ON BERTRAND SUPERCURVES IN SUPER-EUCLIDEAN SPACE

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Abstract. Using Banach Grassmann algebra, given by Rogers, a new scalar product, a new definition of the orthogonality and of Frenet frame associated to supersmooth supercurve are introduced on the \((m, n)\)-dimensional total super-Euclidean space. It is well known that a characteristic property of Bertrand curve is the existence of a linear relation between its curvature and torsion. In this study, definition of Bertrand supercurve in \(B_{L}^{m+n}\) is given and also some theorems for Bertrand supercurve in \(B_{L}^{4+4}\) are obtained.

1. Introduction

In recent years, much conventional differential geometry has been extended to include anticommuting variables; objects in this extended field of study are distinguishable by the prefix "super" which derives from the same prefix in supersymmetry, the fermi-base symmetry which is under such intense study by elementary particle physicist. Historically, the consideration of supermanifolds has a dual origin. Due to the first origin the earliest work is that Berezin and Leites [6] and Konstant [19] arose from the study of the mathematics of fermi field quantisation, their approach was sheaf theoretic, extending the sheaf of \(C^{\infty}\) functions on a manifold, rather than the manifold itself. Afterwards, a supermanifold was developed with a lot of study such as [7], [20]. Secondly, a more geometric approach grew directly from the physicists’ superspace [18] as a space with points labelled by even elements \((x^{\mu})\) and odd elements \((\theta^{\alpha})\) of a Grassmann algebra; a supermanifold is a topological space with local coordinates \((x^{\mu}, \theta^{\alpha})\) of this nature [11], [23]. Alternatively, the the best relationships between them have been made by Rogers [23], Bartocci et al. [4] and Batchelor [5]. Then, \((m, n)\)-dimensional total super-Euclidean space \(B_{L}^{m+n}\) is studied by Rogers [23]. Using Banach Grassmann algebra \(B_{L}\), a new superscalar product, a new definition of the orthogonality and Frenet frame associated to a supersmooth supercurve in general position are given by Cristea [11]. Also, Inoque and Maeda define super-Euclidean space with a different algebra, called a Frechet-Grassmann algebra [17].

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French mathematician Saint-Venant proposed in 1845 [26] the question whether upon the ruled surface generated by the principal normal of a curve in the three-dimensional Euclidean space \( \mathbb{R}^3 \) and a second curve can exist which has for this principal normals of the given curve. The second question was answered by Bertrand [8] in a paper in which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the curvature and torsion of the given original curve. Since the publication of Bertrand’s paper, a pair of curves of this kind has been called Conjugate Bertrand curves or, more commonly, Bertrand curves. Bertrand curves have attracted many mathematicians since the beginning. Later, the relations between Frenet frames of Bertrand couple in the space \( \mathbb{R}^n \) were given in [16].

Also, Bertrand couple is studied by many researchers in Euclidean 3-space \( \mathbb{R}^3 \) [10], [12], [27], [28]. In [22], Pears extended the well-known properties of Bertrand curves in Euclidean 3-space \( \mathbb{R}^3 \) to the curves in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( n > 3 \). However, in the last case, he found that either \( k_2 \) or \( k_3 \) must be zero; in other words, Bertrand curves in \( \mathbb{R}^n \) (\( n > 3 \)) are degenerate, i.e. a Bertrand curve in \( \mathbb{R}^n \) must belong to a three-dimensional subspace \( \mathbb{R}^3 \subseteq \mathbb{R}^n \). The same result has been obtained recently in [1] and [21]. As a natural consequence, some extensions of that concept have been proposed [16], [25], and more recently have been generalized in [9]. Many authors have studied Bertrand curves in other ambient spaces: in the three-dimensional Lorentz-Minkowski space \( \mathbb{R}^3_{1} \) [2], [3], [14], in semi-Euclidean spaces \( \mathbb{R}^n_{+1} \) [15], etc.

In this paper, we firstly define Bertrand supercurve, a one dimensional supermanifold, couples in definition of Bertrand supercurve on total super-Euclidean space \( B^m+n_L \). Later, using the methods expressed in [16] and Frenet frame we calculate some theorems for Bertrand supercurve in \( B^4+4_L \).

