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ROUGH $\mathcal{I}$-CONVERGENCE IN INTUITIONISTIC FUZZY NORMED SPACES

NESAR HOSSAIN, AMAR KUMAR BANERJEE

Abstract. In this paper we have introduced and studied the notion of rough $\mathcal{I}$-convergence in intuitionistic fuzzy normed spaces. Also we have defined rough $\mathcal{I}$-cluster point of a sequence and proved some related results in the same space.

1. Introduction

In 1951, the idea of ordinary convergence of real sequences was extended to statistical convergence of real sequences independently by Fast [12], Steinhaus [32] and Schoenberg [33]. After long 50 years, in 2000 Kostyrko et al. [18] introduced the concept of $\mathcal{I}$-convergence of sequences as a generalization of statistical convergence where $\mathcal{I}$ is an ideal of subsets of the set of natural numbers. Since then this idea has been nurtured by several authors in different directions e.g. [6, 9, 22, 24, 37, 31].

In 2001, Phu [27] first introduced the notion of rough convergence of sequences in finite dimensional normed spaces and in the same paper he investigated that r-limit set is bounded, closed and convex and some interesting results were studied by Phu [27, 28]. In 2003, Phu [29] extended this concept to infinite dimensional normed spaces. Later, this notion was extended into rough statistical convergence [3], rough ideal convergence [10, 30] and this idea was studied by many authors in different directions and different spaces as in [2, 7, 11, 15, 16, 21]. The reader may refer to the textbooks [8] and [25] for summability theory, sequence spaces and related topics.

In 1965 Zadeh [38] introduced the concept of fuzzy sets as an extension of classical set theoritical concept which has wide and extensive applications in various branches of science and engineering [5, 13, 14, 17, 23]. In 1986, Atanassov [1] defined the idea of intuitionistic fuzzy sets and later on, using this idea, in 2004, Park [26] introduced the notion of intuitionistic fuzzy metric spaces. Furthermore, Saadati and Park [35] extended this concept to the theory of intuitionistic fuzzy normed spaces which is, nowadays, a well motivated area of research in science. In this

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paper we study the concept of rough $\mathcal{I}$-convergence in intuitionistic fuzzy normed spaces.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of reals respectively. First we recall some basic definitions and notations.

Definition 2.1. [18] *A family $\mathcal{I}$ of subsets of a non empty set Y is said to be an ideal in Y if*
(1) $\emptyset \in \mathcal{I}$;
(2) $A, B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$;
(3) $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non trivial if $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non trivial ideal $\mathcal{I}$ is called admissible if $\{\{x\} : x \in X\} \subset \mathcal{I}$.

Definition 2.2. [18] *A non empty family $\mathcal{F}$ of subsets of a non empty set Y is called a filter in Y if the following properties hold.*
(1) $\emptyset \notin \mathcal{F}$;
(2) $A, B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$;
(3) $A \in \mathcal{F}$ *and* $A \subset B$ *implies* $B \in \mathcal{F}$.

Lemma 2.1. [18] *If $\mathcal{I} \subset 2^Y$ is a non trivial ideal then the class $\mathcal{F}(\mathcal{I}) = \{Y \setminus A : A \in \mathcal{I}\}$ is a filter on Y which is called filter associated with the ideal $\mathcal{I}$.*

Definition 2.3. *Let $K \subset \mathbb{N}$. Then the natural density $\delta(K)$ of K is defined by*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|,$$

provided the limit exists.

It is clear that if K is finite then $\delta(K) = 0$.

Now we recall some basic definitions and notations which will be useful in the sequal.

Definition 2.4. [34] *A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions hold.*
(1) $\star$ *is associative and commutative;*
(2) $\star$ *is continuous;*
(3) $x \star 1 = x$ *for all* $x \in [0,1]$;
(4) $x \star y \leq z \star w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0,1]$.

Definition 2.5. [34] *A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if the following conditions are satisfied.*
(1) $\circ$ *is associative and commutative;*
(2) $\circ$ *is continuous;*
(3) $x \circ 0 = x$ *for all* $x \in [0,1]$;
(4) $x \circ y \leq z \circ w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0,1]$.

Example 2.1. [19] *The following are the examples of t-norms:*
(1) $x \star y = min\{x, y\}$,
(2) $x \star y = x.y$,
(3) $x \star y = max\{x + y - 1, 0\}$. *This t-norm is known as Lukasiewicz t-norm.*

Example 2.2. [19] *The following are the examples of t-conorms:*
(1) $x \circ y = max\{x, y\}$,
(2) $x \circ y = x + y - x.y$,
(3) $x \circ y = min\{x + y, 1\}$. *This is known as Lukasiewicz t-conorm.*

