SUMMATION FORMULA FOR GENERALIZED DISCRETE 
$q$-HERMITE II POLYNOMIALS

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ABSTRACT. In this paper, we provide a family of generalized discrete $q$-Hermite II polynomials denoted by $\tilde{h}_{n,\alpha}(x, y|q)$. An explicit relations connecting them with the $q$-Laguerre and Stieltjes-Wigert polynomials are obtained. Summation formula is derived by using different analytical means on their generating functions.

1. INTRODUCTION

In their paper, Álvarez-Nodarse et al [2], have introduced a $q$-extension of the discrete $q$-Hermite II polynomials as:

$$H^{(\mu)}_{2n}(x; q) := (-1)^n(q; q)_n L^{(\mu-1/2)}_n(x^2; q)$$

$$H^{(\mu)}_{2n+1}(x; q) := (-1)^n(q; q)_n x L^{(\mu+1/2)}_n(x^2; q)$$

where $\mu > -1/2$, $L^{(\alpha)}_n(x; q)$ are the $q$-Laguerre polynomials given by

$$L^{(\alpha)}_n(x; q) := \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \Phi_1 \left( \frac{q^{-n}}{q^{\alpha+1}} \bigg| q; q^{\alpha+1+1}x \right)$$

with $(a; q)_0 = 1$, $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$, $n = 1, 2, \cdots$, the $q$-shifted factorial, and

$$\Phi_s \left( \frac{q^{-n}, a_2, \cdots, a_r}{b_1, b_2, \cdots, b_s} \bigg| q; x \right) =$$

$$\sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (a_2; q)_k \cdots (a_r; q)_k}{(b_1; q)_k (b_2; q)_k \cdots (b_s; q)_k} \frac{x^k}{(q; q)_k} \left[ (-1)^k q^{k(k-1)/2} \right]^{1+s-r}$$

2000 Mathematics Subject Classification. 33C45, 33D15, 33D50.

Key words and phrases. Basic orthogonal polynomials; discrete $q$-Hermite II polynomials; connection formula.

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Communicated by H. M. Srivastava.
the usual generalized basic or q-hypergeometric function of degree $n$ in the variable $x$ (see Slater [10] Chap. 3, Srivastava and Karlsson [11] p.347, Eq. (272)) for details). For $\mu = 0$ in (1.1), the polynomials $H_n^{(0)}(x;q)$ correspond to the discrete $q$-Hermite II polynomials [1,8], i.e., $H_n^{(0)}(x;q^2) = q^{n(n-1)}\tilde{h}_n(x;q)$. They show that the polynomials $H_n^{(\mu)}(x;q)$ satisfy the orthogonality relation

$$
\int_{-\infty}^{\infty} H_n^{(\mu)}(x;q) H_m^{(\mu)}(x;q) \omega(x) dx = \pi q^{-n/2}(q^{1/2};q^{1/2})_n(q^{1/2};q)_{1/2} \delta_{nm} \tag{1.4}
$$
on the whole real line $\mathbb{R}$ with respect to the positive weight function $\omega(x) = 1/(-x^2;q)_\infty$. A detailed discussion of the properties of the polynomials $H_n^{(\mu)}(x;q)$ can be found in [2].

Recently, Saley Jazmat et al [7], introduced a novel extension of discrete $q$-Hermite II polynomials by using new $q$-operators. This extension is defined as:

$$
\tilde{h}_{2n,\alpha}(x;q) = (-1)^n q^{-n(2n-1)} \frac{(q;q)_{2n}}{(q^{2n+2};q^2)_n} L_n^{(\alpha)}(x^2 q^{-2\alpha-1};q^2) \tag{1.5}
$$

$$
\tilde{h}_{2n+1,\alpha}(x;q) = (-1)^n q^{-n(2n+1)} \frac{(q;q)_{2n+1}}{(q^{2n+4};q^2)_{n+1}} x L_n^{(\alpha)}(x^2 q^{-2\alpha-1};q^2) .
$$

