PARACOMPACTNESS IN A BISPACE

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Abstract. The idea of pairwise paracompactness was studied by many authors in a bitopological space. Here we study the same in the setting of more general structure of a bispace using the thoughts of the same given by Bose et al. [2].

1. Introduction

The idea of paracompactness given by Dieudonné in the year 1944 came out as a generalization of the notion of compactness. It has many implication in field of differential geometry and it plays important roll in metrization theory. The concept of the Alexandroff space [1] (i.e., a σ-space or simply a space) was introduced by A. D. Alexandroff in the year 1940 as a generalization of a topological space where the union of open sets were taken to be open for only countable collection of open sets instead of arbitrary collection. Another kind of generalization of a topological space is the idea of a bitopological space introduced by J.C. Kelly in [13]. Using these ideas Lahiri and Das [17] introduced the idea of a bispace as a generalization of a σ-space. Many works on topological properties were carried out by many authors ([21], [22], [25] etc.) in the setting of a bitopological space. Datta [11] studied the idea of paracompactness in a bitopological space and tried to get analogous results of topological properties given by Michael [19] in respect of paracompactness. In 1986 Raghavan and Reilly [23] gave the idea of paracompactness in a bitopological space in another way. Later in 2008 M. K. Bose et al. [2] studied the same in a bitopological space as a generalization of pairwise compactness. Here we have studied pairwise paracompactness using the thoughts given by Bose et al. [2] in a bispace and discussed some its results in the setting of a bispace, which was firstly introduced by Lahiri and Das [17] as a generalization of the notion of bitopological spaces in 2001.

2. Preliminaries

Definition 2.1. [1] A set $X$ is called an Alexandroff space or σ-space or simply space if it is chosen a system $\mathcal{F}$ of subsets of $X$, satisfying the following axioms
(i) The intersection of countable number sets in \( F \) is a set belonging to \( F \).
(ii) The union of finite number of sets from \( F \) is a set belonging to \( F \).
(iii) The empty set and \( X \) is a set belonging to \( F \).

Sets of \( F \) are called closed sets. There complementary sets are called open. It is clear that instead of closed sets in the definition of a space, one may put open sets with subject to the conditions of countable summability, finite intersectability and the condition that \( X \) and the void set should be open. The collection of such open will sometimes be denoted by \( \mathcal{P} \) and the space by \((X, \mathcal{P})\). It is noted that \( \mathcal{P} \) is not a topology in general as can be seen by taking \( X = \mathbb{R} \), the set of real numbers and \( \tau \) as the collection of all \( F_\sigma \) sets in \( \mathbb{R} \).

**Definition 2.2.** \([1]\)** To every set \( M \) we correlate its closure \( \overline{M} = \text{the intersection of all closed sets containing } M \).

Generally the closure of a set in a \( \sigma \)-space is not a closed set. We denote the closure of a set \( M \) in a space \((X, \mathcal{P})\) by \( \mathcal{P}\text{-cl}(M) \) or simply \( M \) when there is no confusion about \( \mathcal{P} \). The idea of limit points, derived set, interior of a set etc. in a space are similar as in the case of a topological space which have been thoroughly discussed in \([16]\).

**Definition 2.3.** \([3]\)** Let \((X, \mathcal{P})\) be a space. A family of open sets \( B \) is said to form a base (open) for \( \mathcal{P} \) if and only if every open set can be expressed as countable union of members of \( B \).

**Theorem 2.1.** \([3]\)** A collection of subsets \( B \) of a set \( X \) forms an open base of a suitable space structure \( \mathcal{P} \) of \( X \) if and only if
1) the empty set \( \emptyset \) belongs to \( B \)
2) \( X \) is the countable union of some sets belonging to \( B \).
3) intersection of any two sets belonging to \( B \) is expressible as countable union of some sets belonging to \( B \).

**Definition 2.4.** \([17]\)** Let \( X \) be a non-empty set. If \( \mathcal{P} \) and \( \mathcal{Q} \) be two collection of subsets of \( X \) such that \((X, \mathcal{P})\) and \((X, \mathcal{Q})\) are two spaces, then \( X \) is called a bispace.

**Definition 2.5.** \([17]\)** A bispace \((X, \mathcal{P}, \mathcal{Q})\) is called pairwise \( T_1 \) if for any two distinct points \( x, y \) of \( X \), there exist \( U \in \mathcal{P} \) and \( V \in \mathcal{Q} \) such that \( x \in U \), \( y \notin U \) and \( y \in V \), \( x \notin V \).

**Definition 2.6.** \([17]\)** A bispace \((X, \mathcal{P}, \mathcal{Q})\) is called pairwise Hausdorff if for any two distinct points \( x, y \) of \( X \), there exist \( U \in \mathcal{P} \) and \( V \in \mathcal{Q} \) such that \( x \in U \), \( y \in V \), \( U \cap V = \emptyset \).

