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INTERPOLATORY MINIMAL SERIES FOR RECONSTRUCTING AN INFINITE FOURIER SERIES

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Abstract. Based on the continuity of functions of two variables, we provide a new qualitative proof of the well known fast convergence of Fourier series representations of most continuous periodic functions. Our proof is based on a representation of this infinite Fourier series, even when it diverges, by a five-term, with 17 interpolation points, minimal harmonic series with a new minimal series interpolation (MSI) algorithm for an iterative approximant. A smoothing linear summation minimal series is also demonstrated to be constructible by the same algorithm.

1. Introduction

Pointwise convergence of partial Fourier sums for continuous functions, $f(x) \in \mathcal{C}(R)$, was ruled out in 1873, [6], by the du Bois-Reymond counterexample of a 2π-periodic continuous function with a Fourier series that diverges at a given point, $x = 0$. It was not possible, however, to represent this function by a curve or to explain geometrically the divergence of its series at 0. In the following years many simpler similar examples were constructed. One of these, due to Fejér, is for an even 2π-periodic everywhere continuous, $f \in \mathcal{C}(R)$, but nowhere differentiable, $f \notin \mathcal{C}^1(R)$, function defined on $[0, \pi]$ by

$$f(x) = \sum_{p=1}^{\infty} \tfrac{1}{p^2} \sin\left[(2^{\,p^3} + 1)\tfrac{x}{2}\right], \tag{1.1}$$

with a co-sinusoidal Fourier series

$$\tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx = f(x), \forall x \neq 0, \tag{1.2}$$

which diverges at $x = 0$.

It was finally in 1966 when Carleson proved, [3], a conjecture by Luzin, that the Fourier series of $f(x) \in \mathcal{C}(R)$ converges to $f(x)$, *a.e.*(everywhere with the exception of a set of measure zero). Moreover, a fairly easy Baire-category argument, [2], shows that the Fourier series of "most" functions in $\mathcal{C}(R)$ are not everywhere convergent.

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Therefore, although continuity, $f \in \mathcal{C}(R)$, is not enough for the associated Fourier series to converge, Weierstrass [2, 17] proved that if $f \in \mathcal{C}^1(R)$,i.e. both continuous and differentiable, this convergence is assured. This is the reason for the phrase: "most" functions, to stand in the abstract of this paper.

Young & Young [20] showed how the subtle relationship between the continuity of functions of several variables and continuity of their traces can be a platform for research into various questions in functional analysis. This paper is a contribution to such research on the qualitative convergence of Fourier series for most continuous periodic functions. A contribution that invokes the Kolmogorov-Arnold theorem, [10], to develop a representation of such infinite series, even when it diverges, by a five-term, with 17 interpolation points, minimal series with a new algorithm for its iterative approximation.

The paper is organized as follows. Section 2 contains a focused study on continuity of pointwise traces of functions of two variables. Section 3 introduces the Kolmogorov-Arnold theorem for representing such traces, in their infinite Fourier series, by a finite minimal series. The main result of this paper is that the infinite Fourier series of $f \in \mathcal{C}(R)$ can be reconstructed by varying only its first few harmonics, of a minimal series representation, via a newly devised iterative algorithm. Section 4 addresses the question of possible divergence of such series. The question of $\mathcal{C}_1$ linear summability of this series and construction of its minimal smoothing summation series is entertained in section 5. Here a conjecture the relation between the direct and summation minimal series is stated. The conclusion, in section 6 reports on two emerging open problems.

2. TRACES OF FUNCTIONS OF TWO VARIABLES

Definition 2.1. Let $\phi : \Omega \to R$,where $\Omega \subset R^2$, be a real function $z = \phi(x, y)$ in a Banach space $B(\Omega)$.The trace of $\phi(x, y)$ on a vertical surface $y = \beta(x)$, $(x, \beta(x)) \in \mathcal{V} \subset \Omega$, is a restriction $\phi \mid \mathcal{V}$, on the surface $z = \phi(x, y)$, representing a curve $\mathcal{Q}$ defined as the set

$\mathcal{Q} = \left\{(x, \beta(x),\ z(x, \beta(x))) \in R^3 : (x, \beta(x)) \in \mathcal{V}\right\}$.

Accordingly, $\phi(x, 0)$ and $\phi(0, y)$ are respectively the $x-$trace and $y-$trace of ϕ. Moreover, of particular interest in this work, is the trace of ϕ corresponding to $\beta(x) = x$, $\mathcal{V} = \mathcal{M}$(median line), and $\mathcal{Q} = \left\{[x,\ x,\ z(x, x)] \in R^3 : (x,\ x) \in \mathcal{M}\right\} \triangleq \phi(x, x)$, which is called the pointwise trace of ϕ, in accord with the trace notation : tr $\phi = \displaystyle\int \phi(x, x)\ dx$, which is often used in function space theory.

In 1821, Augustin Cauchy made a historic wrong statement, see e.g. [20], that a function of several variables which is continuous in each variable separately is continuous as a function of all vriables. The first counterexample, [20], appeared in 1873 as follows.

Example 1. The function $\phi : R \times R \to R$,defined by

$$z = \phi(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}\ , & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} , \tag{2.1}$$

is continuous separately in its $\phi(x, 0)$ and $\phi(0,\ y)$ traces, but is jointly discontinuous at $(0, 0)$. This example is a contradiction to the previous Cauchy's assertion.

Indeed, this z, along any straight line $y = \beta(x) = kx$ passing through the origin, remains a constant value $\frac{2k}{1 + k^2}$ that depends on k. Thus $\phi(x, y)$ approaches $(0, 0)$ along different paths with different limits. I.e. it is discontinuous at $(0, 0)$.