### 2. Preliminary Notes

In this section, we refer to a few basic definitions for the so-called geometric theory of supernumbers, supermanifolds, total super-Euclidean space, supervector space and operators, initialized by Dewitt, Rogers and Cristea. For further developments of the theory, which eliminated some drawbacks of research topic, the reader may utilize [11], [13], [23], [24].

**Definition 2.1.** For each positive integer \( L \), \( B_L \) will denote Grassmann algebra over the real numbers with generators \( 1^{(L)}, \beta^{(L)}_1, ..., \beta^{(L)}_L \) and relations

\[
1^{(L)} \beta^{(L)}_i = \beta^{(L)}_i 1^{(L)} = \beta^{(L)}_i, \quad i = 1, 2, ..., L
\]

\[
\beta^{(L)}_i \beta^{(L)}_j = -\beta^{(L)}_j \beta^{(L)}_i, \quad i, j = 1, 2, ..., L.
\]

\( B_L \) is a gradded algebra and can be written as

\[
B_L = (B_L)_0 \oplus (B_L)_1
\]

where \( \oplus \) be the direct sum and \( (B_L)_0 \) and \( (B_L)_1 \) be the even and odd part of \( B_L \), respectively [23].

**Definition 2.2.** Let \( M_L \) denote the set of finite sequences of positive integers \( \mu = (\mu_1, \mu_2, ..., \mu_k) \) with \( 1 \leq \mu_1 < \mu_2 < ... < \mu_k \leq L \) [19]. \( M_L \) includes the sequence...
with no elements, denoted $\phi$. As it follows in [24] for each $\mu$ in $M_L$,  
\begin{equation}
\beta^{(L)}_\mu = \beta^{(L)}_{\mu_1} \cdots \beta^{(L)}_{\mu_k}
\end{equation}
and typical element $b$ of $B_L$ may be expressed as  
\begin{equation}
b = \sum_{\mu \in M_L} b^\mu \beta^{(L)}_\mu
\end{equation}
where the coefficients $b^\mu$ are real numbers. We consider the body map [12]  
\begin{equation}
\varepsilon^{(L)}(b) = b^\phi
\end{equation}
is given by  
\begin{equation}
\varepsilon : B_L \to \mathbb{R}.
\end{equation}
with the norm of $B_L$ is defined by  
\begin{equation}
\|b\| = \sum_{\mu \in M_L} |b^\mu|.
\end{equation}

**Definition 2.3.** $B_L$ is Banach algebra, considering $L'$ also a positive integer, with $L \geq L'$, there is a natural projection  
\begin{equation}
i_{L',L} : B_{L'} \to B_L
\end{equation}
which is the unique algebra homomorphism satisfying  
\begin{equation}
i_{L',L} \left( \beta^{(L')}_i \right) = \beta^{(L)}_i, \quad i = 1, 2, \ldots, L
\end{equation}
i_{L',L} \left( 1^{(L')} \right) = 1^{(L)}.

$B_L$ naturally has a $B_{L'}$ module structure with  
\begin{equation}
ab = i_{L',L}(a)b, \quad a \in B_{L'}, \quad b \in B_L
\end{equation}

[24].

**Definition 2.4.** The $(m,n)$-dimensional total super-Euclidean space $B^{m+n}_L$ as the space, which is the cartesian product of $m+n$ copies of $B_L$, is defined by  
\begin{equation}
B^{m+n}_L = (B^{m+n}_L)_0 \oplus (B^{m+n}_L)_1.
\end{equation}
A typical element of $B^{m+n}_L$ is written $(x^1, x^2, \ldots, x^m, \theta^1, \theta^2, \ldots, \theta^n)$ or simply $(x, \theta)$, an element of $(B^{m+n}_L)_0$ is called c-type or even element and is written in the form 
\begin{equation}
(x^{1}, x^{2}, \ldots, x^{m}, \theta^{1}, \theta^{2}, \ldots, \theta^{n})
\end{equation}
with $x^1, x^2, \ldots, x^m \in (B_L)_0$. Also, $\theta^1, \theta^2, \ldots, \theta^n \in (B_L)_1$, an element of $(B^{m+n}_L)_1$ is called a-type or odd element is written in the form  
\begin{equation}
(x'^1, x'^2, \ldots, x'^m, \theta'^1, \theta'^2, \ldots, \theta'^n)
\end{equation}
and $\theta'^1, \theta'^2, \ldots, \theta'^n \in (B_L)_0$. An even element has the parity 0 and an odd element has the parity 1 [13].