For $\alpha = -1/2$ in (1.5), the polynomials $\tilde{h}_{n,-1/2}(x;q)$ correspond to the discrete $q$-Hermite II polynomials, i.e., $\tilde{h}_{n,-1/2}(x;q) = \tilde{h}_n(x;q)$. The generalized discrete $q$-Hermite II polynomials (1.5) satisfy the orthogonality relation

$$
\int_{-\infty}^{+\infty} \tilde{h}_{n,\alpha}(x;q) \tilde{h}_{m,\alpha}(x;q) \omega_{\alpha}(x;q) |x|^{2\alpha+1} d_q x \tag{1.6}
$$

$$
= \frac{2q^{-n^2} (1-q)(-q, -q, q^2; q^2)_\infty}{(q^{-2\alpha-1}, q^{2\alpha+3}, q^{2\alpha+2}; q^2)_\infty} \frac{(q;q)_n^2}{(q;q)_{n+1}} \delta_{n,m}
$$
on the whole real line $\mathbb{R}$ with respect to the positive weight function $\omega_{\alpha}(x) = 1/(-q^{-2\alpha-1} x^2;q^2)_\infty$. A detailed discussion of the properties of the polynomials $\tilde{h}_{n,\alpha}(x;q)$ can be found in [7].

Srivastava and Jain [12] [6], investigated multilinear generating functions for $q$-Hermite, $q$-Laguerre polynomials and other special functions. Relevant connections of these multilinear generating functions with various known results for the classical or $q$-Hermite polynomials are also indicated. They also proved many combinatorial $q$-series identities by applying the theory of $q$-hypergeometric functions (see [6], for more details).

Motivated by Saley Jazmat’s [7] and Srivastava et al [12] [5] works, our interest in this paper is to introduce new family of “generalized discrete $q$-Hermite II polynomials (in short gdq-H2P) $\tilde{h}_{n,\alpha}(x,y;q)$” which is an extension of the generalized discrete $q$-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ and investigate summation formulæ.

The paper is organized as follows. In Section 2, we recall notations to be used in the sequel. In Section 3, we define a gdq-H2P $\tilde{h}_{n,\alpha}(x,y;q)$ and investigate several properties. In Section 4, we derive summation and inversion formulæ for gdq-H2P $\tilde{h}_{n,\alpha}(x,y;q)$. In Section 5, concluding remarks are given.
2. Notations and Preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer to the general references [4, 8] and [7] for the definitions and notations. Throughout this paper, we assume that $0 < q < 1$, $\alpha > -1$.

For a complex number $a$, the $q$-shifted factorials are defined by:

\[
(a; q)_0 = 1; \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \cdots; \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) \tag{2.1}
\]

and the $q$-number is defined by:

\[
[n]_q = \frac{1 - q^n}{1 - q}, \quad [n]_q! := \prod_{k=1}^{n} [k]_q, \quad 0!_q := 1, \quad n \in \mathbb{N}. \tag{2.2}
\]

Let $x$ and $y$ be two real or complex numbers, the Hahn [5] $q$-addition $\oplus_q$ of $x$ and $y$ is given by:

\[
(x \oplus_q y)^n := (x + y)(x + qy) \cdots (x + q^{n-1}y) = (q; q)_n \sum_{k=0}^{n} \frac{q^{k^2}x^{n-k}y^k}{(q; q)_k(q; q)_{n-k}}, \quad n \geq 1, \quad (x \oplus_q y)^0 := 1, \tag{2.3}
\]

while the $q$-subtraction $\ominus_q$ is given by

\[
(x \ominus_q y)^n := (x \oplus_q (-y))^n. \tag{2.4}
\]

The generalized $q$-shifted factorials [7] are defined by the recursion relations

\[
[n + 1]_{q, \alpha}! = [n + 1 + \theta_n(2\alpha + 1)]_q [n]_{q, \alpha}!, \tag{2.5}
\]

and

\[
(q; q)_{n+1, \alpha} = (1 - q)[n + 1 + \theta_n(2\alpha + 1)]_q(q; q)_{n, \alpha}, \tag{2.6}
\]

where

\[
\theta_n = \begin{cases} 
1 & \text{if } n \text{ even} \\
0 & \text{if } n \text{ odd}.
\end{cases} \tag{2.7}
\]