In the example above, $\phi(x,x)=1$, with $\phi(0,0)=0\neq 1$. So, distinctively from the $x-$and $y-$traces, this pointwise trace is discontinuous at $(0,0)$, and regardless of the commutativity, $\phi(x,y)=\phi(y,x)$, or symmetry, of this $\phi(x,y)$.

On another note, the Dirichlet formula, which is of crucial importance in the theory of Volterra integral equations, see e.g. [18] or [1], and in fractional calculus, [14], is:

Dirichlet's Formula [18, 14]. If $\phi(x,y)$ is jointly continuous over $[a,b]\times[a,b]$, then its double integral over an isosceles triangle $\mathfrak{D}$ of side $[a,b]\subset R$ and with $\mathcal{M}$, as a hypotenuse, satisfies

$$\iint_{\mathfrak{D}} \phi(x,y)\ d\mathfrak{D} = \int_a^b \int_x^b \phi(x,y)\ dydx = \int_a^b \int_a^y \phi(x,y)\ dxdy. \tag{2.2}$$

The proof of this formula can be made geometrical, by reversing the order of integration in any side of (2.2), on the assumption of its validity, which implies joint continuity by Fubini's theorem.

Corollary 2.1. *If $\phi(x,y)$ is jointly continuous over $[a,b]\times[a,b]$, then its pointwise trace $\phi(x,x)$ is necessarily continuous.*

Proof. There exist several proofs for this corollary. A global proof [11] states that if $\phi\in\mathcal{C}(\Omega)$, $\Omega\subset R^2$ and $\phi:\Omega\to R$, then $(\phi(x,x):\mathcal{V}\to R)\in\mathcal{C}(\mathcal{V}), \forall\ \mathcal{V}\subset\Omega.\square$

The previous result is obviously irreversible. Indeed, if $\phi(x,y)$ is discontinuous, then $\phi(x,x)$ may or may not be continuous. Anyway, Corollary 2.1 guarantees validity of the Dirichlet formula whenever $\phi(x,y)$ is absolutely integrable, [17], over $\mathfrak{D}$.

Remark 2.1. Hence, conversely, if the point trace $\phi(x,x)$ is jointly discontinuous over $\mathfrak{D}$,then the Dirichlet formula may (or may not) be satisfied by $\phi(x,y)$ pending to its absolute integrability (or nonintegrability) over $\mathfrak{D}$.

Example 2. To illustrate this remark, consider

$$\phi(x,y)=\begin{cases} y^{-2}+\ x\ , & 0<x<y<1 \\ 0, & elsewhere \end{cases},$$

in which $\phi(x,x)=0$, $\forall x\in\mathcal{M}\subset R$. Note, however, that $\phi(x,x)$ is discontinuous across $\mathcal{M}$ over $\mathfrak{D}$ despite the fact that $\phi(x,x)\in\mathcal{C}(\mathcal{M})$. Moreover, it can, straightforwardly, be shown that

$$\int_0^1\int_0^y (y^{-2}+\ x)\ dxdy - \int_0^1\int_x^1 (y^{-2}+\ x)\ dydx = 1\neq 0,$$

in violation of the Dirichlet formula. This happens to take place because $\int_0^1\int_0^y \left|y^{-2}+\ x\right|\ dxdy=\infty$, i.e. this $\phi(x,y)$ does not satisfy the condition for Fubini's theorem, [18], [11], over $\mathfrak{D}$, with $[a,b]=[0,1]$.

Example 3. Consider the function $\phi: R\times R\to R$, defined by

$$z=\phi(x,y)=\cos\ \frac{xy}{x^2+1}, \tag{2.3}$$

which is continuous at every finite $(x,y)\in R\times R$. Asymptotically, however, $\phi(x,\infty)$ is undefined while $\phi(\infty,y)=1$. Moreover, $\phi(y,x)=\cos\ \frac{xy}{y^2+1}\neq\phi(x,y)$, nonsymmetric, with $\phi(y,\infty)=1$ while $\phi(\infty,x)$ is undefined.

As for the traces of this z,we have:

i) the $x-$trace $\phi(x,0)=1$, is continuous$\forall x$.

ii) the $y-$trace $\phi(0,y)=1$, is continuous$\forall y$.

iii) the pointwise trace $\phi(x,x)=\cos\ \frac{x^2}{x^2+1}$ is also continuous, with $\phi(0,0)=1$, and $\phi(\pm\infty,\pm\infty)=\cos 1$.

The continuity of this $\phi(x,x)$ is a demonstration of the power of Corollary 2.1.

Example 4. Let $\phi : R \times R \to R$,be defined by

$$z = \phi(x,y) = \begin{cases} \frac{2xy}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} . \tag{2.4}$$

Unlike the z of example 1, this $\phi(x,y) \neq \phi(y,x)$ is nonsymmetric, but is also discontinuous at $(0,0)$ and continuous elsewhere.

Here along any parabola $y = \beta(x) = k\ x^2$ passing through the origin, z retains a constant value $\frac{k}{1+k^2}$ that depends on k.

As for the basic traces of this z, we have:

i) the $x-$trace $\phi(x,0) = 0, \forall x$, including $\phi(0,0) = 0$. So it is continuous.

ii) the $y-$trace $\phi(0,y) = 0, \forall y$, including $\phi(0,0) = 0$,continuous.

iii) the trace corresponding to $y = \beta(x) = x^2$, i.e. $\phi(x,x^2) = \frac{1}{2}, \forall x$, including $\phi(0,0) = \frac{1}{2} \neq 0$,is discontinuous at $(0,0)$.

iv) the pointwise trace $\phi(x,x) = \frac{x}{x^2+1}$, with $\phi(0,0) = 0$, is continuous everywhere and despite the nonsymmetry of this $\phi(x,y)$.