Definition 2.6. [35] *The 5-tuple $(X, \mu, \nu, \star, \circ)$ is said to be an intuitionistic fuzzy normed space (in short, IFNS) if X is a normed linear space, $\star$ is a continuous t-norm, $\circ$ is a continuous t-conorm and μ and ν are the fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:*
1. $\mu(x, t) + \nu(x, t) \leq 1$,
2. $\mu(x, t) > 0$,
3. $\mu(x, t) = 1$ *if and only if* $x = 0$,
4. $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
5. $\mu(x, t) \star \mu(y, s) \leq \mu(x + y, t + s)$,
6. $\mu(x, t) : (0, \infty) \to [0, 1]$ *is continuous in* t,
7. $\lim_{t\to\infty} \mu(x, t) = 1$ *and* $\lim_{t\to 0} \mu(x, t) = 0$,
8. $\nu(x, t) < 1$,
9. $\nu(x, t) = 0$ *if and only if* $x = 0$,
10. $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$,
11. $\nu(x, t) \circ \nu(y, s) \geq \nu(x + y, s + t)$,
12. $\nu(x, t) : (0, \infty) \to [0, 1]$ *is continuous in* t,
13. $\lim_{t\to\infty} \nu(x, t) = 0$ *and* $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm on X.

Example 2.3. *Let $(X, \|\cdot\|)$ be a normed space. Denote $a \star b = ab$ and $a \circ b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$ and let μ and ν be fuzzy sets on $X \times (0, \infty)$ defined as follows:*

$$\mu(x, t) = \frac{t}{t + \|x\|}, \ \nu(x, t) = \frac{\|x\|}{t + \|x\|}.$$

Then $(X, \mu, \nu, \star, \circ)$ is an intuitionistic fuzzy normed space.

Definition 2.7. [35] *Let $(X, \mu, \nu, \star, \circ)$ be an IFNS. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for each $\varepsilon > 0$ and $t > 0$ there exists a positive integer m such that $\mu(x_n - \xi, t) > 1 - \varepsilon$ and $\nu(x_n - \xi, t) < \varepsilon$ whenever $n \geq m$. The element ξ is called ordinary limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ with respect to the intuitionistic fuzzy norm (μ, ν) and we shall write (μ, ν)-$\lim x_n = \xi$.*

Definition 2.8. [4] *Let $(X, \mu, \nu, \star, \circ)$ be an IFNS with intuitionistic fuzzy norm (μ, ν). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be rough convergent to $\xi \in X$ with respect to the norm (μ, ν) for some non-negative number r if there exists $k_0 \in \mathbb{N}$ for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $\mu(x_n - \xi, r + \varepsilon) > 1 - \lambda$ and $\nu(x_n - \xi, r + \varepsilon) < \lambda$ for all $k \geq k_0$. In this case ξ is called $r_{(\mu,\nu)}$-limit of $\{x_n\}_{n\in\mathbb{N}}$ and we write $r_{(\mu,\nu)}$-$\lim x_n = \xi$ or $x_n \xrightarrow{r_{(\mu,\nu)}} \xi$.*

Definition 2.9. [20] *Let $\mathcal{I} \subset P(\mathbb{N})$ and $(X, \mu, \nu, \star, \circ)$ be an IFNS. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in X is said to be $\mathcal{I}$-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for each $\varepsilon > 0$ and $t > 0$, the set $\{n \in \mathbb{N} : \mu(x_n - L, t) \leq 1 - \varepsilon$ or $\nu(x_n - L, t) \geq \varepsilon\} \in \mathcal{I}$. In this case L is called $\mathcal{I}$-limit of the* *sequnce* $\{x_n\}$ *with respect to the fuzzy norm* (μ,ν) *and we write* $\mathcal{I}_{(\mu,\nu)}\text{-}\lim x_n = L$ *or* $x_n \xrightarrow{\mathcal{I}_{(\mu,\nu)}} L$.

Definition 2.10. [4] *Let* $(X,\mu,\nu,\star,\circ)$ *be an IFNS with intuitionistic fuzzy norm* (μ,ν). *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to be rough statistical convergent to* $\xi \in X$ *with respect to the norm* (μ,ν) *for some non-negative number* r *if for every* $\varepsilon > 0$ *and* $\lambda \in (0,1)$, $\delta(\{n \in \mathbb{N} : \mu(x_n - \xi, r+\varepsilon) \leq 1-\lambda \text{ or } \nu(x_n - \xi, r+\varepsilon) \geq \lambda\}) = 0$.

Definition 2.11. [36] *Let* $(X,\mu,\nu,\star,\circ)$ *be an IFNS with intuitionistic fuzzy norm* (μ,ν). *For* $r > 0$, *we define open ball* $B(x,\lambda,r)$ *with center* $x \in X$ *and radius* $0 < \lambda < 1$, *as*

$$B(x,\lambda,r) = \{y \in X : \mu(x-y,r) > 1-\lambda,\ \nu(x-y,r) < \lambda\}.$$

Similarly, we define closed ball $\overline{B(x,\lambda,r)} = \{y \in X : \mu(x-y,r) \geq 1-\lambda,\ \nu(x-y,r) \leq \lambda\}$.