**Definition 2.5.** The body map $\varepsilon$ is defined by [11]  
\begin{equation}
\varepsilon^{(L)}_{(m,n)} : (B^{m+n}_L)_0 \to \mathbb{R}^m
\end{equation}
\begin{equation}
(x, \theta) \mapsto \varepsilon^{(L)}_{(m,n)}(x, \theta) = (\varepsilon^{(L)}(x^1), \ldots, \varepsilon^{(L)}(x^m))
\end{equation}
where \((x', \theta') = (x'^1, x'^2, ..., x'^m, \theta'^1, \theta'^2, ..., \theta'^n) \in (B_L^{m+n})_0\) and the body map \(\varepsilon'\)

\[
\varepsilon'(L): (B_L^{m+n}),_1 \to \mathbb{R}^n
\]

\[
(x'', \theta'') \mapsto \varepsilon'(L)(x'', \theta'') = (\varepsilon(L)(\theta'^1), ..., \varepsilon(L)(\theta'^n))
\]

where \((x'', \theta'') = (x'^1, x'^2, ..., x'^m, \theta'^1, \theta'^2, ..., \theta'^n) \in (B_L^{m+n})_1\).

**Definition 2.6.** Suppose that \(V \subset B_L^{m+n}\) is open and that \(U = \varepsilon'(L)(V)\). Let \(L > 2n\) and \(L = \lfloor \frac{1}{2} \rfloor \) be the least integer not less than \(\frac{1}{2} L\). \(GH^\infty(V)\) denotes the set of functions,

\[
f : V \to B_L
\]

for which there exists \(f_m \in C^\infty(U, B'L)\) such that

\[
f(x, \theta) = \sum_{\mu \in M_n} Z_{L',L}(\partial_i f_m)(x) \theta^\mu
\]

that the map \(Z_{L',L}: C^\infty(U, B'L) \to \left[\varepsilon'(L)(0) \right]_{B_L} \) is defined by

\[
Z_{L',L}(f)(X) = \sum_{i=0,...,m=0} \frac{1}{\mu! \cdot \nu!} (\partial_x^i \cdot \theta_{\mu} m (\varepsilon(L)(x^1), ..., \varepsilon(L)(x^m)))
\]

\[
\times s(x^1)^\nu ... s(x^m)^\mu \left( \partial_x^i, L' \right)
\] (2.13)

where \((X) = (x^1, ..., x^m)\) and \(s(x^i) = x^i - \varepsilon(L)(x^i) 1\) for \(i = 1, 2, ..., m\). Here \(\theta^\mu = \theta^\mu_1 ... \theta^\mu_n\) and \(\theta^\theta = 1^L\) [24].

**Definition 2.7.** Suppose \(n = 2r\) and the supervectors

\[
v = (x^1, x^2, ..., x^m, \theta^1, \theta^2, ..., \theta^n), w = (y^1, y^2, ..., y^m, \theta_1^1, \theta_1^2, ..., \theta_1^n)
\] (2.14)

are the elements of \(B_L^{m+n}\). Superscalar product is defined by

\[
\langle v, w \rangle = x^1 y^1 + ... + x^m y^m + \theta^1 \theta_{r+1}^1 + ... + \theta^r \theta_r^1 - \theta^{r+1} \theta_1^1 - ... - \theta^n \theta_1^r
\] (2.15)

\[
\langle v, w \rangle f = \sum_{k=1}^{m} x^k y^k + \sum_{j=1}^{r} \left( \theta_j^1 \theta_{f(j)}^1 - \theta_{f(j)}^1 \theta_j^1 \right)
\]

\[
=x^1 y^1 + ... + x^m y^m + \theta^1 \theta_{r+1}^1 + ... + \theta^r \theta_r^1 - \theta^{r+1} \theta_1^1 - ... - \theta^n \theta_1^r
\] (2.16)

where \(f : \{1, ..., r\} \to \{r+1, ..., 2r\}\) is one-to-one function [11].