Remark that, for $\alpha = -1/2$, we have

\[
(q; q)_{n, -1/2}! = (q; q)_n, \quad [n]_{q, -1/2}! = (1 - q)^n(q; q)_n. \tag{2.8}
\]

We denote

\[
(q; q)_{2n, \alpha} = (q^2; q^2)_n(q^{2\alpha + 2}; q^2)_n, \tag{2.9}
\]

and

\[
(q; q)_{2n+1, \alpha} = (q^2; q^2)_n(q^{2\alpha + 2}; q^2)_{n+1}. \tag{2.10}
\]

The two Euler’s $q$-analogues of the exponential functions are given by [4]

\[
E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k^2}x^k}{(q; q)_k} = (-x; q)_\infty \tag{2.11}
\]

and

\[
e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k} = \frac{1}{(x; q)_\infty}, \quad |x| < 1. \tag{2.12}
\]
For \( m \geq 1 \) and by means of the generalized \( q \)-shifted factorials, we define two generalized \( q \)-exponential functions as follows

\[
\hat{E}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{q^{mk(k-1)/2}x^k}{(q^m; q^m)_{k,\alpha}},
\]  

(2.13)

and

\[
\hat{e}_{q^m,\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{(q^m; q^m)_{k,\alpha}}, \quad |x| < 1.
\]  

(2.14)

Remark that, for \( m = 1 \) and \( \alpha = -\frac{1}{2} \), we have:

\[
\hat{E}_{q,\alpha}(x) = E_q(x), \quad \hat{e}_{q,\alpha}(x) = e_q(x).
\]  

(2.15)

For \( m = 2 \), the following elementary result is useful in the sequel to establish the summation formulae for gdq-H2P:

\[
\hat{e}_{q^2,\frac{1}{2}}(x)\tilde{E}_{q^2,\frac{1}{2}}(y) = \hat{e}_{q^2,\frac{1}{2}}(x \oplus q^2 y),
\]  

(2.16)

\[
\hat{e}_{q^2,\frac{1}{2}}(x)\tilde{E}_{q^2,\frac{1}{2}}(-y) = \hat{e}_q(x \ominus q^2 y), \quad \hat{e}_{q^2,\frac{1}{2}}(x)\tilde{E}_{q^2,\frac{1}{2}}(-x) = 1,
\]  

(2.17)

where

\[
(a \ominus q^2 b)^n := n!q^{n/2} \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)}}{(n-k)!q^{k/2}} a^{n-k} b^k, \quad (a \ominus q^2 b)^0 := 1.
\]  

(2.18)

### 3. Generalized discrete \( q \)-Hermite II polynomials

In this section, we introduce a sequence of gdq-H2P \( \{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty} \). Several properties related to these polynomials are derived.

**Definition 3.1.** For \( x, y \in \mathbb{R} \), the gdq-H2P \( \{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty} \) are defined by:

\[
\tilde{h}_{n,\alpha}(x, y|q) := (q; q)_n \sum_{k=0}^{[n/2]} \frac{(-1)^k q^{-2nk+k(2k+1)} x^{n-2k} y^k}{(q; q)_{n-2k,\alpha} (q^2; q^2)_k}
\]  

(3.1)

and

\[
\tilde{h}_{n,\alpha}(x, 0|q) := \frac{(q; q)_n}{(q; q)_{n,\alpha}} x^n.
\]  

(3.2)

Remark that,

1. for \( y = 1 \), we get

\[
\tilde{h}_{n,\alpha}(x, 1|q) = \tilde{h}_{n,\alpha}(x; q)
\]  

(3.3)

where \( \tilde{h}_{n,\alpha}(x; q) \) is the generalized discrete \( q \)-Hermite II polynomial \[7\];

2. for \( \alpha = -\frac{1}{2} \) and \( y = 1 \), we have

\[
\tilde{h}_{n, -1/2}(x, 1|q) = \tilde{h}_{n}(x; q).
\]  

(3.4)

where \( \tilde{h}_{n}(x; q) \) is the discrete \( q \)-Hermite II polynomial \[8\] \[9\].