3. **Main Results**

Throughout the paper $\mathcal{I}$ denotes a non-trivial admissible ideal and r denotes a non-negative real number unless otherwise stated. First we introduce the definition of rough $\mathcal{I}$-convergence in an IFNS $(X,\mu,\nu,\star,\circ)$.

Definition 3.1. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in an IFNS* $(X,\mu,\nu,\star,\circ)$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *is said to be rough* $\mathcal{I}$*-convergent to* $\xi \in X$ *with respect to the intuitionistics fuzzy norm* (μ,ν) *if for every* $\varepsilon > 0$ *and* $\lambda \in (0,1)$, $\{n \in \mathbb{N} : \mu(x_n - \xi, r+\varepsilon) \leq 1-\lambda \text{ or } \nu(x_n - \xi, r+\varepsilon) \geq \lambda\} \in \mathcal{I}$. *In this case* ξ *is called* $r\text{-}\mathcal{I}_{(\mu,\nu)}$*-limit of* $\{x_n\}_{n\in\mathbb{N}}$ *and we write* $r\text{-}\mathcal{I}_{(\mu,\nu)}\text{-}\lim_{n\to\infty} x_n = \xi$ *or* $x_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \xi$.

Remark 3.1. (*a*) *Suppose* $\mathcal{I}_f$ *is the class of all finite subsets of* $\mathbb{N}$. *Then clearly* $\mathcal{I}_f$ *is a non-trivial admissible ideal. So rough* $\mathcal{I}_f$*-convergence with respect to the intuitionistic fuzzy norm* (μ,ν) *agrees with the rough convergence with respect to the intuitionistic fuzzy norm* (μ,ν) *in an IFNS* $(X,\mu,\nu,\star,\circ)$.
(*b*) *If we take* $\mathcal{I}_\delta$ *as the class of all subsets of* $\mathbb{N}$ *whose natural density are zero. Then* $\mathcal{I}_\delta$ *will be a non-trivial admissible ideal. So rough* $\mathcal{I}_\delta$*-convergence with respect to the intuitionistic fuzzy norm* (μ,ν) *coincides with the rough statistical convergence with respect to the intuitionistic fuzzy norm* (μ,ν) *in an IFNS* $(X,\mu,\nu,\star,\circ)$.

If $r = 0$, the notion of rough $\mathcal{I}$-convergence with respect to the intuitionistic fuzzy norm (μ,ν) coincides with the $\mathcal{I}$-convergence with respect to the intuitionistic fuzzy norm (μ,ν) in an IFNS $(X,\mu,\nu,\star,\circ)$. From the Definition 3.1 it is clear that $r\text{-}\mathcal{I}_{(\mu,\nu)}$-limit of $\{x_n\}_{n\in\mathbb{N}}$ is not unique. Here we use the notation $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ and $LIM^{r_{(\mu,\nu)}}_{x_n}$ to denote the set of all $r\text{-}\mathcal{I}_{(\mu,\nu)}$-limits and $r_{(\mu,\nu)}$-limits of the sequence $\{x_n\}_{n\in\mathbb{N}}$ respectively. For an unbounded sequence, $LIM^{r_{(\mu,\nu)}}_{x_n}$ is always empty. But for such a sequence, $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$ would happen as shown in the following example.

Example 3.1. *Let* $(X, \|\cdot\|)$ *be a real normed linear space with the usual norm and let* $\mu(x,t) = \frac{t}{t+\|x\|}$ *and* $\nu(x,t) = \frac{\|x\|}{t+\|x\|}$ *for all* $x \in X$ *and* $t > 0$. *Also let* $a \star b = ab$ *and* $a \circ b = min\{a+b, 1\}$. *Then* $(X,\mu,\nu,\star,\circ)$ *is an IFNS. Now let us consider the ideal* $\mathcal{I}$ *consisting of all those subsets of* $\mathbb{N}$ *whose natural density are zero. Then* $\mathcal{I}$ *is a non-trivial admissible ideal of* $\mathbb{N}$. *Let us* *take the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *as* $x_n = \begin{cases} (-1)^n, & \text{if } n \neq k^2, k \in \mathbb{N} \\ n, & \text{otherwise} \end{cases}$. *Then* $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} = \begin{cases} \emptyset, & r < 1 \\ [1-r, r-1], & \text{otherwise} \end{cases}$ *and* $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} = \emptyset$ *when* $r = 0$. *Also since the sequence is unbounded,* $LIM^{r(\mu,\nu)}_{x_n} = \emptyset$ *for any* r.

We obtain by Example 3.1 that $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$ does not imply $LIM^{r(\mu,\nu)}_{x_n} \neq \emptyset$, but when $\mathcal{I}$ is an admissible ideal, $LIM^{r(\mu,\nu)}_{x_n} \neq \emptyset$ implies $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$.