It should be underlined that $\phi(x,y)$ in examples 1, 2 & 4 has been discontinuous over $R \times R$. While, in example 3, $\phi(x,x)$ has been continuous. Remarkably, this variety in the continuity of these pointwise traces does not constitute any violation of Corollary 2.1.

3. Qualitative Convergence Of Fourier Series

Let $\mathcal{C}(R)$ be the space of continuous functions over R. Assume $f(x) \in \mathcal{C}(R)$ to be periodic with a period $T = 2L$. Then it is representable in the Fourier series

$$f(x) := S(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\tfrac{\pi}{L}x + b_k \sin k\tfrac{\pi}{L}x \right). \tag{3.1}$$

The speed of convergence of this series crucially depends on the nature of the infinite set $\{a_k,\ b_k\}_{k=0}^{\infty}$ of Fourier coefficients,

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \tfrac{1}{L}\int_{-L}^{L} f(x) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} k\tfrac{\pi}{L}x\ dx,$$

even when the Fourier series (7) diverges.

Theorem 3.1. *If $f(x) \in \mathcal{C}(R)$ is periodic with a period $T = 2L$, then its infinite Fourier series $S(x)$, when it converges, is always representable by a finite five-term functional series:*

$$S(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\tfrac{\pi}{L}x + b_k \sin k\tfrac{\pi}{L}x \right) = \sum_{k=0}^{4} U_k[u_k(x)], \tag{3.2}$$

consisting of a composition of some outer, $U_k[\cdot]$, and inner $u_k(x)$ continuous functions.

Proof. Let $f(x) = \phi(x,x)$ be the pointwise trace of a particular jointly continuous $\phi(x,y)$. Then according to the Kolmogorov-Arnold representation theorem, [10], highlighted in the Appendix. Such a function can always be represented as

$$\phi(x,y) = \sum_{k=0}^{4} U_k[\ g_k(x) + h_k(y)], \tag{3.3}$$

where the g_k's and h_k's are some inner continuous functions of a single variable and the $U_k[\cdot]$'s are some outer functions of the inner ones. The corresponding pointwise trace

$$\phi(x,x) = \sum_{k=0}^{4} U_k[\, g_k(x) + h_k(x)], \tag{3.4}$$

should, by virtue of corollary 1, also be continuous and satisfy (6) when $u_k(x) = g_k(x) + h_k(x)$. □

The previous result motivates the following proposition.

Proposition 3.1. *For Fourier series representation of a 2L- periodic signal $f(x) \in \mathcal{C}(R)$, variation of only the first 5 harmonics, forming a minimal series representation, $\hat{S}(x)$, suffices to reproduce the entire signal.*

Quantitatively,

$$\hat{S}(x) = \sum_{k=0}^{4} U_k[u_k(x)] = \sum_{k=0}^{4} A(a_k,b_k)\cos k\tfrac{\pi}{L}\mu_k(x) + B(a_k,b_k)\sin k\tfrac{\pi}{L}w_k(x) \tag{3.5}$$
$$= \tfrac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos k\tfrac{\pi}{L}x + b_k \sin k\tfrac{\pi}{L}x\right) = S(x) = f(x),$$

in which

$$A(a_k,b_k) = \left(p(a_k) = \begin{cases} \tfrac{1}{2}(a_0+\alpha_0), k=0 \\ (a_k+\alpha_k), k\geq 1 \end{cases}, 0\right),$$
$$B(a_k,b_k) = \left(0, \quad q(b_k) = \begin{cases} 0, k=0 \\ (b_k+\nu_k), k\geq 1 \end{cases}\right),$$
$$u_k(x) = a_k\cos 2k\tfrac{\pi}{L}\mu_k(x) + b_k\cos 2k\tfrac{\pi}{L}w_k(x),$$
$$\mu_k(x) = x+\delta_k \text{ and } w_k(x) = x+\varepsilon_k\ , \tag{3.6}$$

holds with the $U_k[\cdot]$ operator's set

$$\mathfrak{F} = \Big\{\ U_k[\cdot] = A(a_k,b_k)\ \sqrt{\tfrac{1}{4k\pi\ \mu_k(x)}\cos^{-1}(\cdot) + \tfrac{1}{2}(\cdot)}$$
$$+ B(a_k,b_k)\ \sqrt{\tfrac{1}{4k\pi\ w_k(x)}\cos^{-1}(\cdot) - \tfrac{1}{2}(\cdot)}\ \Big\}_{k=0}^{4}. \tag{3.7}$$

Proof. The perturbational relations (3.5)-(3.7) happen to represent the only parametrized $(U_k,\ u_k)$ pair that preserves its harmonicity and can tend to $\frac{a_0}{2}$ or $\left(a_k\cos k\frac{\pi}{L}x + b_k\sin k\frac{\pi}{L}x\right)$ by varying (or indexing) its α_k, ν_k , δ_k, and ε_k parameters.□

To demonstrate the utility of the previous theorem and proposition, we construct a new minimal series interpolation (MSI) algorithm which turns out to be iterative.

Algorithm (MSI) **1.** Let $f(x) \in \mathcal{C}(R)$ be a periodic function with a period $T = 2L$. The set $\mathfrak{F} = \{U_k[\cdot]\ ,\ u_k(x)\}_{k=0}^{4}$ for its five-term minimal series representation, $\hat{S}(x)$,can be constructed via the following two steps.