**Definition 2.8.** Supervector \(v \in B_L^{m+n}\) is orthogonal to supervector \(w \in B_L^{m+n}\) if and only if \(\varepsilon(L)(\langle v, w \rangle) = 0\). The standart base vectors on \((B_L^{m+n})_0\) form as

\[
E_1 = (1, 0, ..., 0), \ E_2 = (0, 1, ..., 0), \ ..., \ E_m = (0, ..., 1, 0, ..., 0)
\]

\[
E_{m+1} = (0, 0, ..., -1, 0, ..., 0), \ ..., \ E_{m+r} = (0, 0, ..., -1)
\] (2.17)

\[
E_{m+r+1} = (0, 0, 1, 0, ..., 0), \ ..., \ E_{m+n} = (0, 0, 1, 0, ..., 0)
\]

where the first \(m\) supervectors are even or \(c\)-type and the last \(n\) supervectors are odd or \(a\)-type [11].

**Definition 2.9.** Let \(f\) be an element of \(GH^\infty(V)\). Then, for \(i = 1, 2, ..., m\),

\[
G_i f : V \to B_L
\]

\[
(x, \theta) \mapsto G_i f(x, \theta) = \sum_{\mu \in M_n} Z_{L',L}(\partial_i f_m)(x) \theta^\mu
\] (2.18)
is defined. Also, for \(j = 1, 2, \ldots, n\)

\[
g_{j+m} : V \rightarrow B_L
\]

\[
(x, \theta) \mapsto g_{j+m}(x, \theta) = \sum_{\mu \in M_n} Z_{L', \mu}(f_{\mu})(x, \theta)^{\mu/j} \times (-1)^{j-1}f_{\mu}(x) \tag{2.19}
\]

is defined where \(|f_{\mu}(x)|\) is parity of \(f_{\mu}(x)\) and \(\theta^\mu/j = \theta^{\mu_1} \cdots \theta^{\mu_k} (-1)^{j-1}\), if \(j = \mu_i\) for some \(i\), \(1 \leq i \leq k\) and \(\theta^{\mu/j} = 0\), otherwise \(\text{[24]}\).

**Definition 2.10.** Let \(B_{L}^{m+n}\) be an \((m, n)\)-dimensional total super-Euclidean space for \(L > 2n\) and \(V \subset B_{L}^{1, 1}\) be an open set. Assume that

\[
c : V \subset B_{L}^{1, 1} \rightarrow B_{L}^{m+n}
\]

is a function and for \(\forall \theta \in V \cap (B_L)_1\) and \(\forall t \in V \cap (B_L)_0\)

\[
c_{0, 0} : V \cap (B_L)_0 \rightarrow (B_{L}^{m+n})_0
t \mapsto c_{0, 0}(t) = (c(t, \theta))_0
\]

\[
c_{0, \theta} : V \cap (B_L)_0 \rightarrow R^m
t \mapsto c_{0, \theta}(t) = (\varepsilon^{(L)})(t)
\]

are given where \((c(t, \theta))_0\) is the even part of the supervector \(c(t, \theta)\). The function \(c\) is to be supercurve if and only if \(c_{0, \theta} |_{V \cap R} \) is a curve. The function \(c\) is called supersmooth supercurve if and only if

\[
c^i \in GH^\infty(V) \quad i \in \{1, 2, \ldots, m\}
\]

\[
c^{j+m} \in GH^\infty(V) \quad j \in \{1, 2, \ldots, n\}\tag{2.21}
\]

where

\[
c^i = x^i \circ c \forall i \in \{1, 2, \ldots, m\}
\]

\[
c^{j+m} = \theta^j \circ c \forall j \in \{1, 2, \ldots, n\}\tag{2.22}
\]

\[\text{[24].}\]