**Definition 2.7.** \([17]\)** In a bispace \((X, \mathcal{P}, \mathcal{Q})\), \( \mathcal{P} \) is said to be regular with respect to \( \mathcal{Q} \) if for any \( x \in X \) and a \( \mathcal{P} \)-closed set \( F \) not containing \( x \), there exist \( U \in \mathcal{P} \), \( V \in \mathcal{Q} \) such that \( x \in U \), \( F \subseteq V \), \( U \cap V = \emptyset \). \((X, \mathcal{P}, \mathcal{Q})\) is said to be pairwise regular if \( \mathcal{P} \) and \( \mathcal{Q} \) are regular with respect to each other.

**Definition 2.8.** \([17]\)** A bispace \((X, \mathcal{P}, \mathcal{Q})\) is said to be pairwise normal if for any \( \mathcal{P} \)-closed set \( F_1 \) and \( \mathcal{Q} \)-closed set \( F_2 \) satisfying \( F_1 \cap F_2 = \emptyset \), there exist \( G_1 \in \mathcal{P} \), \( G_2 \in \mathcal{Q} \) such that \( F_1 \subseteq G_2 \), \( F_2 \subseteq G_1 \), \( G_1 \cap G_2 = \emptyset \).
3. Pairwise paracompactness

We called a space (or a set) is bicom pact [17] if every open cover of it has a finite subcover. Also similarly as [17] a cover B of \((X, \mathcal{P}, \mathcal{Q})\) is said to be pairwise open if \(B \subseteq \mathcal{P} \cup \mathcal{Q}\) and B contains at least one nonempty member from each of \(\mathcal{P}\) and \(\mathcal{Q}\). Bourbaki and many authors defined the term paracompactness in a topological space including the requirement that the space is Hausdorff. Also in a bitopological space some authors follow this idea. But in our discussion we shall follow the convention as adopted in Munkresh[20] to define the following terminologies as in the case of a topological space.

Definition 3.1. cf. [20] In a space \(X\) a collection of subsets \(A\) is said to be locally finite in \(X\) if every point has a neighborhood that intersects only a finitely many elements of \(A\).

Similarly a collection of subsets \(B\) in a space \(X\) is said to be countably locally finite in \(X\) if \(B\) can be expressed as a countable union of locally finite collection.

Definition 3.2. cf. [20] Let \(A\) and \(B\) be two covers of a space \(X\). Then \(B\) is said to be a refinement of \(A\) if for \(B \in B\) there exists a \(A \in A\) containing \(B\).

We call \(B\) is an open refinement of \(A\) if the elements of \(B\) are open and similarly we call \(B\) is an closed refinement if the elements of \(B\) are closed.

Definition 3.3. cf. [20] A space \(X\) is said to be paracompact if every open covering \(A\) of \(X\) has a locally finite open refinement \(B\) that covers \(X\).

As in the case of a topological space [11, 2] we define the following terminologies. Let \(A\) and \(B\) be two pairwise open covers of a bispace \((X, \mathcal{P}_1, \mathcal{P}_2)\). Then \(B\) is said to be a parallel refinement [11] of \(A\) if for any \(\mathcal{P}\)-open set (respectively \(\mathcal{Q}\)-open set) \(B\) in \(B\) there exists a \(\mathcal{P}\)-open set (respectively \(\mathcal{Q}\)-open set) \(A\) in \(A\) containing \(B\). Let \(U\) be a pairwise open cover in a bispace \((X, \mathcal{P}_1, \mathcal{P}_2)\). If \(x\) belongs to \(X\) and \(M\) be a subset of \(X\), then by “\(M\) is \(\mathcal{P}_{lt}\)-open” we mean \(M\) is \(\mathcal{P}_1\)-open (respectively \(\mathcal{P}_2\)-open set) if \(x\) belongs to a \(\mathcal{P}_1\)-open set (respectively \(\mathcal{P}_2\)-open set) in \(U\).

Definition 3.4. cf. [2] Let \(A\) and \(B\) be two pairwise open covers of a bispace \((X, \mathcal{P}_1, \mathcal{P}_2)\). Then \(B\) is said to be a locally finite refinement of \(A\) if for each \(x\) belonging to \(X\), there exists a \(\mathcal{P}_{lt}\)-open open neighborhood of \(x\) intersecting only a finite number of sets of \(B\).

Definition 3.5. cf. [2] A bispace \((X, \mathcal{P}_1, \mathcal{P}_2)\) is said to be pairwise paracompact if every pairwise open cover of \(X\) has a locally finite parallel refinement.

To study the notion of paracompactness in a bispace the idea of pairwise regular and strongly pairwise regular spaces play significant roll as discussed below.

As in the case of a bitopological space a bispace \((X, \mathcal{P}_1, \mathcal{P}_2)\) is said to be strongly pairwise regular[2] if \((X, \mathcal{P}_1, \mathcal{P}_2)\) is pairwise regular and both the spaces \((X, \mathcal{P}_1)\) and \((X, \mathcal{P}_2)\) are regular.

Now we present two examples, the first one is of a strongly regular bispace and the second one is of a pairwise regular bispace without being a strongly pairwise regular bispace.