Step 1: Approximate solution. Reconsider the proposition with some iterative superindexing of the $\mathfrak{F}$ set to $\mathfrak{F}^i, i = 1,2,3,.....,N$. Accordingly, we define

$$Q^0(x) = \tfrac{a_0}{2} + \sum_{k=1}^{4} a_k\cos k\tfrac{\pi}{L}x + b_k\sin k\tfrac{\pi}{L}x \approx f(x), \tag{3.8}$$

associated with $\mathfrak{F}^0 \triangleq\ \delta_k^0 = \varepsilon_k^0 = \alpha_k^0 = \sigma_k^0 = 0, \forall k$. Then

$$Q^1(x) = \tfrac{1}{2}(a_0+\alpha_0^1) + \sum_{k=1}^{4}\left[(a_k+\alpha_k^1)\cos k\tfrac{\pi}{L}(x+\delta_k^1) + (b_k+\nu_k^1)\sin k\tfrac{\pi}{L}(x+\varepsilon_k^1)\right] \approx f(x)\ , \tag{3.9}$$

with the set $\mathfrak{F}^1 = \{\alpha_0^1,\ \delta_k^1,\ \varepsilon_k^1, \alpha_k^1,\ \nu_k^1\}_{k\,=\,1}^4$ of 17 unknown parameters, which may be determined numerically by solving the system

$$\tfrac{1}{2}(a_0+\alpha_0^1)+\sum_{k\,=\,1}^{4}[(a_k+\alpha_k^1)\cos k\tfrac{\pi}{L}(x_j^1+\delta_k^1)+(b_k+\nu_k^1)\sin k\tfrac{\pi}{L}(x_j^1+\varepsilon_k^1)]\approx f(x_j^1),$$
$$j=1,2,3,.....,17, \tag{3.10}$$

of 17 interpolation equations. Here the set $\{x_j^1\}_{j=1}^{17}$ corresponds to the 1-st randomized selection of 17 numbers from the interval [0, 2L].

We proceed in this manner to arrive at the $i-$th set $\{x_j^i\}_{j=1}^{17}\subset[0,2L]$,in the sense of [8], $i=1,2,3,.....,N$, where N is a certain iteration termination number. Hence

$$Q^i(x)=\tfrac{1}{2}(a_0+\alpha_0^i)+\sum_{k\,=\,1}^{4}[(a_k+\alpha_k^i)\cos k\tfrac{\pi}{L}(x+\delta_k^i)+(b_k+\nu_k^i)\sin k\tfrac{\pi}{L}(x+\varepsilon_k^i)]\approx f(x)\ , \tag{3.11}$$

with the set $\mathfrak{F}^i = \{\alpha_0^i,\ \delta_k^i,\ \varepsilon_k^i, \alpha_k^i,\ \nu_k^i\}_{k\,=\,1}^4$ determined via solving the system

$$\tfrac{1}{2}(a_0+\alpha_0^i)+\sum_{k\,=\,1}^{4}[(a_k+\alpha_k^i)\cos k\tfrac{\pi}{L}(x_j^i+\delta_k^i)+(b_k+\nu_k^i)\sin k\tfrac{\pi}{L}(x_j^i+\varepsilon_k^i)]\approx f(x_j^i),$$
$$j=1,2,3,.....,17. \tag{3.12}$$

The stopping rule, $i=N$, for the iterations is when

$$\left\|Q^i(x)-Q^{i-1}(x)\right\|\le\epsilon_s\ ,$$

where the tolerance ϵ is defined essentially by the numerical accuracy of solving the system (3.12) of transcendental equations. The quality of the approximant $Q^N(x)$ is expected, [20], moreover, to be intimately related to the choice of the norm $\|.\|$.

Step 2: Fine tuning. The term $(a_k+\alpha_k^i)\cos k\frac{\pi}{L}(x+\delta_k^i)$ of (3.11) is the same as

$$(a_k+\alpha_k^i)\left[(\cos k\tfrac{\pi}{L}\delta_k^i)\cos k\tfrac{\pi}{L}x-(\sin k\tfrac{\pi}{L}\delta_k^i)\sin k\tfrac{\pi}{L}x\right].$$

It should be noted that $a_k\cos k\frac{\pi}{L}x$ are even terms in the $Q^i(x)$ functional approximant to the associated $f(x)$. But here the term $\eta^i(x)=-(a_k+\alpha_k^i)(\sin k\frac{\pi}{L}\delta_k^i)\sin k\frac{\pi}{L}x$ happens to serve as a measure of the induced, by the increments δ_k^i and α_k^i , odd symmetry into the previous even terms. Similarly $b_k\sin k\frac{\pi}{L}x$ are odd terms in $Q^i(x)$,while $(b_k+\nu_k^i)\sin k\frac{\pi}{L}(x+\varepsilon_k^i)$ of (3.11) is the same as

$$(b_k+\nu_k^i)\left[(\cos k\tfrac{\pi}{L}\varepsilon_k^i)\sin k\tfrac{\pi}{L}x+(\sin k\tfrac{\pi}{L}\varepsilon_k^i)\cos k\tfrac{\pi}{L}x\right],$$

and the term $\zeta^i(x)=(b_k+\nu_k^i)(\sin k\frac{\pi}{L}\varepsilon_k^i)\cos k\frac{\pi}{L}x$ serves as a measure of the induced, by the increments $\frac{1}{2}\alpha_0^i$, ε_k^i and ν_k^i, even symmetry into the odd terms. Accordingly, the process $\mathfrak{F}^0\to\mathfrak{F}^i\to\mathfrak{F}^N$ can be modified to minimize the sum of certain norms of $\eta^i(x)$ and $\zeta^i(x)$ such as

$$\left\|\ \eta^i(x)\right\|=\sum_{k\,=\,0}^{4}\int_{-L}^{L}\left|(a_k+\alpha_k^i)(\sin k\tfrac{\pi}{L}\delta_k^i)\sin k\tfrac{\pi}{L}x\right|\,dx,$$
$$\left\|\ \zeta^i(x)\right\|=\left|\tfrac{1}{2}\alpha_0^i\right|+\sum_{k\,=\,1}^{4}\int_{-L}^{L}\left|(b_k+\nu_k^i)(\sin k\tfrac{\pi}{L}\varepsilon_k^i)\cos k\tfrac{\pi}{L}x\right|\,dx. \tag{3.13}$$