**Definition 2.11.** Let \(c\) be a regular smooth curve in Euclidean 4-space \(E^4\) defined by

\[x : s \in L \rightarrow x(s) \in E^4\]

where \(L\) denotes a subset of the set \(R\) of all real numbers, and \(s\) is the arc-length parameter of \(c\). The curve \(c\) is called a special Frenet curve if there exist three smooth functions \(k_1, k_2, k_3\) on \(c\) and smooth frame field \(\{e_1, e_2, e_3, e_4\}\) along the curve \(c\). The formulas of Frenet-Serret hold:

\[
\begin{bmatrix}
  e'_1 \\
  e'_2 \\
  e'_3 \\
  e'_4
\end{bmatrix} =
\begin{bmatrix}
  0 & k_1 e_2 & 0 & 0 \\
  -k_1 e_2 & 0 & k_2 e_3 & 0 \\
  0 & -k_2 e_3 & 0 & k_3 e_4 \\
  0 & 0 & -k_3 e_4 & 0
\end{bmatrix}
\begin{bmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  e_4
\end{bmatrix} \tag{2.23}
\]

for \(s \in L\), where the prime (‘) denotes differentiation with respect to \(s\). The frame field \(\{e_1, e_2, e_3, e_4\}\) is of orthonormal positive orientation. The functions \(k_1\) and \(k_2\) are of positive, and the function \(k_3\) doesn’t vanish. Also, the functions \(k_1, k_2, k_3\) are called the first, the second, and the third curvature function of \(c\), respectively. The frame field \(\{e_1, e_2, e_3, e_4\}\) is called Frenet frame field on \(c\) \[10\]. We refer this notion to \[28\].
3. Frenet frame associated to a supersmooth supercurve in general position

In this part, using definition of supersmooth supercurve and Frenet frame associated to a supersmooth supercurve in general position \[ \text{[11]} \]. Frenet frame associated to a supersmooth supercurve in general position of even and odd part of super-Euclidean space \( B_{L}^{m+n} \) is given.

**Definition 3.1.** Let \( (B_{L}^{m+n})_{0} \) be an even part of \( (m,n) \)-dimensional total super-Euclidean space for \( L > 2n \) and \( V \subset B_{L}^{1,1} \) be an open set and \( c : V \subset B_{L}^{1,1} \rightarrow (B_{L}^{m+n})_{0} \) be supersmooth supercurve. The supercurve \( c \) is in general position if and only if

\[
\left\{ G_{1}c(t,\theta), G_{1}(m-1)c(t,\theta), G_{2}c(t,\theta), G_{1}G_{2}c(t,\theta), \ldots, G_{1}^{(n-1)}G_{2}c(t,\theta) \right\}
\]

are linearly independent where \( G_{1}c(t,\theta) \) is a supervector which is expressed by

\[
(G_{1}c^{1}(t,\theta), \ldots, G_{1}c^{m}(t,\theta), G_{1}c^{m+1}(t,\theta), \ldots, G_{1}c^{m+n}(t,\theta))
\]

and same as \( G_{2}c(t,\theta) \) is a supervector which is expressed by

\[
(G_{2}c^{1}(t,\theta), \ldots, G_{2}c^{m}(t,\theta), G_{2}c^{m+1}(t,\theta), \ldots, G_{2}c^{m+n}(t,\theta))
\]  

(3.1)

with

\[
G_{1}^{(s)}c(t,\theta) = c(t,\theta), G_{1}^{(1)}c(t,\theta) = G_{1}c(t,\theta), \ldots, G_{1}^{(s)}c(t,\theta) = G_{1}\ldots G_{1}c(t,\theta)
\]

where \( \forall (t,\theta) \in V \subset B_{L}^{1,1} \) \[ \text{[11]} \].

**Definition 3.2.** Let \( (B_{L}^{m+n})_{0} \) be an even part of \( (m,n) \)-dimensional total super-Euclidean space for \( L > 2n \) and \( V \subset B_{L}^{1,1} \) be an open set. Consider that

\[ c : V \subset B_{L}^{1,1} \rightarrow (B_{L}^{m+n})_{0} \]

is a supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve \( c \) we shall mean a system of \( m+n \) supervector fields \{\( e_{1}, \ldots, e_{m+n} \)\} along to the supersmooth supercurve \( c \) for \( \forall (t,\theta) \in V \subset B_{L}^{1,1} \), we have the following properties:

\[
\langle e_{k}(t,\theta), e_{h}(t,\theta) \rangle = \delta_{kh} \quad \forall k, h \in \{1, 2, \ldots, m\} \]

\[
\langle e_{m+j}(t,\theta), e_{m+j}(t,\theta) \rangle = -\delta_{jj} \quad \forall j \in \{1, 2, \ldots, r\}, \quad \delta_{jj} \in \{0, 1\},
\]