Example 3.1. Let \(X = \mathbb{R}\) and \((x, y)\) be an open interval in \(X\). We consider the collection \(\tau_1\) with sets \(A\) in \(\mathbb{R}\) such that either \((x, y) \subset \mathbb{R} \setminus A\) or \(A \cap (x, y)\) can
be expressed as some union of open subintervals of \((x, y)\) and \(\tau_2\) be the collection of all countable subsets in \((x, y)\). Also if \(\tau\) be the collection of all countable union of members of \(\tau_1 \cup \tau_2\) then clearly \((X, \tau)\) is a \(\sigma\)-space but not a topological space. Also consider the bispace \((X, \tau, \sigma)\), where \(\sigma\) is the usual topology on \(X\).

We first show that \((X, \tau)\) is regular. Let \(p \in X\) and \(p\) be any \(\tau\)-closed set not containing \(p\). Then \(A = \{p\}\) is a \(\tau\)-open set containing \(p\). Also \(A = \{p\}\) is closed in \((X, \tau)\) because if \(p \notin (x, y)\) then \(A^c \cap (x, y) = (x, y)\) and if \(p \in (x, y)\) then \(A^c \cap (x, y) = (x, p) \cup (p, y)\) and hence \(A^c\) is a \(\tau\)-open set containing \(p\).

Now we show that the bispace \((X, \tau, \sigma)\) is pairwise regular. Let \(p \in X\) and \(M\) be a \(\tau\)-closed set not containing \(p\). Then \(A = \{p\}\) is a \(\tau\)-open set containing \(p\) and also as every singleton set is closed in \((X, \sigma)\), \(A^c\) is a \(\sigma\)-open set containing \(M\).

Now let \(p \in X\) and \(P\) be a \(\sigma\)-closed set not containing \(p\). Now consider the case when \(P \cap (x, y) = \emptyset\) then \(P\) is a \(\tau\)-open set containing \(P\) and \(P^c\) is a \(\sigma\)-open set containing \(p\).

Now we consider the case when \(P \cap (x, y) \neq \emptyset\). Since \(p \notin P\), \(P^c\) is a \(\sigma\)-open set containing \(P\) and hence there exists an open interval \(I\) containing \(p\) such that \(p \in I \subset P^c\) and \(p \in \bar{I} \subset P^c\), where \(\bar{I}\) denotes the closer of \(I\) with respect to \(\sigma\). If \(I\) intersects \((x, y)\) then let \(I_1 = (x, y) \setminus \bar{I}\). Clearly \(I_1\) is non empty because \(P \cap (x, y) \neq \emptyset\). Also \(I \subset P^c\) and hence \((x, y) \setminus P^c \subset (x, y) \setminus \bar{I}\) and its follows that \(P \cap (x, y) \subset I_1\). So clearly \(P \cup I_1\) is a \(\tau\)-open set containing \(P\) and \(I_1\) is a \(\sigma\)-open set containing \(p\) and which are disjoint. Again if \(I\) does not intersect \((x, y)\) then \(P \cup (x, y)\) is a \(\tau\)-open set containing \(P\) and \(I\) itself a \(\sigma\)-open set containing \(p\) and which are disjoint. Therefore the bispace \((X, \tau, \sigma)\) is strongly pairwise regular.

**Example 3.2.** Let \(X = \mathbb{R}\) and \((X, \tau_1, \tau_2)\) be a bispace where \((X, \tau_1)\) is cocompact topological space and \(\tau_2 = \{X, \emptyset\} \cup \{\text{countable subsets of real numbers}\}\). Clearly \(\tau_2\) is not a topology and hence \((X, \tau_1, \tau_2)\) is not a bitopological space. We show that \((X, \tau_1, \tau_2)\) is a pairwise regular bispace but not a strongly pairwise regular bispace.

Let \(p \in X\) and \(A\) be a \(\tau_1\)-closed set not containing \(p\). Then clearly \(A\) itself a \(\tau_2\)-open set containing \(A\) and \(A^c\) is a \(\tau_1\)-open set containing \(p\) and clearly they are disjoint.

Similarly if \(B\) is a \(\tau_2\)-closed set such that \(p \notin B\), then \(B\) being a complement of a countable set is \(\tau_1\)-open set containing \(B\). Also \(B^c\) being countable is \(\tau_2\)-open set containing \(p\).

Now let \(p \in X\) and \(P\) be a closed set in \((X, \tau_2)\) such that \(p \notin P\). Then \(P\) must be a complement of a countable set in \(\mathbb{R}\) and hence it must be an uncountable set. So clearly the only open set containing \(P\) is \(\mathbb{R}\) itself. Therefore \((X, \tau_2)\) is not regular and hence \((X, \tau_1, \tau_2)\) can not be strongly pairwise regular.

**Remark 3.1.** In a bitopological space, pairwise Hausdorffness and pairwise paracompactness together imply pairwise normality but similar result holds in a bispace if an additional condition \(C(1)\) holds.

**Theorem 3.1.** Let \((X, P, Q)\) be a bispace, which is pairwise Hausdorff and pairwise paracompact and satisfies the condition \(C(1)\) as stated below then it is pairwise normal.