Definition 3.2. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in an IFNS* $(X, \mu, \nu, \star, \circ)$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *is said to be* $\mathcal{I}$*-bounded with respect to the intuitionistic fuzzy norm* (μ, ν) *if for every* $\lambda \in (0, 1)$ *there exists a positive real numgber* G *such that the set* $\{n \in \mathbb{N} : \mu(x_n, G) \leq 1 - \lambda \text{ or } \nu(x_n, G) \geq \lambda\} \in \mathcal{I}$.

Theorem 3.1. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in an IFNS* $(X, \mu, \nu, \star, \circ)$. *Then* $\{x_n\}_{n\in\mathbb{N}}$ *is* $\mathcal{I}$*-bounded if and only if* $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$ *for some* $r > 0$.

Proof. First suppose that $\{x_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-bounded sequence. Then, for every $\lambda \in (0, 1)$ there exists a non-negative real number G such that $\{n \in \mathbb{N} : \mu(x_n, G) \leq 1 - \lambda \text{ or } \nu(x_n, G) \geq \lambda\} \in \mathcal{I}$. Now, let $A = \{n \in \mathbb{N} : \mu(x_n, G) \leq 1 - \lambda \text{ or } \nu(x_n, G) \geq \lambda\}$. Then for $k \in A^c$, $\mu(x_k, G) > 1-\lambda$ and $\nu(x_k, G) < \lambda$. Now $\mu(x_k, r+G) \geq \mu(x_k, G) \star \mu(\theta, r) = \mu(x_k, G) \star 1 = \mu(x_k, G) > 1 - \lambda$ and $\nu(x_k, r + G) \leq \nu(x_k, G) \circ \nu(\theta, r) = \nu(x_k, G) \circ 0 = \nu(x_k, G) < \lambda$. Hence $\theta \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$. Therefore $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$ for some $r > 0$.

Conversely suppose that $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$ for some $r > 0$. Then for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, $\{n \in \mathbb{N} : \mu(x_n - \xi, r + \varepsilon) \leq 1 - \lambda \text{ or } \nu(x_n - \xi, r + \varepsilon) \geq \lambda\} \in \mathcal{I}$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-bounded sequence in the IFNS $(X, \mu, \nu, \star, \circ)$. □

Theorem 3.2. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be two sequences in an IFNS* $(X, \mu, \nu, \star, \circ)$. *Then, the following statements hold:*

(i) *If* $x_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \xi$ *and* $\alpha \in \mathbb{R}$ *then* $\alpha x_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \alpha\xi$.

(ii) *If* $x_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \xi$ *and If* $y_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \eta$ *then* $x_n + y_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \xi + \eta$.

Proof. This is easy. So, we omit details. □

Theorem 3.3. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in an IFNS* $(X, \mu, \nu, \star, \circ)$. *Then the set* $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ *is a closed set.*

Proof. If $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} = \emptyset$, then we have nothing to prove. So let $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n} \neq \emptyset$. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ such that $(\mu, \nu)\text{-}\lim y_n = \xi$. For given $\lambda \in (0, 1)$, choose $s \in (0, 1)$ such that $(1 - s) \star (1 - s) > 1 - \lambda$ and $s \circ s < \lambda$. Let $\varepsilon > 0$ be given. Then there exists a $k_0 \in \mathbb{N}$ such that $\mu(y_n - \xi, \frac{\varepsilon}{2}) > 1 - s$ and $\nu(y_n - \xi, \frac{\varepsilon}{2}) < s$ for all $n \geq k_0$. Suppose $y_m \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ where $m > k_0$. Consequently the set $A = \{n \in \mathbb{N} : \mu(x_n - y_m, r + \frac{\varepsilon}{2}) \leq 1 - s \text{ or } \nu(x_n - y_m, r + \frac{\varepsilon}{2}) \geq s\} \in \mathcal{I}$. Now we have $M = \mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$. So $M \neq \emptyset$. Let $j \in M$. So we have $\mu(x_j - y_m, r + \frac{\varepsilon}{2}) > 1 - s$ and $\nu(x_j - y_m, r + \frac{\varepsilon}{2}) < s$. Again we get, for $m > k_0$, $\mu(y_m - \xi, \frac{\varepsilon}{2}) > 1 - s$ and $\nu(y_m - \xi, \frac{\varepsilon}{2}) < s$. Now $\mu(x_j - \xi, r + \varepsilon) \geq \mu(x_j - y_m, r + \frac{\varepsilon}{2}) \star \mu(y_m - \xi, \frac{\varepsilon}{2}) > (1 - s) \star (1 - s) > 1 - \lambda$ and $\nu(x_j - \xi, r + \varepsilon) \leq \nu(x_j - y_m, r + \frac{\varepsilon}{2}) \circ \nu(y_m - \xi, \frac{\varepsilon}{2}) < s \circ s < \lambda$. Therefore $M \subset \{n \in \mathbb{N} : \mu(x_n - \xi, r + \varepsilon) > 1 - \lambda \text{ or } \nu(x_n - \xi, r + \varepsilon) < \lambda\}$. Consequently