\[
\langle e_{m+j}(t,\theta), e_{m+j}(t,\theta) \rangle = \delta_{jj} \quad \forall j \in \{1, 2, \ldots, r-1\}, \quad \delta_{jj} \in \{0, 1\},
\]

\[
\langle e_{m+j}(t,\theta), e_{m+j}(t,\theta) \rangle = 0 \quad \forall j \in \{1, 2, \ldots, r\}
\]

\[
\langle e_{m+j}(t,\theta), e_{m+j}(t,\theta) \rangle = 0 \quad \forall j \in \{1, 2, \ldots, r\}, \quad \delta_{jj} \in \{0, 1\}
\]

\[
\langle e_{1}(t,\theta), e_{m+j}(t,\theta) \rangle = 0 \quad \forall i \in \{1, 2, \ldots, m\}, \forall j \in \{1, 2, \ldots, n\}
\]

where

\[ Sp\left(G_{1}c(t,\theta), \ldots, G_{1}^{(k)}c(t,\theta)\right) = Sp\left(e_{1}(t,\theta), \ldots, e_{k}(t,\theta)\right) \quad \forall k \in \{1, 2, \ldots, m-1\}, \]

(3.3)

and

\[ Sp\left(G_{2}c(t,\theta), G_{1}G_{2}c(t,\theta), \ldots, G_{1}^{(j-1)}G_{2}c(t,\theta)\right) = Sp\left(e_{m+1}(t,\theta), \ldots, e_{m+j}(t,\theta)\right) \quad \forall i \in \{1, 2, \ldots, m\} \text{ and } \forall j \in \{1, 2, \ldots, n\} \]

(3.4)
Theorem 3.3. Let \((B_{L}^{m+n})_0\) be an even part of \((m,n)\)-dimensional total super-Euclidean space \(B_{L}^{m+n}\) for \(L > 2n\) and \(V \subset B_{L}^{1,1}\) be an open set and 
\[
c : V \subset B_{L}^{1,1} \rightarrow (B_{L}^{m+n})_0
\]
be a supersmooth supercurve in general position which is satisfied following relation:
For \(\forall (t, \theta) \in V \subset B_{L}^{1,1}\),
\[
\begin{align*}
\varepsilon^{(L)}( \langle G_1 c(t, \theta), G_1^{(r)} G_2 c(t, \theta) \rangle ) & > 0 \\
\varepsilon^{(L)}( \langle G_1^{(r)} c(t, \theta), G_1^{(r+j_1)} G_2 c(t, \theta) \rangle ) & > 0 \\
\varepsilon^{(L)}( \langle G_2 c(t, \theta), G_1^{(j)} G_2 c(t, \theta) \rangle ) & = 0 \\
\varepsilon^{(L)}( \langle G_1^{(r)} G_2 c(t, \theta), G_1^{(r+j_1)} G_2 c(t, \theta) \rangle ) & = 0
\end{align*}
\]
(3.5)
Then there exists a unique Frenet frame \(\{e_1, ..., e_{m+n}\}\) associated to the supercurve \(c\) and for \(\forall (t, \theta) \in V \subset B_{L}^{1,1}\),
\[
\begin{align*}
G_1 e_k(t, \theta) & = \sum_{h=1}^{m} a_{kh}(t, \theta) e_h(t, \theta) \\
G_1 e_{m+j}(t, \theta) & = \sum_{l=1}^{m} a_{m+j_l}(t, \theta) e_{m+l}(t, \theta)
\end{align*}
\]
(3.6)
where
\[
\begin{align*}
a_{kh}(t, \theta) + a_{hk}(t, \theta) & = 0 \\
a_{kh}(t, \theta) & = 0 \\
a_{m+j_l} m+j_l(t, \theta) + a_{m+r+j_2} m+r+j_2(t, \theta) & = 0 \\
a_{m+r+j_2} m+j_2(t, \theta) - a_{m+j_2} m+r+j_2(t, \theta) & = 0 \\
a_{i} m+j_i(t, \theta) & = 0 \\
a_{m+j_i} m+i_l(t, \theta) & = 0 \\
a_{m+j} m+l(t, \theta) & = 0
\end{align*}
\]
(3.7)
and
\[
\begin{align*}
a_{m+j_1} m+j_1(t, \theta) & = \langle G_1 e_{m+j_1}(t, \theta), e_{m+r+j_2}(t, \theta) \rangle \\
a_{m+j_1} m+r+j_2(t, \theta) & = - \langle G_1 e_{m+j_1}(t, \theta), e_{m+j_2}(t, \theta) \rangle \\
a_{m+r+j_1} m+j_2(t, \theta) & = \langle G_1 e_{m+r+j_1}(t, \theta), e_{m+r+j_2}(t, \theta) \rangle \\
a_{m+r+j_2} m+r+j_2(t, \theta) & = - \langle G_1 e_{m+r+j_2}(t, \theta), e_{m+j_2}(t, \theta) \rangle \\
a_{k} m+j_k(t, \theta) & = \langle G_1 e_k(t, \theta), e_{m+j}(t, \theta) \rangle \\
a_{m+j} k(t, \theta) & = \langle G_1 e_{m+j}(t, \theta), e_k(t, \theta) \rangle
\end{align*}
\]
(3.8)
are obtained \([11]\).