\(C(1):\) If \(A \subset X\) is expressible as an arbitrary union of \(P\)-open sets and \(A \subset B\), \(B\) is an arbitrary intersection of \(Q\)-closed sets, then there exists a \(P\)-open set \(K\), such that \(A \subset K \subset B\), the role of \(P\) and \(Q\) can be interchangeable.
Proof. We first show that $X$ is pairwise regular. So let us suppose $F$ be a $\mathcal{P}$-closed set not containing $x \in X$. Since $X$ is pairwise Hausdorff for $\xi \in F$, there exists a $U_\xi \in \mathcal{P}$ and $V_\xi \in \mathcal{Q}$, such that $x \in U_\xi$ and $\xi \in V_\xi$ and $U_\xi \cap V_\xi = \emptyset$. Then the collection $\{V_\xi : \xi \in F\} \cup (X \setminus F)$ forms a pairwise open cover of $X$. Therefore it has a locally finite parallel refinement $W$. Let $H = \cup \{W \in \mathcal{W} : W \cap F \neq \emptyset\}$. Now $x \in X \setminus F$ and $X \setminus F$ is $\mathcal{P}$-open set and hence there exists a $\mathcal{P}$-open neighborhood $D$ of $x$ intersecting only a finite number of members $W_1, W_2, \ldots, W_n$ of $W$. Now if $W_i \cap F = \emptyset$ for all $n = 1, 2, \ldots, n$, then $H \cap D = \emptyset$. Therefore by C(1) we must have a $\mathcal{Q}$-open set $K$ such that $F \subset H \subset K \subset D^c$. Hence we have a $\mathcal{Q}$-open set $K$ containing $F$ and $\mathcal{P}$-open set $D$ containing $x$ with $D \cap K = \emptyset$. If there exists a finite number of elements $W_{p_1}, W_{p_2}, \ldots, W_{p_k}$ from the collection $\{W_1, W_2, \ldots, W_n\}$ such that $W_{p_i} \cap F \neq \emptyset$, then we consider $V_{\xi_{p_i}}$ such that $W_{p_i} \subset V_{\xi_{p_i}}, \xi_{p_i} \in F$ and $i = 1, 2, \ldots, k$, since $W$ is a locally finite parallel refinement of $\{V_\xi : \xi \in F\} \cup (X \setminus F)$. Now, if $U_{\xi_{p_i}}$’s are the corresponding member of $V_{\xi_{p_i}}$, then $x \in D \cap (\bigcap_{i=1}^k U_{\xi_{p_i}}) = G$(say) $\in \mathcal{P}$. Since $W$ is a cover of $X$ it covers also $D$ and since $D$ intersects only finite number of members $W_1, W_2, \ldots, W_n$, these $n$ sets covers $D$. Now since the members $W_{p_1}, W_{p_2}, \ldots, W_{p_k}$ be such that $W_{p_i} \cap F = \emptyset$, we have $D \cap F \subset \bigcup_{i=1}^k W_{p_i}$. Now let $W_{p_i} \subset V_{\xi_{p_i}}$ for some $\xi_{p_i} \in F$ and consider $U_{\xi_{p_i}}$ corresponding to $V_{\xi_{p_i}}$ be such that $U_{\xi_{p_i}} \cap V_{\xi_{p_i}} = \emptyset$. Now we claim that $G \cap F = \emptyset$. If not let $y \in G \cap F = [D \cap (\bigcap_{i=1}^k U_{\xi_{p_i}})] \cap F = [D \cap F] \cap (\bigcap_{i=1}^k U_{\xi_{p_i}})$. Then $y \in D \cap F$ and hence there exists $W_{p_i}$ for some $i = 1, 2, \ldots, k$ such that $y \in W_{p_i} \subset V_{\xi_{p_i}}$. Also $y \in (\bigcap_{i=1}^k U_{\xi_{p_i}}) \subset U_{\xi_{p_i}}$ and hence $y \in U_{\xi_{p_i}} \cap V_{\xi_{p_i}}$, which is a contradiction. So $G \cap F = \emptyset$. Now we have a $\mathcal{P}$-open neighborhood $G$ of $x$ intersecting only a finite number of members $W_{r_1}, W_{r_2}, \ldots, W_{r_k}$ of $W$ where $W_{r_i} \cap F = \emptyset$. So by similar argument there exists a $\mathcal{Q}$-open set $K$ such that $F \subset H \subset K \subset G^c$. Thus we have a $\mathcal{Q}$-open set $K$ containing $F$ and a $\mathcal{P}$-open set $G$ containing $x$ such that $G \cap K = \emptyset$.