3. Indeed since

\[
\lim_{q \to 1} (q^n; q)_n = (a)_n
\]  

(3.5)

one readily verifies that

\[
\lim_{q \to 1} \frac{\tilde{h}_{n, -1/2}(\sqrt{1-q^2} x, 1|q)}{(1-q^2)^{n/2}} = \frac{\tilde{h}_{n}^{\alpha+\frac{1}{2}}(x)}{2^n}
\]  

(3.6)
where $h_n^{\alpha+\frac{1}{2}}(x)$ is the Rosenblums generalized Hermite polynomial [9].

**Lemma 3.2.** The following recursion relation for gdq-H2P \{$\tilde{h}_{n,\alpha}(x, y|q)$\}$_{n=0}^\infty$ holds true.

$$\frac{1 - q^{n+1} + \theta_n(2\alpha + 1)}{1 - q^{n+1}} \tilde{h}_{n+1,\alpha}(x, y|q) = x\tilde{h}_{n,\alpha}(x, y|q) - y q^{-2n+1}(1 - q^n)\tilde{h}_{n-1,\alpha}(x, y|q).$$  \hspace{1cm} (3.7)

**Proof.** To prove the assertion (3.7), we consider separately even and odd cases of the expression

$$x\tilde{h}_{n,\alpha}(x, y|q) - y q^{-2n+1}(1 - q^n)\tilde{h}_{n-1,\alpha}(x, y|q).$$  \hspace{1cm} (3.8)

For $n$ even, we have:

$$x\tilde{h}_{2n,\alpha}(x, y|q) = (q; q)_{2n+1}^{2n+1} + (q; q)_{2n+1} \sum_{k=1}^{n} \frac{(-1)^k q^{-2n+k+k(2k+1)x+2n+k} y^k}{(q; q)_{2n+1-k}\alpha (q^2; q^2)_k}.$$  \hspace{1cm} (3.9)

The right-hand side of the last relation can be written as

$$\frac{(q; q)_{2n+1}^{2n+1} + (q; q)_{2n+1}}{(q; q)_{2n+1-k}\alpha (q^2; q^2)_k} \left[ q^{2k}(1 - q^{2n+2+2\alpha-2k}) \right].$$  \hspace{1cm} (3.10)

In the same way,

$$- y q^{-4n+1}(1 - q^{2n})\tilde{h}_{2n-1,\alpha}(x, y|q) = -y q^{-4n+1}(q; q)_{2n} \sum_{k=0}^{n-1} \frac{(-1)^k q^{-2n+1+k(2k+1)x} y^k}{(q; q)_{2n+1-k}\alpha (q^2; q^2)_k}.$$  \hspace{1cm} (3.11)

Change $k$ to $k - 1$ in (3.10), we obtain

$$\frac{(q; q)_{2n}^{2n+1}}{(q; q)_{2n+1-k}\alpha (q^2; q^2)_k} \left[ q^{2k}(1 - q^{2n+2+2\alpha-2k}) + (1 - q^{2k}) \right].$$  \hspace{1cm} (3.12)

Then combining (3.9) and (3.11), we have

$$x\tilde{h}_{2n,\alpha}(x, y|q) - y q^{-4n+1}(1 - q^{2n})\tilde{h}_{2n-1,\alpha}(x, y|q) = \frac{(q; q)_{2n}^{2n+1}}{(q; q)_{2n+1-k}\alpha (q^2; q^2)_k} \left[ q^{2k}(1 - q^{2n+2+2\alpha-2k}) + (1 - q^{2k}) \right].$$  \hspace{1cm} (3.13)
Summarizing the above calculations in (3.12)-(3.13), we get the assertion (3.17) for \( n \) even. In the odd case, the proof follows the same steps as the even case.