Definition 3.4. Let \((B_{L}^{m+n})_1\) be an odd part of \((m,n)\)-dimensional total super-Euclidean space for \(L > 2n\) and \(V \subset B_{L}^{1,1}\) be an open set and 
\[
c : V \subset B_{L}^{1,1} \rightarrow (B_{L}^{m+n})_1
\]
be supersmooth supercurve. The supercurve \(c\) is in general position if and only if
\[
\left\{ G_2 c(t, \theta), G_1 G_2 c(t, \theta), ..., G_1^{(m-1)} G_2 c(t, \theta), G_1 c(t, \theta), ..., G_1^{(n-1)} c(t, \theta) \right\}
\]
are linearly independent where \( G_1c(t, \theta) \) is a supervector which is expressed by
\[
\left( G_1c^1(t, \theta), \ldots, G_1c^m(t, \theta), G_1c^{m+1}(t, \theta), \ldots, G_1c^{m+n}(t, \theta) \right)
\]
and same as \( G_2c(t, \theta) \) is a supervector which is expressed by
\[
\left( G_2c^1(t, \theta), \ldots, G_2c^m(t, \theta), G_2c^{m+1}(t, \theta), \ldots, G_2c^{m+n}(t, \theta) \right)
\]
with
\[
G_1^{(s)}c(t, \theta) = c(t, \theta), G_1^{(1)}c(t, \theta) = G_1c(t, \theta), \ldots, G_1^{(s)}c(t, \theta) = G_1^{\ldots}G_1c(t, \theta).
\]
where for \( \forall (t, \theta) \in V \subset B_L^{1,1} \).

**Definition 3.5.** Let \( (B_L^{m+n})_1 \) be an odd part of \((m, n)\)-dimensional total super-Euclidean space for \( L > 2n \) and \( V \subset B_L^{1,1} \) be an open set. Consider
\[
c : V \subset B_L^{1,1} \rightarrow (B_L^{m+n})_1
\]
supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve
\[
c : V \subset B_L^{1,1} \rightarrow (B_L^{m+n})_1
\]
we shall mean a system of \( m + n \) supervector fields \( \{e_1, \ldots, e_{m+n}\} \) along to the supersmooth supercurve \( c \) such that for \( \forall (t, \theta) \in V \subset B_L^{1,1} \) we have the following properties:
\[
\begin{align*}
\langle e_k(t, \theta), e_h(t, \theta) \rangle &= 0 & \forall k, h \in \{1, 2, \ldots, r\} \\
\langle e_{r+j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle &= 0 & \forall j_1, j_2 \in \{1, 2, \ldots, r\} \\
\langle e_{j_1}(t, \theta), e_{j_2}(t, \theta) \rangle &= \delta_{j_1j_2} & \forall j_1, j_2 \in \{1, 2, \ldots, r\} \\
\langle e_{r+j_1}(t, \theta), e_{j_2}(t, \theta) \rangle &= -\delta_{j_1j_2} & \forall j_1, j_2 \in \{1, 2, \ldots, r\} \\
\langle e_{m+j_1}(t, \theta), e_{m+j_2}(t, \theta) \rangle &= \delta_{j_1j_2} & \forall j_1, j_2 \in \{1, 2, \ldots, n\} \\
\langle e_{j_1}(t, \theta), e_{m+j_2}(t, \theta) \rangle &= 0 & \forall j_1 \in \{1, 2, \ldots, r\}, j_2 \in \{1, 2, \ldots, n\} \\
\langle e_{r+j_1}(t, \theta), e_{m+j_2}(t, \theta) \rangle &= 0 & \forall j_2 \in \{1, 2, \ldots, n\}
\end{align*}
\]
where
\[
Sp \left( G_2c(t, \theta), G_1G_2c(t, \theta), \ldots, G_1^{(j-1)}G_2c(t, \theta) \right) = Sp(e_1(t, \theta), \ldots, e_j(t, \theta))
\]  
\[\forall i \in \{1, 2, \ldots, m\}, \forall j \in \{1, 2, \ldots, n\} \text{ and} \]
\[
Sp \left( G_1c(t, \theta), G_1^{(k)}c(t, \theta) \right) = Sp(e_{m+1}(t, \theta), \ldots, e_{m+k}(t, \theta))
\]  
\[\forall k \in \{1, 2, \ldots, m-1\}.
\]