Next let $A$ be a $\mathcal{Q}$-closed set and $B$ be a $\mathcal{P}$-closed set and $A \cap B = \emptyset$. Then for every $x \in A$ and $\mathcal{Q}$-closed set $A$ there exists $\mathcal{P}$-open set $U_x$ containing $A$ and $\mathcal{Q}$-open set $V_x$ containing $x$ with $U_x \cap V_x = \emptyset$. Now the collection $U = (X \setminus B) \cup \{V_x : x \in B\}$ forms a pairwise open cover of $X$. Hence there exists a locally finite parallel refinement $\mathcal{M}$ of $U$. Clearly $B \subset Q$ where $Q = \cup \{M \in \mathcal{M} : M \cap B = \emptyset\}$. Now for $x \in X \setminus B$, a $\mathcal{P}$-open set there exists a $\mathcal{P}$-open neighborhood of $x$ intersecting only a finite number of elements of $\mathcal{M}$. Since $A \subset X \setminus B$, so for $x \in A$ there exists a $\mathcal{P}$-open neighborhood $D_x$ of $x$ intersecting only a finite number of elements $M_{x_1}, M_{x_2}, \ldots, M_{x_n}$ of $\mathcal{M}$ with $M_{x_i} \cap B \neq \emptyset$ for some $i = 1, 2, \ldots, n$. Suppose if $M_{x_i} \cap V_{x_i} = \emptyset$, then $D_x = D_x \cap (\bigcap_{i=1}^n U_{x_i})$ where $U_{x_i} \cap V_{x_i} = \emptyset$. If $M_{x_i} \cap B = \emptyset$ for all $i = 1, 2, \ldots, n$, then we consider $D_x = D_x$. Now if $P = \cup \{P_x : x \in A\}$ then $A \subset P$ and $P \subset Q^c$.

Now by the given condition C(1) there exists a $\mathcal{P}$-open set $R$ be such that $A \subset P \subset R \subset Q^c$. Again by the same argument there exists a $\mathcal{Q}$-open set $S$ be such that $B \subset Q \subset S \subset R^c$. Hence there exists a $\mathcal{P}$-open set $R$ containing $A$ and $\mathcal{Q}$-open set $S$ containing $B$ with $R \cap S = \emptyset$. \hfill \qed

\textbf{Theorem 3.2.} If the bispace $(X, \mathcal{P}_1, \mathcal{P}_2)$ is strongly pairwise regular and satisfies the condition C(2) given below, then the following statements are equivalent:

(i) $X$ is pairwise paracompact.

(ii) Each pairwise open cover $C$ of $X$ has a countably locally finite parallel refinement.
(iii) Each pairwise open cover $\mathcal{C}$ of $X$ has a locally finite refinement.
(iv) Each pairwise open cover $\mathcal{C}$ of $X$ has a locally finite refinement $\mathcal{B}$ such that if
$B \subset C$ where $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then $\mathcal{P}_1\text{-cl}(B) \cup \mathcal{P}_2\text{-cl}(B) \subset C$.

$C(2)$ : If $M \subset X$ and $\mathcal{B}$ is a subfamily of $\mathcal{P}_1 \cup \mathcal{P}_2$ such that $\mathcal{P}_1\text{-cl}(B) \cap M = \emptyset$, for
all $B \in \mathcal{B}$, then there exists a $\mathcal{P}_1$-open set $S$ such that $M \subset S \subset \bigcup_{B \in \mathcal{B}} \mathcal{P}_1\text{-cl}(B)^c$.

**Proof.** (i) $\Rightarrow$ (ii)

Let $\mathcal{C}$ be a pairwise open cover of $X$. Let $\mathcal{U}$ be a locally finite parallel refinement of $\mathcal{C}$. Then the collection $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where $\mathcal{V}_n = \mathcal{U}$ for all $n \in \mathbb{N}$, becomes the
countably locally finite parallel refinement of $\mathcal{C}$.

(ii) $\Rightarrow$ (iii)

We consider a pairwise open cover $\mathcal{C}$ of $X$. Let $\mathcal{V}$ be a parallel refinement of $\mathcal{C}$, such that $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where for each $n$ and for each $x$ there exists a $\mathcal{P}_\mathcal{C}_x$-open neighborhood $x$ intersecting only a finite number of members of $\mathcal{V}_n$. For
each $n \in \mathbb{N}$, let us agree to write $\mathcal{V}_n$ as $\mathcal{V}_n = \{\mathcal{V}_{n\alpha} : \alpha \in \Lambda_n\}$ and we consider
$M_n = \bigcup_{\alpha \in \Lambda_n} \mathcal{V}_{n\alpha}$, $n \in \mathbb{N}$. Clearly the collection $\{M_n\}_{n \in \mathbb{N}}$ is a cover of $X$.
Let $N_n = M_n - \bigcup_{k<n} M_k$. Clearly for $x \in X$ if $x \in M_n$, where $n$ is the least positive
integer then $x \in N_n$ and hence $\{N_n : n \in \mathbb{N}\}$ covers $X$. Also $\mathcal{N}_n \subset M_n$ for every
$n$, so $\{N_n : n \in \mathbb{N}\}$ is a refinement of $\{M_n : n \in \mathbb{N}\}$. The family $\{N_n : n \in \mathbb{N}\}$ is
locally finite because for $x \in X$ there exists a $\mathcal{V}_{n\alpha} \in \mathcal{V}$ which intersects only
some or all of $N_1, N_2, \ldots, N_n$. Now the collection $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \Lambda_n, n \in \mathbb{N}\}$
covers $X$ as if $x \in \mathcal{V}_{p\alpha}$ for the least positive integer $p$ then $x \in N_{p\alpha}$ and hence
$x \in \mathcal{V}_{n\alpha} \cap N_n$. So clearly $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \Lambda_n, n \in \mathbb{N}\}$ is a refinement of $\mathcal{V}$ and hence
of $\mathcal{C}$. Also for $x \in X$ there exists a $\mathcal{P}_\mathcal{C}_x$-open neighborhood $\mathcal{V}_{k\alpha}$ intersecting only
a finite number of members of $\{N_n : n \in \mathbb{N}\}$ and hence it intersects only a finite
number of members of $\{\mathcal{V}_{n\alpha} \cap N_n : \alpha \in \Lambda_n, n \in \mathbb{N}\}$.