**Theorem 3.3.** We have:

\[
\lim_{\alpha \rightarrow +\infty} \tilde{h}_{2n,\alpha}(x, y|q) = q^{-n(2n-1)}(q; q)_{2n}(y|q)_{2n}(x^{2}y^{-1}q^{-1}; q^{2})
\]

(3.14)

and

\[
\lim_{\alpha \rightarrow +\infty} \tilde{h}_{2n+1,\alpha}(x, y|q) = q^{-n(2n+1)}(q; q)_{2n+1}(y|q)_{2n+1}(x^{2}y^{-1}q^{-1}; q^{2})
\]

(3.15)

where \( S_n(x; q) \) are the Stieltjes-Wigert polynomials \([8]\).

In order to prove Theorem 3.3, we need the following Lemma.

**Lemma 3.4.** For \( \alpha > -1 \), the sequence of gdq-H2P \( \{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty} \) can be written in terms of q-Laguerre polynomials \( L_n^{(\alpha)}(x; q) \) as

\[
\tilde{h}_{2n,\alpha}(x, y|q) = q^{-n(2n-1)}(q; q)_{2n}(y|q)_{2n}(x^{2}y^{-1}q^{-2\alpha-1}; q^{2})
\]

(3.16)

and

\[
\tilde{h}_{2n+1,\alpha}(x, y|q) = q^{-n(2n+1)}(q; q)_{2n+1}(y|q)_{2n+1}(x^{2}y^{-1}q^{-2\alpha-1}; q^{2})
\]

(3.17)

In order to prove Lemma 3.4, we need the following Proposition.

**Proposition 3.5.** For \( \alpha > -1 \), the sequence of gdq-H2P \( \{\tilde{h}_{n,\alpha}(x, y|q)\}_{n=0}^{\infty} \) can be written in terms of basic hypergeometric functions as

\[
\tilde{h}_{n,\alpha}(x, y|q) = (q; q)_{n}(q; q)_{\alpha, n} x^{n} \Phi_{1}\left(\begin{array}{c}
q^{-\alpha}, q^{-n-2\alpha} \\
n, 0 \end{array}\right)|q^{-1}x^{2}; -y q^{2\alpha+3}/x^{2}\right)
\]

(3.18)

Proof. In fact, for \( n \) even, and by using

\[
(q; q)_{2n-2k,\alpha} = (q^{2}; q^{2})_{n-k}(q^{2\alpha+2}; q^{2})_{n-k},
\]

(3.19)

the gdq-H2P \( \tilde{h}_{n,\alpha}(x, y|q) \) defined in (3.1) can be rewritten as

\[
\tilde{h}_{2n,\alpha}(x, y|q) = (q; q)_{2n} \sum_{k=0}^{n} (-1)^{k} q^{4nk+k(k+1)} x^{2n-2k} y^{k}
\]

(3.20)

From the formula [8] p.9, Eq. (0.2.12)

\[
(a; q)_{n-k} = \frac{(a; q)_{n}}{(a^{1-n}; q)_{k}} \left(\frac{q}{a}\right)^{k} q^{\binom{k}{2}} \sum_{k=0}^{2n} \frac{n}{(q^{2}; q^{2})_{n-k}(q^{2\alpha+2}; q^{2})_{n-k}(q^{2}; q^{2})_{k}}
\]

(3.21)

we have for \( a = q^{2} \) and \( q^{2\alpha+2} \),

\[
\tilde{h}_{2n,\alpha}(x, y|q) = (q; q)_{2n} x^{2n} \sum_{k=0}^{n} (-1)^{k} q^{-4nk+k(k+1)} q^{-2n-2\alpha} q^{2\alpha+3} \frac{y^{k}}{x^{2}}
\]

(3.22)

After simplification, the last equation reads

\[
\tilde{h}_{2n,\alpha}(x, y|q) = (q; q)_{2n} x^{2n} \sum_{k=0}^{n} \frac{(q^{2}; q^{2})_{k} q^{4nk+k(k+1)} (y^{2} q^{2\alpha+3})^{k}}{(x^{2})^{k}}
\]

(3.22)

In the odd case, the proof follows the same steps as the even case.