$\{n \in \mathbb{N} : \mu(x_n - \xi, r+\varepsilon) \leq 1-\lambda$ or $\nu(x_n - \xi, r+\varepsilon) \geq \lambda\} \in \mathcal{I}$. Hence $\xi \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$. Therefore $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ is closed. □

Theorem 3.4. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then the set $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ is convex.*

Proof. Let $\xi_1,\ \xi_2 \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ and $\alpha \in (0,1)$. Suppose $\lambda \in (0,1)$. Choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$ and $s \circ s < \lambda$. Then for any $\varepsilon > 0$, $A_1 = \{n \in \mathbb{N} : \mu(x_n - \xi_1, \frac{r+\varepsilon}{2(1-\alpha)}) \leq 1-s$ or $\nu(x_n - \xi_1, \frac{r+\varepsilon}{2(1-\alpha)}) \geq s\} \in \mathcal{I}$ and $A_2 = \{n \in \mathbb{N} : \mu(x_n - \xi_2, \frac{r+\varepsilon}{2\alpha}) \leq 1-s$ or $\nu(x_n - \xi_2, \frac{r+\varepsilon}{2\alpha}) \geq s\} \in \mathcal{I}$. Now for $k \in A_1^c \cap A_2^c$, we have $\mu(x_k - [(1-\alpha)\xi_1 + \alpha\xi_2], r+\varepsilon) \geq \mu\{(1-\alpha)(x_k - \xi_1), \frac{r+\varepsilon}{2}\} \star \mu\{\alpha(x_k - \xi_2), \frac{r+\varepsilon}{2}\} = \mu(x_k - \xi_1, \frac{r+\varepsilon}{2(1-\alpha)}) \star \mu(x_k - \xi_2, \frac{r+\varepsilon}{2\alpha}) > (1-s) \star (1-s) > 1-\lambda$ and $\nu(x_k - [(1-\alpha)\xi_1 + \alpha\xi_2], r+\varepsilon) \leq \nu\{(1-\alpha)(x_k - \xi_1), \frac{r+\varepsilon}{2}\} \circ \nu\{\alpha(x_k - \xi_2), \frac{r+\varepsilon}{2}\} = \nu(x_k - \xi_1, \frac{r+\varepsilon}{2(1-\alpha)}) \circ \nu(x_k - \xi_2, \frac{r+\varepsilon}{2\alpha}) < s \circ s < \lambda$. Thus $\{n \in \mathbb{N} : \mu(x_n - [(1-\alpha)\xi_1 + \alpha\xi_2], r+\varepsilon) \leq 1-\lambda$ or $\nu(x_n - [(1-\alpha)\xi_1 + \alpha\xi_2], r+\varepsilon) \geq \lambda\} \in \mathcal{I}$. Therefore $(1-\alpha)\xi_1 + \alpha\xi_2 \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ i.e. $\mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ is a convex set. □

Theorem 3.5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an IFNS $(X, \mu, \nu, \star, \circ)$ rough $\mathcal{I}$-convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) for some $r > 0$ if there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \in X$ such that $\mathcal{I}_{(\mu,\nu)}\text{-}\lim y_n = \xi$ and for every $\lambda \in (0,1)$, $\mu(x_n - y_n, r) > 1-\lambda$ and $\nu(x_n - y_n, r) < \lambda$ for all $n \in \mathbb{N}$.*

Proof. Let $\varepsilon > 0$ be given. For a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$ and $s \circ s < \lambda$. First suppose that $\mathcal{I}_{(\mu,\nu)}\text{-}\lim y_n = \xi$ and $\mu(x_n - y_n, r) > 1-s$ and $\nu(x_n - y_n, r) < s$ for all $n \in \mathbb{N}$. Then the set $A = \{n \in \mathbb{N} : \mu(y_n - \xi, \varepsilon) \leq 1-s$ or $\nu(y_n - \xi, \varepsilon) \geq s\} \in \mathcal{I}$. Then there exists $M \in \mathcal{F}(\mathcal{I})$ such that $M = \mathbb{N} \setminus A$. Now for $n \in M$, we have $\mu(x_n - \xi, r+\varepsilon) \geq \mu(x_n - y_n, r) \star \mu(y_n - \xi, \varepsilon) > (1-s) \star (1-s) > 1-\lambda$ and $\nu(x_n - \xi, r+\varepsilon) \leq (x_n - y_n, r) \circ (y_n - \xi, \varepsilon) < s \circ s < \lambda$. Consequently $\{n \in \mathbb{N} : \mu(x_n - \xi, r+\varepsilon) \leq 1-\lambda$ or $\nu(x_n - \xi, r+\varepsilon) \geq \lambda\} \in \mathcal{I}$. Therefore $x_n \xrightarrow{r\text{-}\mathcal{I}_{(\mu,\nu)}} \xi$. This completes the proof. □

