}$ and $z = \frac{1}{\mathfrak{m}}$, when $\mathfrak{m} \geq \mathfrak{n} \geq 2$.

Since $\Gamma_{\varpi}=\frac{1}{\mathbf{n}^4},\Gamma_{\vartheta}=\frac{1}{\mathbf{m}^4}$ and $\Gamma_{\xi}=\frac{1}{\mathbf{m}^4}$ then,

$$\mathfrak{K}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right)=\min\left\{\begin{array}{c}\mathfrak{G}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right),\mathfrak{G}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{G}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right),\frac{1}{2}\left[\mathfrak{G}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{G}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{G}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=1,$$

$$\mathfrak{L}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right)=\max\left\{\begin{array}{c}\mathfrak{H}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right),\mathfrak{H}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{H}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right),\frac{1}{2}\left[\mathfrak{H}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{H}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{H}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=0,$$

$$\mathfrak{M}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right)=\max\left\{\begin{array}{c}\mathfrak{I}\left(\frac{1}{n},\frac{1}{m},\frac{1}{m},\varphi\right),\mathfrak{I}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{I}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right),\\ \mathfrak{I}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right),\frac{1}{2}\left[\mathfrak{I}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{I}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{I}\left(\frac{1}{n},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{I}\left(\frac{1}{m},\frac{1}{m^4},\frac{1}{m^4},\varphi\right)+\mathfrak{I}\left(\frac{1}{m},\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=0.$$

Hence the LHS of (3.1)
$\phi(\mathfrak{G}((\Gamma_{\varpi},,\Gamma_{\vartheta},\Gamma_{\xi},\varphi))=1,\phi(\mathfrak{H}((\Gamma_{\varpi},,\Gamma_{\vartheta},\Gamma_{\xi},\varphi))=0,\phi(\mathfrak{I}((\Gamma_{\varpi},,\Gamma_{\vartheta},\Gamma_{\xi},\varphi))=0,$
and the RHS of (3.1)
$\psi[\phi(\mathfrak{K}(\varpi,\vartheta,\xi))]=1,\psi[\phi(\mathfrak{L}(\varpi,\vartheta,\xi))]=0,\psi[\phi(\mathfrak{M}(\varpi,\vartheta,\xi))]=0.$
Therefore,
$\phi(\mathfrak{G}(\Gamma_{\varpi},\Gamma_{\vartheta},\Gamma_{\xi},\varphi))\ \geq\ \psi[\phi(\mathfrak{K}(\varpi,\vartheta,\xi))],\phi(\mathfrak{H}((\Gamma_{\varpi},,\Gamma_{\vartheta},\Gamma_{\xi},\varphi))\ \leq\ \psi[\phi(\mathfrak{L}(\varpi,\vartheta,\xi))]$
and
$\phi(\mathfrak{I}((\Gamma_{\varpi},,\Gamma_{\vartheta},\Gamma_{\xi},\varphi))\leq\psi[\phi(\mathfrak{M}(\varpi,\vartheta,\xi))].$

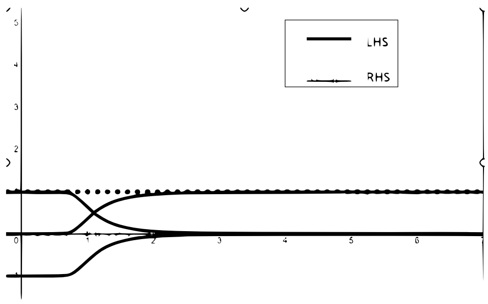

FIGURE 1. Comparison of L.H.S and R.H.S of Case I and Case II of (3.1) in 2D view