Now, we are in position to prove Lemma 3.3.
Proof. (of Lemma 3.3) For \( n \) even, the relation (3.18) becomes:
\[
\tilde{h}_{2n,\alpha}(x,y|q) = \frac{(q; q)_{2n}}{(q; q)_{2n,\alpha}} x^{2n} 2\Phi_1 \left( \begin{array}{c} q^{-2n}, q^{-2n-2\alpha} \\ 0 \end{array} \middle| q^{-2n-2\alpha}; -y q^{2n+3} x^2 \right). \quad (3.23)
\]
By taking \( a^{-1} = q^{-2\alpha-2} \) and \( z = -q^{2n+1} x^2 y^{-1} \) and the formula [8, p.17, Eq. (0.6.17)]
\[
2\Phi_1 \left( \begin{array}{c} q^{-n}, a^{-1}, q^{1-n} \\ 0 \end{array} \middle| q^{-n}; -y a^{n+1} \right) = (a; q)_n (q z^{-1})^n_1 \Phi_1 \left( \begin{array}{c} q^{-n} \\ a \end{array} \middle| q; z \right) \quad (3.24)
\]
we have
\[
2\Phi_1 \left( \begin{array}{c} q^{-2n}, q^{-2n-2\alpha} \\ 0 \end{array} \middle| q^{2\alpha+3}; -y \right) = (q^{2n+2}; q^2)_n \Phi_1 \left( \begin{array}{c} q^{-2n}, q^{2\alpha+3} \\ q^{2\alpha+2}; \right) \quad (3.25)
\]
By using (1.2), the relation (3.25) can be written as
\[
q^{-2n^2} (q^2; q^2)_n \left( -\frac{y}{x^2} \right)^n L_n^{(\alpha)} (x^2 q^{-1} q^{-2\alpha-1}; q^2). \quad (3.26)
\]
The assertion (3.16) of Lemma 3.3 follows by summarizing the above calculations in \((3.23)-(3.26)\). In the odd case, the proof follows the same steps as the even case. \( \square \)

Proof. (of Theorem 3.4) By taking the limit \( \alpha \rightarrow +\infty \) in the assertions (3.16) and (3.17) of Lemma 3.3, respectively, we get the assertions (3.14) and (3.15) of Theorem 3.4. \( \square \)

4. Connection formulae for the generalized discrete \( q \)-Hermite II polynomials \( \{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty} \)

We begin this section with the following theorem:

**Theorem 4.1.** The sequence of gdq-H2P \( \{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty} \), which is defined by the relation (3.7), satisfies the connection formula
\[
\tilde{h}_{n,\alpha}(x,\omega|q) = (q; q)_n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{-2nk+k(2k+1)} (-\omega \otimes q^2 y)^k}{(q^2; q^2)_k (q; q)_{n-2k}} \tilde{h}_{n-2k,\alpha}(x,y|q). \quad (4.1)
\]

To prove Theorem 4.1, we need the following Lemma.

**Lemma 4.2.** The following generating function for gdq-H2P \( \{\tilde{h}_{n,\alpha}(x,y|q)\}_{n=0}^{\infty} \) holds true.
\[
\tilde{e}_{q^2,-\frac{1}{2}} \left( -yt^2 \right) \tilde{E}_{q,\alpha}(xt) = \sum_{n=0}^{\infty} \frac{q^{(\alpha)}_n t^n}{(q; q)_n} \tilde{h}_{n,\alpha}(x,y|q), \quad |yt| < 1. \quad (4.2)
\]

**Proof.** Let us consider the function
\[
f_q(t;x,y) := \sum_{n=0}^{\infty} \frac{q^{(\alpha)}_n t^n}{(q; q)_n} \tilde{h}_{n,\alpha}(x,y|q). \quad (4.3)
\]
By replacing in (4.3) gdq-H2P \( \tilde{h}_{n,\alpha}(x,y|q) \) by its explicit expression (3.1) we obtain
\[
f_q(t;x,y) = \sum_{n=0}^{\infty} t^n \left( \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(\alpha)}_n - 2nk+k(2k+1)x^n-2k y^k \right). \quad (4.4)
\]
The right-hand side of (4.4) also reads
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{(-2k)} (yt^2)^k (xt)^{n-2k} (q^2; q^2)_k (q; q)_n (q^2; q^2)^2.
\] (4.5)

Next, changing \( n-2k \) by \( r \), \( r = 0, 1, \cdots \), the last relation becomes
\[
\sum_{n=0}^{\infty} \frac{(-yt^2)^n}{(q^2; q^2)_n} \frac{q^{(2)}(xt)^r}{(q; q)_r}.
\] (4.6)

Hence,
\[
f_q(t; x, y) = \tilde{e}_{q^2, -\frac{1}{2}} (-yt^2) \tilde{E}_{q, \alpha}(xt).
\] (4.7)

Now, we are in position to prove Theorem 4.1.