Theorem 3.6. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then there does not exist $y, z \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ for some $r > 0$ and every $\lambda \in (0,1)$ such that $\mu(y - z, mr) \leq 1-\lambda$ and $\nu(y - z, mr) \geq \lambda$ for $m(\in \mathbb{R}) > 2$.*

Proof. Suppose on the contrary that there exist the elements $y, z \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$ for which

$$\mu(y - z, mr) \leq 1-\lambda \text{ and } \nu(y - z, mr) \geq \lambda \text{ for } m(\in \mathbb{R}) > 2. \tag{3.1}$$

For a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \star (1-s) > 1-\lambda$ and $s \circ s < \lambda$. Since $y, z \in \mathcal{I}_{(\mu,\nu)}\text{-}LIM^r_{x_n}$, then for every $\varepsilon > 0$ we have $A_1 = \{n \in \mathbb{N} : \mu(x_n - y, r + \frac{\varepsilon}{2}) \leq 1-\lambda$ or $\nu(x_n - y, r + \frac{\varepsilon}{2}) \geq \lambda\} \in \mathcal{I}$ and $A_2 = \{n \in \mathbb{N} : \mu(x_n - z, r + \frac{\varepsilon}{2}) \leq 1-\lambda$ or $\nu(x_n - z, r + \frac{\varepsilon}{2}) \geq \lambda\} \in \mathcal{I}$. Now for $n \in A_1^c \cap A_2^c$, we have $\mu(y - z, 2r+\varepsilon) \geq \mu(x_n - z, r + \frac{\varepsilon}{2}) \star \mu(x_n - y, r + \frac{\varepsilon}{2}) > (1-s) \star (1-s) > 1-\lambda$ and $\nu(y - z, 2r+\varepsilon) < \nu(x_n - y, r + \frac{\varepsilon}{2}) \circ \nu(x_n - z, r + \frac{\varepsilon}{2}) < s \circ s < \lambda$. Therefore

$$\mu(y - z, 2r+\varepsilon) > 1-\lambda \text{ and } \nu(y - z, 2r+\varepsilon) < \lambda. \tag{3.2}$$

Now if we choose $\varepsilon = mr - 2r$, $m(\in \mathbb{R}) > 2$, then from (3.2) we get $\mu(y - z, mr) > 1-\lambda$ and $\nu(y - z, mr) < \lambda$. This contradicts (3.1). This completes the proof. □

Definition 3.3. *Let $(X, \mu, \nu, \star, \circ)$ be an IFNS. Then a point $\eta \in X$ is called rough $\mathcal{I}$-cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\{n \in \mathbb{N} : \mu(x_n - \eta, r+\varepsilon) > 1 - \lambda$ and $\nu(x_n - \eta, r+\varepsilon) < \lambda\} \notin \mathcal{I}$. The set of all rough $\mathcal{I}$-cluster points of $\{x_n\}_{n\in\mathbb{N}}$ is denoted as $\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$.*

We denote by $\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ to mean the set of all ordinary $\mathcal{I}$-cluster points of $\{x_n\}_{n\in\mathbb{N}}$ with respect to the fuzzy norm (μ, ν). If $r = 0$, then we have $\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)}) = \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$.

Theorem 3.7. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then $\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ is a closed set.*

Proof. The proof is an analogue to Theorem 3.3. So it is omitted. □

Theorem 3.8. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X, \mu, \nu, \star, \circ)$. Then, for an arbitrary $\beta \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ and $\lambda \in (0,1)$ we have $\mu(\eta - \beta, r) > 1 - \lambda$ and $\nu(\eta - \beta, r) < \lambda$ for all $\eta \in \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$.*

Proof. Let $w \in (0,1)$. Now choose $\lambda \in (0,1)$ such that $(1-\lambda) \star (1-\lambda) > 1 - w$ and $\lambda \circ \lambda < w$. Let $\beta \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. Then, for every $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : \mu(x_n - \beta, \varepsilon) > 1 - \lambda \text{ and } \nu(x_n - \beta, \varepsilon) < \lambda\} \notin \mathcal{I}. \tag{3.3}$$