Hence, instead of (3.12) for determining the $\mathfrak{F}^i$ set in (3.11), we may solve the nonlinear programming problem

$$\underset{\mathfrak{F}^i}{Minimize}\left\|\ \eta^i(x)\right\|+\left\|\ \zeta^i(x)\right\|, \tag{3.14}$$
$$Subject\ to:(3.12),\ j=1,2,3,.....,17. \tag{3.15}$$

The faster the convergence of this $\mathfrak{F}^0 \to \mathfrak{F}^N$ process, the smaller N should be ; a fact that can be proved by contradiction. For large N, however, because of the coercive nature of the representation (3.2), one can always modify the $\mathfrak{F}^0 \to \mathfrak{F}^i \to \mathfrak{F}^N$ process to a $\widetilde{\mathfrak{F}}^0 \to \widetilde{\mathfrak{F}}^i = \left\{ \widetilde{\alpha}_0^i,\ \widetilde{\delta}_k^i,\ \widetilde{\varepsilon}_k^i\ ,\ \widetilde{\alpha}_k^i,\ \widetilde{\nu}_k^i) \right\}_{k\,=\,1}^4 \to \widetilde{\mathfrak{F}}^N$ process, comprising a replacement of (3.11)-(3.12) by

$$\widetilde{Q}^i(x) = \tfrac{1}{2}(a_0 + \widetilde{\alpha}_0^i) + \sum_{k\,=\,1}^{4} [(a_k + \widetilde{\alpha}_k^i)\cos k\tfrac{\pi}{L}(x + \widetilde{\delta}_k^i) + (b_k + \widetilde{\nu}_k^i)\sin k\tfrac{\pi}{L}(x + \widetilde{\varepsilon}_k^i)] \approx f(x)\ , \tag{3.16}$$

with

$$\tfrac{1}{2}(a_0 + \widetilde{\alpha}_0^i) + \sum_{k\,=\,1}^{4} [(a_k + \widetilde{\alpha}_k^i)\cos k\tfrac{\pi}{L}(x_j^i + \widetilde{\delta}_k^i) + (b_k + \widetilde{\nu}_k^i)\sin k\tfrac{\pi}{L}(x_j^i + \widetilde{\varepsilon}_k^i)] \approx \widetilde{Q}^{i-1}(x_j^i),$$
$$j = 1, 2, 3, \ldots.., 17. \tag{3.17}$$

The stopping rule obviously transforms then to

$$\left\| \widetilde{Q}^i(x) - \widetilde{Q}^{i-1} \right\| \le \epsilon_s,$$

but the subtle relationship between $Q^N(x)$ and $\widetilde{Q}^N(x)$ seems to remain as an open question.

Also here, instead of (3.17) for determining the $\widetilde{\mathfrak{F}}^i$ set in (3.16), we may solve the nonlinear programming problem

$$\underset{\widetilde{\mathfrak{F}}^i}{Minimize} \left\| \ \widetilde{\eta}^i \right\| + \left\| \ \widetilde{\zeta}^i \right\| \tag{3.18}$$

$$Subject\ to : (3.17),\ j = 1, 2, 3, \ldots.., 17. \tag{3.19}$$

Definition 3.1. The five-terms harmonic series $\hat{S}(x) = Q^N(x)$, or $\widetilde{Q}^N(x)$, generated by the MSI algorithm, is called the minimal series representation of the infinite series $S(x)$.

Remark 3.1. The superscript 4 in (3.3) appears remarkably to be rather like a magical number. Indeed why 4 ? and not 75, for example. Also why the MSI algorithm requires only 17 interpolation points? A magic that perhaps stems from the underlying Hilbert's Thirteenth problem, [11], solvable by a more general form of (3.3). The validity of the previous result is unquestionably a qualitative indicator of the expected fast convergence of the Fourier series for most periodic $f(x) \in \mathcal{C}(R)$.

Remark 3.2. Our theorem, proposition and algorithm do not hold, of course, for discontinuous periodic functions, i.e. when $f(x) \notin \mathcal{C}(R)$. A consequence of the well known slow convergence of the pertaining Fourier series associated with the accompanying Gibbs effects [9].

4. Diverging Fourier Series

Let us revisit Fejer's example of $f \in \mathcal{C}(R)$ with a divergent, at $x = 0$, Fourier series (1.2). Its Fourier coefficients can easily be shown to satisfy

$$a_k = \tfrac{2}{\pi} \sum_{p\,=1}^{\infty} \tfrac{1}{p^2}\ \ \mu_{k,\ 2^{\,p^3 - 1}}, \tag{4.1}$$

with

$$\mu_{k,\ m} = \int_0^{\pi} \sin\left[(2m\ + 1)\tfrac{t}{2}\right] \cos kt\ dt. \tag{4.2}$$

Trying to draw $f(x)$ via (1.2) using (4.1)-(4.2) with basic software could be too slow, even for the first few terms. However, a celebrated theorem by Fejér [2] says that when $f \in \mathcal{C}(R)$, the Cesàro means $\sigma_n(x) = \sigma_n(f)$ converge to f, not only pointwise, but uniformly, [12]. Hence although the series (1.2) diverges at $x = 0$, it is $\mathcal{C}_1$ summable. Moreover, if the Cesàro means turn out to be computationally too expensive, then one can resort instead to Poisson-Abel summability [2].