Proof. (of Theorem 4.1) Replacing \( t \) by \( u \oplus_q t \) in (4.2), we find the following generating function
\[
\tilde{E}_{q, \alpha}[(u \oplus_q t)x] = \tilde{E}_{q^2, -\frac{1}{2}} \sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, y|q).
\] (4.8)

which by using (2.17), becomes
\[
\tilde{E}_{q, \alpha}[(u \oplus_q t)x] = \tilde{E}_{q^2, -\frac{1}{2}} \sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, y|q).
\] (4.9)

Replacing \( y \) by \( \omega \) and (4.9), respectively, in (4.8), we get
\[
\sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, \omega|q) = \sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, y|q).
\] (4.10)

By using (2.17), the last relation reads
\[
\sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, \omega|q)
\] (4.11)

According to (2.12), the right-hand side of (4.11) can be written as
\[
\sum_{r=0}^{\infty} \frac{(-\omega \oplus_q y)^r (u \oplus_q t)^{2r}}{(q^2; q^2)_r} \sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \tilde{h}_{n, \alpha}(x, y|q).
\] (4.12)

Let us substitute \( n + 2r = k \) \( \implies r \leq \lfloor k/2 \rfloor \) in (4.12), then we have:
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{q^{(n-2k)} (-\omega \oplus_q y)^k}{(q^2; q^2)_k (q; q)^{n-2k}} \tilde{h}_{n-2k, \alpha}(x, y|q) \right) (u \oplus_q t)^n.
\] (4.13)
Next, replacing (4.13) in (4.11), we obtain
\[
\sum_{n=0}^{\infty} \frac{q^{(2)}(u \oplus_q t)^n}{(q; q)_n} \hat{h}_{n, \alpha}(x, \omega|q) = (4.14)
\]
\[
\sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \frac{(-1)^k q^{-2nk+k(2k+1)} x^{n-2k} \omega^k}{(q^2; q^2)_k (q; q)_{n-2k}} \right) (u \oplus_q t)^n.
\]
Finally, on equating the coefficients of like powers of \((u \oplus_q t)^n/(q; q)_n\) in (4.14), we get the desired identity.

We have the following special cases of Theorem 4.1 of particular interest.

**Corollary 4.3.** Letting:

(i) \(y = 0\) in the assertion \(4.1\) of Theorem 4.1, we get the definition of gdq-H2P \((3.1)\), i.e.,
\[
\tilde{h}_{n, \alpha}(x, \omega|q) = (q; q)_n \sum_{k=0}^{[n/2]} \frac{(-1)^k q^{-2nk+k(2k+1)} x^{n-2k} \omega^k}{(q^2; q^2)_k (q; q)_{n-2k}}; (4.15)
\]

(ii) \(\omega = 0\) in the assertion \(4.1\) of Theorem 4.1 and using \((3.3)\), we get the inversion formula for gdq-H2P
\[
x^n = (q; q)_{n, \alpha} \sum_{k=0}^{[n/2]} q^{-2nk+3k^2} y^k (q^2; q^2)_k (q; q)_{n-2k} \tilde{h}_{n, -2k+\alpha}(x, y|q). (4.16)
\]

(iii) For \(y = 1\), the summation formulae \((4.1)\) can be expressed in terms of generalized discrete q-Hermite II polynomials \(h_{n, \alpha}(x; q)\). Also, the summation formulae \((4.1)\) can be written in terms of discrete q-Hermite II polynomials \(\tilde{h}_{n}(x; q)\) by choosing \(y = 1\) and \(\alpha = -1/2\).