(iii) $\Rightarrow$ (iv)

Let $\mathcal{C}$ be a pairwise open cover of $X$. Let $x \in X$ and suppose that $x \in C_x$ for
some $C_x \in \mathcal{C}$. Without any loss of generality let $C_x \in \mathcal{P}_1$. Then $x \notin C_x^c$ and hence
by using the condition of strongly pairwise regularity of $X$ there exists a $\mathcal{P}_1$-open
set $D_1$ containing $x$ and a $\mathcal{P}_1$-open set $D_1'$ containing $C_x^c$ with $D_1 \cap D_1' = \emptyset$. Now
$D_1^c \subset C_x$ and hence $(D_1')^c$ is a $\mathcal{P}_1$-closed set such that $x \in (D_1')^c \subset C_x$. Therefore
$\mathcal{P}_1\text{-cl}(D_1) \subset C_x$ and $D_1 \subset (D_1')^c \subset C_x$. Again $x \notin D_1'$, a $\mathcal{P}_1$-closed set and hence
by pairwise regularity of $X$ there exists a $\mathcal{P}_1$-open set $D_2$ containing $x$ and a $\mathcal{P}_2$-open
set $D_2'$ containing $D_1'$ with $D_2 \cap D_2' = \emptyset$. Now $D_2 \subset (D_1')^c$ and then $D_2 \subset (D_1')^c \subset D_1 \subset C_x$. Hence $\mathcal{P}_2\text{-cl}(D_2) \subset C_x$ and also $D_2 \subset D_1$. Therefore $\mathcal{P}_1\text{-cl}(D_2) \subset \mathcal{P}_1\text{-cl}(D_1)$ and
hence $\mathcal{P}_1\text{-cl}(D_2) \cup \mathcal{P}_2\text{-cl}(D_2) \subset C_x$. Similarly if $C_x \in \mathcal{P}_2$ then there exists a $\mathcal{P}_2$-open
set $D_2$ containing $x$ such that $\mathcal{P}_1\text{-cl}(D_2) \cup \mathcal{P}_2\text{-cl}(D_2) \subset C_x$. Let us denote $D_2$
y by a general notation $G_x$ and then we can write $\mathcal{P}_1\text{-cl}(G_x) \cup \mathcal{P}_2\text{-cl}(G_x) \subset C_x$. Then,
since $\mathcal{C}$ be a pairwise open cover $\{G_x : x \in X, C_x \in \mathcal{C}\}$ is a pairwise open cover
of $X$ which refines $\mathcal{C}$. Therefore by (iii) there exists a locally finite refinement $B$ of
$\{G_x : x \in X\}$ and hence $\mathcal{C}$. If $B \in \mathcal{B}$ then for some $G_x$ we have $B \subset G_x \subset C_x$
and so $\mathcal{P}_1\text{-cl}(B) \cup \mathcal{P}_2\text{-cl}(B) \subset \mathcal{P}_1\text{-cl}(G_x) \cup \mathcal{P}_2\text{-cl}(G_x) \subset C_x$.

(iv) $\Rightarrow$ (i)

Let $\mathcal{C}$ be a pairwise open cover of $X$ and without any loss of generality we assume
that there does not exist any element of $\mathcal{C}$ which is both $\mathcal{P}_1$-open and $\mathcal{P}_2$-open. So
there exists a locally finite refinement $\mathcal{A}$ of $\mathcal{C}$. For $x \in X$ we must have $a \in \mathcal{C}$.
containing $x$. Let us suppose $C$ is $\mathcal{P}_1$-open. Let $W_x$ be a $\mathcal{P}_1$-open neighborhood of $x$ intersecting only a finite number of elements of $\mathcal{A}$. So the collection $W = \{W_x : x \in X\}$ is a pairwise open cover of $X$ and let $E = \{E_\lambda : \lambda \in \Lambda\}$ be a locally finite refinement of $W$ such that if $E_\lambda \subset W_x$ then $\mathcal{P}_1\text{-cl}(E_\lambda) \cup \mathcal{P}_2\text{-cl}(E_\lambda) \subset W_x$.