Alternatively, $\lambda-$permutations, [4, 13], may be used to investigate this divergent Fourier series.. Here, incidentally, identification of convergence preserving permutations is a problem to be faced.

In concluding this subsection, it should be mentioned that the class of linear summabilities is not restricted to $\sigma_n(x)$ averaging. The Poisson-Abel, [2], and Riesz-Nörlund, [19], summabilities are also linear but both are defined in the context of a limiting asymptotic process. In particular, a sequence $< S_n >$ is said to be harmonically $\mathcal{H}_1$ summable (in the Riesz-Nörlund sense) if

$$\Omega(x) = \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{S_{n-k}(x)}{(k+1)}$$

exists. Riesz proved, additionally, [19], that every $S(x)$ that is $\mathcal{H}_1$ summable is also $\mathcal{C}_1$ summable.

5. Enhanced Smoothing By summation

Despite the settlement, in section 3, of the question on existence of the minimal series $\hat{S}(x)$, its construction happens to be essentially computational. As in any interpolational procedure, one should expect errors to follow the nature of the set of interpolation points used, see e.g. [5]. This motivates a need for finding an optimal subdivision scheme. A problem that is minimized, however, in the MSI algorithm by resort to random sampling that is governed solely by the assigned tolerance ϵ_s. Moreover, too high degree of interpolation may turn out some times to be pathalogical, [16-7], by inducing virtual rapid oscillations. Such a problem is anticipated to be avoided in the MSI by the fact that is intrinsically a low degree (17 points) interpolation.

Additional smoothness, though may not be essential, can however be guaranteed by processing a summation series e.g. the Cesàro-Fejér series $\sigma(x)$, to find a $\widehat{\sigma}(x)$, instead of the direct Fourier series $S(x)$.It should be noted though that any enhanced smoothing to be brought about by summation is tautologically "forced", and has nothing to do with estimation of the underlying derivatives of $f(x)$.

5.1. Smoothing enhancement by linear summation. It is well known that $S(x)$ of a $2L-$ periodic $f \in \mathcal{C}[0, 2L]$, whose $n-$th partial sum is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos k\frac{\pi}{L}x + b_k \sin k\frac{\pi}{L}x \right), \tag{5.1}$$

can be uniformly summable to $f(x)$ by the Cesàro-Fejér method, see e.g. [17], [2], of arithmetic means

$$\sigma_n(x) = \frac{1}{(n+1)} \sum_{k=0}^{n} S_k(x), \tag{5.2}$$

which are representable , [2], in terms of the Fejér kernel

$$F_n(x) = \frac{1}{(n+1)} \left(\frac{\sin\frac{n+1}{2}\frac{\pi}{L}x}{\sin\frac{1}{2}\frac{\pi}{L}x} \right)^2, \tag{5.3}$$

as the Fejér integrals

$$\sigma_n(x) = \tfrac{1}{2\,L}\int_{-L}^{L} F_n(x)\ f(x+\tau)\ d\tau. \tag{5.4}$$

Furthermore, it can readily be shown , see e.g. [15], that

$$\sigma_n(x) = \tfrac{a_0}{2} + \sum_{k\,=\,1}^{n} \tfrac{n\,+1-k}{n\,+1}\left(a_k \cos k\tfrac{\pi}{L}x + b_k \sin k\tfrac{\pi}{L}x\right), \tag{5.5}$$

from which it follows that

$$\lim_{n\,\to\,\infty}\ \sigma_n(x) = \sigma(x) = S(x). \tag{5.6}$$

This takes place despite the fact that $\sigma_n(x)$ differs from $S_n(x)$ for finite n.

Moreover, $\sigma(x)$, like $S(x)$, can converge in a norm to $f(x)$ viz $\|f-\sigma\| = \sup_{[0,2L]} |f(x)-\sigma(x)| \to 0$.Hence the limit $\sigma(x)$ can exist even when the sequence $< S_n(x) >$ may diverge. A limit that justifies saying that $S(x)$ is $\mathcal{C}_1$ summable to $\sigma(x)$.

In an attempt towards enhanced smoothing of $\hat{S}(x) = Q^N(x)$, it is possible now to conceive an associated $\widehat{\sigma}(x)$, which can be derived starting from $\sigma(x)$ of (3.5).

Minimal Linear Summation Series.

Proposition 5.1. *For Fourier series representation of a 2L- periodic signal $f(x) \in \mathcal{C}(R)$, variation of only the first 5 harmonics, forming a* minimal $\mathcal{C}_1$ summation series representation*, $\widehat{\sigma}(x)$, suffices to reproduce the entire signal.*

Quantitatively,

$$\widehat{\sigma}(x) = \sum_{k\,=\,0}^{4} V_k[u_k(x)] = \sum_{k\,=\,0}^{4} \left[\overline{A}(a_k,b_k)\cos k\tfrac{\pi}{L}\mu_k(x) + \overline{B}(a_k,b_k)\sin k\tfrac{\pi}{L}w_k(x)\right] \tag{5.7}$$

$$= \tfrac{a_0}{2} + \sum_{k\,=\,1}^{\infty} \tfrac{n\,+1-k}{n\,+1}\left(a_k \cos k\tfrac{\pi}{L}x + b_k \sin k\tfrac{\pi}{L}x\right) = \ \sigma(x)\ \ = f(x),$$

in which

$$\overline{A}(a_k,b_k) = \left(\overline{p}(a_k) = \begin{cases} \tfrac{1}{2}(a_0\,+\,\alpha_0), k=0 \\ \left(\tfrac{5\,-\,k}{5}\right)(a_k\,+\,\alpha_k), k\geq 1 \end{cases}, 0\right),$$

$$\overline{B}(a_k,b_k) = \left(0,\ \overline{q}(b_k) = \begin{cases} 0, k=0 \\ \left(\tfrac{5\,-\,k}{5}\right)(b_k\,+\,\nu_k), k\geq 1 \end{cases}\right),$$