5. **Concluding remarks**

In the previous sections, we have introduced gdq-H2P \(\tilde{h}_{n, \alpha}(x, y|q)\) and derived several properties. Also, we have derived implicit summation formula for gdq-H2P \(\tilde{h}_{n, \alpha}(x, y|q)\) by using different analytical means on their generating function. This process can be extended to summation formulae for more generalized forms of q-Hermite polynomials. This study is still in progress.

We note that the generating function of even and odd gdq-H2P \(\tilde{h}_{n, \alpha}(x, y|q)\) are given by
\[
\sum_{n=0}^{\infty} \frac{(-t^2)^n q^n (2n-1)}{(q; q)_{2n}^2} \tilde{h}_{2n, \alpha}(x, y|q) = \frac{q^\alpha(\alpha+\frac{1}{2}) (q^2; q^2)_\infty}{(q^{2\alpha+2}; q^2)_\infty} \frac{x^{-\alpha} J^{(2)}_{\alpha}(2xq^{-\alpha}; q^2)}{(y t^2; q^2)_\infty}
\]
and
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} t^{2n+1}}{(q; q)_{2n+1}^2} \tilde{h}_{2n+1, \alpha}(x, y|q) = \frac{q^{\alpha(\alpha+1)} (q^2; q^2)_\infty}{(q^{2\alpha+2}; q^2)_\infty} \frac{x^{-\alpha} J^{(2)}_{\alpha}(2xq^{-\alpha}; q^2)}{(y t^2; q^2)_\infty}
\]
where \(J^{(2)}_{\alpha}(z; q)\) is the q-analogue of the Bessel function \([8]\).

Indeed, it is well known that from (4.2), the generating function of gdq-H2P
where the generalized $q,h$-editor for their comments that helped us improve this article.

The author would like to thank the anonymous referee and editor for their comments that helped us improve this article.

Acknowledgments. The author would like to thank the anonymous referee and editor for their comments that helped us improve this article.

By using (2.9) and (2.10), respectively, the relations (5.6) and (5.7) can be expressed in terms of basic hypergeometric functions as

$$
\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1)
$$

which on separating the power in the right-hand side into their even and odd terms by using the elementary identity

$$
\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} t^n}{(q;q)_n} h_{n,\alpha}(x, y|q) = \sum_{n=0}^{\infty} t^{2n} h_{2n,\alpha}(x, y|q) + \sum_{n=0}^{\infty} t^{2n+1} h_{2n+1,\alpha}(x, y|q).
$$

Now replacing $t$ by $i t$ in (5.3) and equating the real and imaginary parts of the resultant equation, we get the generating function of even and odd gdq-H$_2$P $h_{n,\alpha}(x, y|q)$ as

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} t^{2n}}{(q;q)_{2n}} h_{2n,\alpha}(x, y|q) = Cos_{q,\alpha}(x|t) h_{q^2, -\frac{1}{2}}(yt^2)
$$

and

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} t^{2n+1}}{(q;q)_{2n+1}} h_{2n+1,\alpha}(x, y|q) = Sin_{q,\alpha}(x|t) h_{q^2, -\frac{1}{2}}(yt^2)
$$

where the generalized $q$-Cosine and $q$-Sine are defined as:

$$
Cos_{q,\alpha}(x) : = \sum_{k=0}^{\infty} \frac{(-1)^n q^{n(n-1)} x^{2n}}{(q;q)_{2n,\alpha}},
$$

$$
Sin_{q,\alpha}(x) : = \sum_{k=0}^{\infty} \frac{(-1)^n q^{n(n-1)} x^{2n+1}}{(q;q)_{2n+1,\alpha}}.
$$

The $q$-analogue of the Bessel function is defined [8] p.20, Eq.(0.7.14)] by

$$
J_{\nu}^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{z}{2}\right)^{\nu} \Phi_1 \left(\frac{1}{q^{\nu+1}} \right| q; -\frac{q^{\nu+1} z^2}{4}
$$

from which the generating functions of (5.8) and (5.9) follow.
References


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