Now for $A \in \mathcal{A}$ we consider $C_A \subset C$ such that $A \subset C_A$. Then if $C_A$ is $\mathcal{P}_1$-open, then we consider the set $F_A = \cup\{\mathcal{P}_1\text{-cl}(E_\lambda) : E_\lambda \in E, \mathcal{P}_1\text{-cl}(E_\lambda) \cap A = \emptyset\}$. Let $G_A = X \setminus F_A$, then by the given condition C(2) there exists a $\mathcal{P}_1$-open set $S_A$ such that $A \subset S_A \subset G_A$. We write $H_A = S_A \cap C_A$ and since $A \subset H_A$, the collection \{H_A : A \in \mathcal{A}\} covers $X$. Also $H_A \subset C_A$ and $H_A$ is $\mathcal{P}_1$-open. Thus \{H_A : A \in \mathcal{A}\} is a parallel refinement of $C$. Now we show that \{H_A : A \in \mathcal{A}\} is a locally finite refinement of $C$.

We show that if $M$ is a $\mathcal{P}_{W_x}$-open set containing $x$ then it is also a $\mathcal{P}_{C_x}$-open set containing $x$. Let $M$ be a $\mathcal{P}_{W_x}$-open set containing $x$ and $M$ is $\mathcal{P}_1$-open set then $x$ must be contained in a $\mathcal{P}_1$-open set $W_x$ in $W$. So there exists a $\mathcal{P}_1$-open set $C$ in $C$ containing $x$. This shows that $M$ is also a $\mathcal{P}_{C_x}$-open set containing $x$.

Now let $x \in X$ and $J_x$ be a $\mathcal{P}_{W_x}$-open neighborhood of $x$ intersecting only a finite numbers of members $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of $E$. Hence $J_x$ is also a $\mathcal{P}_{C_x}$-open neighborhood of $x$ intersecting only a finite numbers of members $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of $E$. Clearly $J_x$ can be covered by these members of $E$. Now each $E_\lambda$ is contained in some $W_x$ with $\mathcal{P}_1\text{-cl}(E_\lambda) \cup \mathcal{P}_2\text{-cl}(E_\lambda) \subset W_x$. Also $W_x$ can intersects only a finite number of members of $\mathcal{A}$. Hence each $\mathcal{P}_1\text{-cl}(E_\lambda)$ or $\mathcal{P}_2\text{-cl}(E_\lambda)$ can intersect only a finite number of sets in $\mathcal{A}$. So each $\mathcal{P}_1\text{-cl}(E_\lambda)$ or $\mathcal{P}_2\text{-cl}(E_\lambda)$ can intersect only a finite number of sets in $\{G_A : A \in \mathcal{A}\}$. Therefore $J_x$ can intersect only a finite number of sets of $\{G_A : A \in \mathcal{A}\}$. Now \{H_A : A \in \mathcal{A}\} covers $X$ and $H_A \subset G_A$, hence $J_x$ can intersect only a finite number of sets in $\{H_A : A \in \mathcal{A}\}$. Also $H_A \subset C_A$ and hence clearly $\{H_A : A \in \mathcal{A}\}$ refines $C$. Therefore $\{H_A : A \in \mathcal{A}\}$ is a locally finite parallel refinement of $C$.

\begin{theorem}
Let $\mathcal{A}$ be a locally finite collection in a $\sigma$-space $X$. Then the collection $\mathcal{B} = \{A\}_{A \in \mathcal{A}}$ is also locally finite.
\end{theorem}

\begin{proof}
Let $x \in X$ and $U$ be a neighborhood of $x$ intersecting only a finite number of members of $\mathcal{A}$. Now if for $A \in \mathcal{A}$, $A \cap U = \emptyset$ then $A \subset U^c$ and hence $A \subset \overline{A} \subset U^c$. Therefore $\overline{A} \subset U^c$ so $\overline{A} \cap U = \emptyset$. Therefore $U$ can intersect only a finite number of members of $\mathcal{B}$.
\end{proof}

\begin{theorem}
In a space any sub collection of a locally finite collection of sets is locally finite.
\end{theorem}

\begin{proof}
Let $\mathcal{A}$ be a locally finite collection of sets in a space $X$ and $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda, \text{an indexing set}\}$ be a sub collection of $\mathcal{A}$. If $x \in X$ then there exists a neighborhood $U$ of $x$ intersecting only a finite number of sets in $\mathcal{A}$. Hence $U$ can not intersect infinite number of sets in $\mathcal{B}$. If $U$ does not intersect any member of $\mathcal{B}$, then consider $B_\alpha \in \mathcal{B}$ such that $M = B_\alpha \setminus \bigcup_{\alpha \in \Lambda} B_\alpha \neq \emptyset$. Then $M \cup U$ is a neighborhood of $x$ intersecting only $B_\alpha$ of $\mathcal{B}$. Hence $\mathcal{B}$ is locally finite.
\end{proof}

It has been discussed in [10] that in a regular topological space $X$ the following four conditions are equivalent:

(i) The space $X$ is paracompact.

(ii) If $\mathcal{U}$ is an open cover of $X$ then it has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} V_n$, where $V_n$ is a locally finite collection in $X$ for each $n$. 
(iii) For every open cover of the space \(X\) there exists its locally finite refinement.

(iv) For every open cover of the space \(X\) there exists its closed locally finite refinement.

In a \(\sigma\)-space it is not true because closure of a set may not be closed. But a similar kind of result has been discussed below.