$u_k(x) = a_k \cos 2k\tfrac{\pi}{L}\mu_k(x) + b_k \cos 2k\tfrac{\pi}{L}w_k(x)$,

$\mu_k(x) = x+\delta_k$ and $w_k(x) = x+\varepsilon_k$,

holds with the $V_k[\cdot]$ operator's set

$$\mathfrak{G} = \Big\{\ V_k[\cdot] = \overline{A}(a_k,b_k)\ \sqrt{\tfrac{1}{4k\pi\ \mu_k(x)}\cos^{-1}(\cdot) +\ \tfrac{1}{2}(\cdot)}$$
$$+\ \overline{B}(a_k,b_k)\ \sqrt{\tfrac{1}{4k\pi\ w_k(x)}\cos^{-1}(\cdot) -\ \tfrac{1}{2}(\cdot)}\ \Big\}_{k\,=\,0}^{4}. \tag{5.8}$$

As with $S(x)$, $\sigma(x)$ can be processed by the MSI algorithm to yield a $\widehat{\sigma}(x)$.

Algorithm (Summation MSI) **2.** The set $\mathfrak{G} = \{U_k[\cdot]\ ,\ u_k(x)\}_{k\,=\,0}^{4}$ for the five-terms minimal summation series representation, $\widehat{\sigma}(x)$,can be found as follows.

Step 1: Approximate solution. Proposition 5.2 motivates superindexing of the $\mathfrak{G}$ set to $\mathfrak{G}^i, i = 1,2,3,.....,N$.The i-th iteration of the interpolation set $\{x_r^i\}_{r\,=1}^{17} \subset [0,2L]$, in the sense of [8], leads to

$$G^i(x) = \tfrac{1}{2}(a_0+\alpha_0^i) + \sum_{k\,=\,1}^{4}\left(\tfrac{5\,-\,k}{5}\right)[(a_k+\alpha_k^i)\cos k\tfrac{\pi}{L}(x+\delta_k^i)$$
$$+(b_k\,+\,\nu_k^i)\sin k\tfrac{\pi}{L}(x+\varepsilon_k^i)] \approx \sigma(x)\ , \tag{5.9}$$

with the set $\mathfrak{G}^i = \left\{\alpha_0^i,\ \delta_k^i,\ \varepsilon_k^i, \alpha_k^i,\ \nu_k^i\right\}_{k=1}^{4}$ determined via solving the system

$\frac{1}{2}(a_0+\alpha_0^i)+\sum_{k=1}^{4}\left(\frac{5-k}{5}\right)[(a_k+\alpha_k^i)\cos k\frac{\pi}{L}(x_r^i+\delta_k^i)+(b_k+\nu_k^i)\sin k\frac{\pi}{L}(x_r^i+\varepsilon_k^i)]\approx$
$\sigma(x_r^i)$,
$r = 1, 2, 3, \ldots\ldots, 17.$ (5.10)

The stopping rule, $i = N$, for the iterations is when

$$\left\|G^i(x) - G^{i-1}(x)\right\| \leq \epsilon_\sigma.$$

Step 2: Fine tuning. The summation series symmetry deformation measures of (3.13) become

$$\left\|\ \varrho^i(x)\right\| = \sum_{k=0}^{4}\int_{-L}^{L}\left(\frac{5-k}{5}\right)\left|(a_k+\alpha_k^i)(\sin k\tfrac{\pi}{L}\delta_k^i)\sin k\tfrac{\pi}{L}x\right|\,dx,$$

$$\left\|\ \vartheta^i(x)\right\| = \left|\tfrac{1}{2}\alpha_0^i\right| + \sum_{k=1}^{4}\int_{-L}^{L}\left(\frac{5-k}{5}\right)\left|(b_k+\nu_k^i)(\sin k\tfrac{\pi}{L}\varepsilon_k^i)\cos k\tfrac{\pi}{L}x\right|\,dx. \quad (5.11)$$

Hence, instead of (5.10) for determining the $\mathfrak{G}^i$ set in (5.9), we may solve the nonlinear programming problem

$\underset{\mathfrak{G}^i}{Minimize}\left\|\ \varrho^i(x)\right\| + \left\|\ \vartheta^i(x)\right\|,$ (5.12)

$Subject\ to$: (5.10), $r = 1, 2, 3, \ldots\ldots, 17.$ (5.13)