Now we prove that if $\eta \in X$ having the properties $\mu(\eta - \beta, r) > 1 - \lambda$ and $\nu(\eta - \beta, r) < \lambda$ then $\eta \in \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. Let $k \in \{n \in \mathbb{N} : \mu(x_n - \beta, \varepsilon) > 1 - \lambda$ and $\nu(x_n - \beta, \varepsilon) < \lambda\}$. Now we have $\mu(x_k - \eta, r+\varepsilon) \geq \mu(x_k - \beta, \varepsilon) \star \mu(\eta - \beta, r) > (1-\lambda) \star (1-\lambda) > 1 - w$ and $\nu(x_k - \eta, r+\varepsilon) \leq \nu(x_k - \beta, \varepsilon) \circ (\eta - \beta, r) < \lambda \circ \lambda < w$. Therefore $\{n \in \mathbb{N} : \mu(x_n - \beta, \varepsilon) > 1 - \lambda$ and $\nu(x_n - \beta, \varepsilon) < \lambda\} \subset \{n \in \mathbb{N} : \mu(x_n - \eta, r+\varepsilon) > 1 - w$ and $\nu(x_n - \eta, r+\varepsilon) < w\}$. Hence from (3.3) we obtain $\{n \in \mathbb{N} : \mu(x_n - \eta, r+\varepsilon) > 1 - w$ and $\nu(x_n - \eta, r+\varepsilon) < w\} \notin \mathcal{I}$. So $\eta \in \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. This completes the proof. □

Theorem 3.9. *Let $(X, \mu, \nu, \star, \circ)$ be an IFNS.Then for some $r > 0$, $\lambda \in (0,1)$ and fixed $c \in X$ we have*

$$\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)}) = \bigcup_{c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})} \overline{B(c, \lambda, r)}.$$

Bar denotes the closure of open ball $B(c, \lambda, r)$.

Proof. Choose $s, w \in (0,1)$ such that $(1-s) \star (1-\lambda) > 1 - w$ and $s \circ \lambda < w$. Let $y_* \in \bigcup_{c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})} \overline{B(c, \lambda, r)}$. Then there is $c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ such that $\mu(c - y_*, r) > 1 - \lambda$ and $\nu(c - y_*, r) < \lambda$. Let $\varepsilon > 0$ be given. Since $c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$, then there exists a set $A = \{n \in \mathbb{N} : \mu(x_n - c, \varepsilon) > 1 - s$ and $\nu(x_n - c, \varepsilon) < s\}$ with $A \notin \mathcal{I}$. Now for $i \in A$, we have $\mu(x_i - y_*, r+\varepsilon) \geq \mu(x_i - c, \varepsilon) \star \mu(c - y_*, r) > (1-s) \star (1-\lambda) > 1 - w$ and $\nu(x_i - y_*, r+\varepsilon) \leq \nu(x_i - c, \varepsilon) \circ \nu(c - y_*, r) < s \circ \lambda < w$. Therefore $A \subset \{n \in \mathbb{N} : \mu(x_n - y_*, r+\varepsilon) > 1 - w$ and $\nu(x_n - y_*, r+\varepsilon) < w\}$. So $\{n \in \mathbb{N} : \mu(x_n - y_*, r+\varepsilon) > 1 - w$ and $\nu(x_n - y_*, r+\varepsilon) < w\} \notin \mathcal{I}$. This implies that $y_* \in \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. Hence $\bigcup_{c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})} \overline{B(c, \lambda, r)} \subseteq \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$.

Conversely suppose that $x_* \in \Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. We shall show that $x_* \in \bigcup_{c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})} \overline{B(c, \lambda, r)}$. On contrary that $x_* \notin \bigcup_{c \in \Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})} \overline{B(c, \lambda, r)}$. Then

$\mu(c-x_*,r)\leq 1-\lambda$ or $\nu(c-x_*,r)\geq\lambda$ for every $c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$. Now, by Theorem 3.8 we get $\mu(x_*-c,r)>1-\lambda$ and $\nu(x_*-c,r)<\lambda$ which is a contradiction. Therefore $\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\bigcup_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}$. This completes the proof. □

Theorem 3.10. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X,\mu,\nu,\star,\circ)$. Then, for any $\lambda\in(0,1)$, the following statements hold:*
(a) if $c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ then $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}\subseteq\overline{B(c,\lambda,r)}$.
(b) $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}=\bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}=\{x_0\in X:\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(x_0,\lambda,r)}\}$.

Proof. (a) First we choose $s,t\in(0,1)$ such that $(1-s)\star(1-t)>1-\lambda$ and $s\circ t<\lambda$. On the contrary we assume that there exist a point $c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ and $\beta\in\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$ such that $\mu(\beta-c,r)<1-\lambda$ and $\nu(\beta-c,r)>\lambda$. Let $\varepsilon>0$ be given. Then we have $P=\{n\in\mathbb{N}:\mu(x_n-c,\varepsilon)>1-t \text{ and } \nu(x_n-c,\varepsilon)<t\}\notin\mathcal{I}$ and $Q=\{n\in\mathbb{N}:\mu(x_n-\beta,r+\varepsilon)\leq 1-s \text{ or } \nu(x_n-\beta,r+\varepsilon)\geq s\}\in\mathcal{I}$. Suppose $Q^c=M\in\mathcal{F}(\mathcal{I})$. Now for $n\in P\cap M$ we get $\mu(\beta-c,r)\geq\mu(x_n-\beta,r+\varepsilon)\star\mu(x_n-c,\varepsilon)>(1-s)\star(1-t)>1-\lambda$ and $\nu(\beta-c,r)\leq\nu(x_n-\beta,r+\varepsilon)\circ\nu(x_n-c,\varepsilon)<s\circ t<\lambda$, which is a contradiction. Therefore we have $\mu(\beta-c,r)\geq 1-\lambda$ and $\nu(\beta-c,r)\leq\lambda$. Hence $\beta\in\overline{B(c,\lambda,r)}$. This completes the proof of Part (a).