Case-III Let $\varpi=\frac{1}{n}$, when $n\geq 2,\vartheta=0$ (or $\vartheta=1$) and $\xi=0$ (or $\xi=1$). Since, $\Gamma\varpi=\frac{1}{n^4},\Gamma\vartheta=\Gamma\xi=0$ then,

$$\mathfrak{K}\left(\frac{1}{n},0,0,\varphi\right)=\min\left\{\begin{array}{c}\mathfrak{G}\left(\frac{1}{n},0,0,\varphi\right),\mathfrak{G}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{G}\left(0,0,0,\varphi\right),\\ \mathfrak{G}\left(0,0,0,\varphi\right),\frac{1}{2}\left[\mathfrak{G}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{G}\left(0,0,0,\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{G}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{G}\left(0,0,0,\varphi\right)+\mathfrak{G}\left(0,\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=1,$$

$$\mathfrak{L}\left(\frac{1}{n},0,0,\varphi\right)=\max\left\{\begin{array}{c}\mathfrak{H}\left(\frac{1}{n},0,0,\varphi\right),\mathfrak{H}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{H}\left(0,0,0,\varphi\right),\\ \mathfrak{H}\left(0,0,0,\varphi\right),\frac{1}{2}\left[\mathfrak{H}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{H}\left(0,0,0,\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{H}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{H}\left(0,0,0,\varphi\right)+\mathfrak{H}\left(0,\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=0,$$

$$\mathfrak{M}\left(\frac{1}{n},0,0,\varphi\right)=\max\left\{\begin{array}{c}\mathfrak{I}\left(\frac{1}{n},0,0,\varphi\right),\mathfrak{I}(\frac{1}{n},\frac{1}{n^4},\frac{1}{n^4},\varphi),\mathfrak{I}\left(0,0,0,\varphi\right),\\ \mathfrak{I}\left(0,0,0,\varphi\right),\frac{1}{2}\left[\mathfrak{I}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{I}\left(0,0,0,\varphi\right)\right],\\ \frac{1}{3}\left[\mathfrak{I}\left(\frac{1}{n},0,0,\varphi\right)+\mathfrak{I}\left(0,0,0,\varphi\right)+\mathfrak{I}\left(0,\frac{1}{n^4},\frac{1}{n^4},\varphi\right)\right]\end{array}\right\}=0.$$

Hence the L.H.S of (3.1), $\phi(\mathfrak{G}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) = e^{\frac{2}{n^4}}$, $\phi(\mathfrak{H}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) = 1 - e^{\frac{2}{n^4}}$ and $\phi(\mathfrak{J}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) = e^{\frac{2}{n^4}} - 1$, the R.H.S of (3.1), $\psi[\phi(\mathfrak{K}(\varpi, \vartheta, \xi))] = 1$, $\psi[\phi(\mathfrak{L}(\varpi, \vartheta, \xi))] = 0$, $\psi[\phi(\mathfrak{M}(\varpi, \vartheta, \xi))] = 0$.
Therefore, $\phi(\mathfrak{G}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) \geq \psi[\phi(\mathfrak{K}(\varpi, \vartheta, \xi))]$, $\phi(\mathfrak{H}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) \leq \psi[\phi(\mathfrak{L}(\varpi, \vartheta, \xi))]$ and
$\phi(\mathfrak{J}(\Gamma\varpi, \Gamma\vartheta, \Gamma\xi)) \leq \psi[\phi(\mathfrak{M}(\varpi, \vartheta, \xi))]$.
As a result, $\varpi = 0$ is a fixed point of Γ, satisfying all the requirements of Theorem (3.2).

4. Conclusion

Theoretical gaming, dynamic programming, financial studies, and research on integral and differential equations share foundations in fixed-point theories. This article elucidates the common fixed point concept within $GNMS$. The findings presented in this study not only generalise but also build upon prior research in NMS, contributing to an enhanced understanding of the subject. Additionally, researchers might explore the connections between generalized fixed-point theories and other mathematical concepts or theories. This could involve investigating how these fixed-point theories relate to existing theorems, lemmas, or mathematical structures, providing a deeper understanding of their place within the broader mathematical landscape.

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