Here also one can always modify the $\mathfrak{G}^0 \to \mathfrak{G}^i \to \mathfrak{G}^N$ process to a $\widetilde{\mathfrak{G}}^0 \to \widetilde{\mathfrak{G}}^i = \left\{\ \widetilde{\alpha}_0^i,\ \widetilde{\delta}_k^i,\ \widetilde{\varepsilon}_k^i\ ,\ \widetilde{\alpha}_k^i,\ \widetilde{\nu}_k^i)\right\}_{k=1}^{4} \to \widetilde{\mathfrak{G}}^N$ process, comprising a replacement of (5.9)-(5.10) by

$$\widetilde{G}^i(x) = \tfrac{1}{2}(a_0+\widetilde{\alpha}_0^i) + \sum_{k=1}^{4}\left(\frac{5-k}{5}\right)[(a_k+\widetilde{\alpha}_k^i)\cos k\tfrac{\pi}{L}(x+\widetilde{\delta}_k^i)$$
$$+(b_k+\widetilde{\nu}_k^i)\sin k\tfrac{\pi}{L}(x+\widetilde{\varepsilon}_k^i)] \approx \sigma(x)\ , \quad (5.14)$$

with

$\frac{1}{2}(a_0+\widetilde{\alpha}_0^i)+\sum_{k=1}^{4}\left(\frac{5-k}{5}\right)[(a_k+\widetilde{\alpha}_k^i)\cos k\frac{\pi}{L}(x_r^i+\widetilde{\delta}_k^i)+(b_k+\widetilde{\nu}_k^i)\sin k\frac{\pi}{L}(x_r^i+\widetilde{\varepsilon}_k^i)]\approx$
$\widetilde{Q}^{i-1}(x_r^i)$,
$r = 1, 2, 3, \ldots\ldots, 17.$ (5.15)

The stopping rule obviously transforms then to

$$\left\|\widetilde{G}^i(x) - \widetilde{G}^{i-1}\right\| \leq \epsilon_\sigma,$$

with the relationship between $G^N(x)$ and $\widetilde{G}^N(x)$ remaining as an open question.

Also here, instead of (5.15) for determining the $\widetilde{\mathfrak{G}}^i$ set in (5.14), we may solve the nonlinear programming problem

$\underset{\widetilde{\mathfrak{G}}^i}{Minimize}\left\|\ \widetilde{\varrho}^i\right\| + \left\|\ \widetilde{\vartheta}^i\right\|$

$Subject\ to$: (5.15), $j = 1, 2, 3, \ldots\ldots, 17.$

Definition 5.1. The five-terms linear summation series $\widehat{\sigma}(x)$, generated by the MSI algorithm, is called the minimal $\mathcal{C}_1$ summation series representation of the infinite series $S(x)$.

Conjecture 5.1. *The reported five-terms iterative minimal series $\widehat{S}(x)$ approximant to a continuous function with a divergent Fourier series $S(x)$ can equate the minimal summation estimate $\widehat{\sigma}(x)$ for such series.*

Only a sketch of the proof to this conjecture can possibly be contemplated in the mean time. On one hand, the Kolmogorov-Arnold theorem, that underlies the five-term representation (3.4) or (3.5), is exact when $\phi(x,x) = f(x) \in \mathcal{C}(R)$. On the other hand, all summability methods eliminate the divergence trend in the infinite Fourier series. This paves the way towards a possibility for a confluence between the two facts. Moreover, both $\hat{S}(x)$ and $\widehat{\sigma}(x)$ are derived, within possibly the same tolerance $\epsilon_s = \epsilon_\sigma$, from the same series $S(x) = \sigma(x)$. A fact that in no way suggests that $\sigma_n(x) = S_n(x)$, for finite n.

6. Conclusion

In this paper, we employ the Kolmogorov-Arnold theorem to represent infinite the Fourier series for most continuous periodic functions. The emerging, from the MSI algorithm, five-term, with 17 interpolation points, minimal series $\hat{S}(x)$ and associated minimal summation series $\widehat{\sigma}(x)$ are respectively defined by the random interpolation sets $\{x_j^i\}_{j=1}^{17}$ and $\{x_r^i\}_{r=1}^{17}$. Sets that are controllable, though indirectly, by the tolerances ϵ_s , ϵ_σ and pose a number of new open problems. One of them, is the relationship between the $Q^N(x)$ and $\widetilde{Q}^N(x)$ (or the $G^N(x)$ and $\widetilde{G}^N(x)$) approximants of the reported MSI algorithm. This also includes the relationship between $Q^N(x)$ and $G^N(x)$, especially for div egent $S(x)$. The other is a rigorous proof for the above summability conjectue.

Appendix. *On the Kolmogorov-Arnold Theorem.*

The special forms of $\phi(x,y)$, most encountered in mathematical physics or engineering, are:

(i) Additively separable : $\phi(x,y) = g(x) + g(y)$.
(ii) Multiplicatively separable : $\phi(x,y) = g(x)h(y)$.
(iii) Travelling waves : $\phi(x,y) = g(x+y) + h(x-y)$.
(iv) Homogeneous of $n-$th degree : $\phi(x,y) = t^{-n}\ \phi(tx,ty)$.
(v) Bose-Einstein particle symmetric amplitude : $\phi(x,y) = g(x)h(y)+g(y)h(x) = \phi(y,x)$,
Fermi particle skew symmetric amplitude : $\phi(x,y) = g(x)h(y)-g(y)h(x) = -\phi(y,x)$.

All these forms, and many others, can be derived as special cases, by the Kolmogorov-Arnold theorem (3.3):

$$\phi(x,y) = \sum_{k=0}^{4} U_k[\ g_k(x) + h_k(y)].$$

Obviously, case (i) corresponds to (3.3) when

$g_1(x) = g(x)$, $h_1(y) = h(y)$, $U_1 = 1$; $U_0 = 0$, $U_2 = U_3 = U_4 = 0$.

Case (ii) derives from (9) via the map $\phi(x,y) = e^{p(x,y)}$ when $p(x,y)$ is identified, like in case (i), with (9). Finally, case (iii) corresponds to (9) when

$g_1(x) = x$, $h_1(y) = y$, $U_1 = g$; $g_2(x) = x$, $h_2(y) = -y$, $U_2 = h$; $U_0 = U_3 = U_4 = 0$.

We stop here and refer the interested reader to [10] for a detailed account on this fundamental theorem.

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