**Theorem 3.5.** In a regular space \(X\) for the following four conditions we have (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv):

(i) The space \(X\) is paracompact.

(ii) If \(U\) is a open cover of \(X\) then it has an open refinement \(V = \bigcup_{n=1}^{\infty} V_n\), where \(V_n\) is a locally finite collection in \(X\) for each \(n\).

(iii) For every open cover of the space \(X\) there exists its locally finite refinement.

(iv) For every open cover \(A\) of the space \(X\) there exists its locally finite refinement \(S = \{S_\alpha : \alpha \in \Lambda\}\) such that \(\overline{S_\alpha} : S_\alpha \in S\) is also its locally finite refinement, \(\Lambda\) being an indexing set.

**Proof.** (i) \(\Rightarrow\) (ii)

The proof is straightforward.

(ii) \(\Rightarrow\) (iii)

Let \(A\) be an open cover of \(X\). Then by (ii) there exists an open refinement \(B = \bigcup_{n=1}^{\infty} B_n\) where \(B_n\) is a locally finite collection in \(X\) for each \(n\). Let \(B_n = \{B_{n\alpha} : \alpha \in \Lambda_n\}\) and \(C_n = \bigcup_{\alpha \in \Lambda_n} B_{n\alpha}\), \(\Lambda_n\) being an indexing set. Now clearly the collection \(\{C_n\}\) covers \(X\). Let us consider \(D_n = C_n \setminus \bigcup_{k<n} C_k\). For \(x \in X\), suppose that \(k\) be the least natural number for which \(x \in B_{kn}\), then \(B_{kn}\) can intersect at most \(k\) members \(D_1, D_2, \ldots, D_k\) of \(\{D_n : n \in \mathbb{N}\}\). Hence \(\{D_n : n \in \mathbb{N}\}\) is a locally finite refinement of \(\{C_n : n \in \mathbb{N}\}\). Now we show that \(M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}\) is a locally finite refinement of \(B\). For \(n \in \mathbb{N}\) we have \(\bigcup_{\alpha \in \Lambda_n} (D_n \cap B_{n\alpha}) = D_n \cap \bigcup_{\alpha \in \Lambda_n} B_{n\alpha} = D_n \cap C_n = D_n \cap D_n = C_n\). Also \(D_n\) covers \(X\) and hence \(\bigcup_{n \in \mathbb{N}} \bigcup_{\alpha \in \Lambda_n} (D_n \cap B_{n\alpha}) = X\). Let \(x \in X\) then there exists an neighborhood \(U\) of \(x\) intersecting only a finite number members \(D_{i_1}, D_{i_2}, \ldots, D_{i_m}\) (say) of \(\{D_n : n \in \mathbb{N}\}\). Also there exists an open set \(U_{i_n}\) intersecting only a finite number of members of \(B_{i_n}\). Now \(U \cap \bigcap_{k=1}^{m} U_{i_k}\) is an neighborhood of \(x\) intersecting only a finite numbers of \(M\) as \(M\) covers \(X\). Also \(D_n \cap B_{n\alpha} \subset B_{n\alpha}\) and hence \(M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}\) is a locally finite refinement of \(B\). And also since \(D_n \cap B_{n\alpha} \subset B_{n\alpha} \subset A\) for some \(A \in A\), \(M = \{D_n \cap B_{n\alpha} : n \in \mathbb{N}, \alpha \in \Lambda_n\}\) is a locally finite refinement of \(A\).

(iii) \(\Rightarrow\) (iv)

Let \(U\) be an open cover of \(X\). Now for \(x \in X\) we have a \(U_x \in U\) such that \(x \in U_x\). So \(x \notin (U_x)^c\) and hence by regularity of \(X\), there exist disjoint open sets \(P_x\) and \(Q_x\) containing \(x\) and \((U_x)^c\) respectively. Hence \(x \in P_x \subset (Q_x)^c \subset U_x\) and clearly \(x \in P_x \subset U_x\). Now \(P = \{P_x : x \in X\}\) is an open cover of \(X\) and by (iii) it has a locally finite refinement \(S = \{S_\alpha : \alpha \in \Lambda,\) an indexing set\(\}\) (say). Also the collection \(\{S_\alpha : S_\alpha \in S\}\) is locally finite by previous lemma. Now for \(\alpha \in \Lambda\), \(S_\alpha \subset P_x \subset U_x\) for some \(P_x \in P\) and hence \(S_\alpha \subset P_x \subset U_x\) for some \(U_x \in U\). Therefore \(S\) is a locally finite refinement of \(U\) such that \(\{S_\alpha : S_\alpha \in S\}\) is also a locally finite refinement of \(U\).
We have discussed some results associated with paracompactness in a \( \sigma \)-space because our motivation was to establish the statement “If \((X, P_1, P_2)\) is a pairwise paracompact bispace with \((X, P_2)\) regular, then every \( P_1 \)-\( F_{\sigma} \) proper subset is \( P_2 \) paracompact”. This has been discussed in a bitopological space [2]. But we failed due to the fact that arbitrary union of open sets in a \( \sigma \)-space may not be open.

REFERENCES

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