(b) Using Part (a), above, we have $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}\subseteq\bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}$. Now let $l\in\bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}$. Then we have $\mu(l-c,r)\geq 1-\lambda$ and $\nu(l-c,r)\leq\lambda$ for all $c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})$ and so $\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(l,\lambda,r)}$, i.e., $\bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}\subseteq\{x_0\in X:\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(x_0,\lambda,r)}\}$. Now assume $l\notin\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$. Then there exists an $\varepsilon>0$ such that $\{n\in\mathbb{N}:\mu(x_n-l,r+\varepsilon)\leq 1-\lambda \text{ or } \nu(x_n-l,r+\varepsilon)\geq\lambda\}\notin\mathcal{I}$, which gives that there exists an $\mathcal{I}$-cluster point c for the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\mu(l-c,r+\varepsilon)\leq 1-\lambda$ and $\nu(l-c,r+\varepsilon)\geq\lambda$. Hence $\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\nsubseteq\overline{B(l,\lambda,r)}$ and $l\notin\{x_0\in X:\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(x_0,\lambda,r)}\}$. This gives $\{x_0\in X:\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(x_0,\lambda,r)}\}\subseteq\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$. Therefore $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}=\bigcap_{c\in\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})}\overline{B(c,\lambda,r)}=\{x_0\in X:\Lambda_{(x_n)}(\mathcal{I}_{(\mu,\nu)})\subseteq\overline{B(x_0,\lambda,r)}\}$. □

Theorem 3.11. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X,\mu,\nu,\star,\circ)$ and $x_n\xrightarrow{\mathcal{I}_{(\mu,\nu)}}x_0$ then $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}=\overline{B(x_0,\lambda,r)}$ for $\lambda\in(0,1)$.*

Proof. Let $w\in(0,1)$. Now choose $s\in(0,1)$ such that $(1-s)\star(1-\lambda)>1-w$ and $s\circ\lambda<w$. Let $\varepsilon>0$ be given. Since $x_n\xrightarrow{\mathcal{I}_{(\mu,\nu)}}x_0$, $E=\{n\in\mathbb{N}:\mu(x_n-x_0,\varepsilon)\leq 1-s \textit{ or } \nu(x_n-x_0,\varepsilon)\geq s\}\in\mathcal{I}$. Let $\zeta\in\overline{B(x_0,\lambda,r)}$. Now for $n\in E^c$ we have $\mu(x_n-\zeta,r+\varepsilon)\geq\mu(x_n-x_0,\varepsilon)\star\mu(x_0-\zeta,r)>(1-s)\star(1-\lambda)>1-w$ and $\nu(x_n-\zeta,r+\varepsilon)\leq\nu(x_n-x_0,\varepsilon)\circ\nu(x_0-\zeta,r)<\circ\lambda<w$. Therefore $\zeta\in\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$. Hence $\overline{B(x_0,\lambda,r)}\subseteq\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$. Again from the Theorem 3.10, $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}\subseteq\overline{B(x_0,\lambda,r)}$. Therefore $\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}=\overline{B(x_0,\lambda,r)}$. This completes the proof. □

Theorem 3.12. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an IFNS $(X,\mu,\nu,\star,\circ)$ such that $x_n\xrightarrow{\mathcal{I}_{(\mu,\nu)}}\eta$ then $\Lambda^r_{(x_n)}(\mathcal{I}_{(\mu,\nu)})=\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$.*

Proof. Since $x_n\xrightarrow{\mathcal{I}_{(\mu,\nu)}}\eta$, therefore $\Lambda_{x_n}(\mathcal{I}_{(\mu,\nu)})=\{\eta\}$. Now, by Theorem 3.9, $\Lambda^r_{x_n}(\mathcal{I}_{(\mu,\nu)})=\overline{B(\eta,\lambda,r)}$. Again using Theorem 3.11, we have $\Lambda^r_{x_n}(\mathcal{I}_{(\mu,\nu)})=\mathcal{I}_{(\mu,\nu)}$-$LIM^r_{x_n}$. This completes the proof. □

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Nesar Hossain
Department of Mathematics, The University of Burdwan, Golapbag, Burdwan - 713104, West Bengal, India.
E-mail address: nesarhossain24@gmail.com

Amar Kumar Banerjee
Department of Mathematics, The University of Burdwan, Golapbag, Burdwan - 713104, West Bengal, India.
E-mail address: akbanerjee1971@gmail.com