**Theorem 3.6.** Let \( (B_L^{m+n})_1 \) be an odd part of \((m, n)\)-dimensional total super-Euclidean space \( B_L^{m+n} \) for \( L > 2n \) and \( V \subset B_L^{1,1} \) be an open set and
\[
c : V \subset B_L^{1,1} \rightarrow (B_L^{m+n})_1
\]
be a supersmooth supercurve in general position which satisfies following relations:
For \( \forall (t, \theta) \in V \subset B_L^{1,1}, \)
\[
\begin{align*}
\varepsilon^{\mathcal{L}} \left( \left( G_2c(t, \theta), G_1^{(r)}G_2c(t, \theta) \right) \right) > 0 & \quad \forall j_1, j_2 \in \{1, 2, \ldots, r-1\} \\
\varepsilon^{\mathcal{L}} \left( \left( G_1^{j_1}G_2c(t, \theta), G_1^{(r+j_1)}G_2c(t, \theta) \right) \right) > 0 & \quad \forall j \in \{1, 2, \ldots, n-1\} \\
\varepsilon^{\mathcal{L}} \left( \left( G_2c(t, \theta), G_1^{(j)}G_2c(t, \theta) \right) \right) = 0 & \quad \forall j \in \{1, 2, \ldots, n-1\} \\
\varepsilon^{\mathcal{L}} \left( \left( G_1^{j_1}G_2c(t, \theta), G_1^{(r+j_1)}G_2c(t, \theta) \right) \right) = 0 & \quad \forall j \neq j, j \neq j, j \neq j.
\end{align*}
\]
Then there exists a unique Frenet frame \( \{e_1, \ldots, e_{m+n}\} \) associated to the supercurve \( c \) and for \( \forall (t, \theta) \in V \subset B_L^{1,1} \)

\[
G_1 e_k(t, \theta) = \sum_{h=1}^{m} a_{kh}(t, \theta) e_h(t, \theta) \quad \forall k \in \{1, 2, \ldots, m\}
\]

\[
G_1 e_{m+j}(t, \theta) = \sum_{l=1}^{m+n} a_{m+j, m+l}(t, \theta) e_{m+l}(t, \theta) \quad \forall j \in \{1, 2, \ldots, n\}
\]

(3.13)

are obtained where

\[
a_{j_1, j_2}(t, \theta) + a_{r+j_2, r+j_1}(t, \theta) = 0 \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{r+j_1, j_2}(t, \theta) - a_{r+j_2, j_1}(t, \theta) = 0 \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{j_1, r+j_2}(t, \theta) - a_{j_2, r+j_1}(t, \theta) = 0 \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{j_1, m+j_2}(t, \theta) = a_{j_1, m+j_2}(t, \theta) = 0 \quad \forall j_1 \in \{1, 2, \ldots, m\}, \forall j_2 \in \{1, 2, \ldots, n\}
\]

\[
a_{kh}(t, \theta) = 0 \quad h \neq k + 1, \forall k, h \in \{1, 2, \ldots, n\}
\]

(3.14)

and

\[
a_{j_1, j_2}(t, \theta) = \langle G_1 e_{j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{j_1, r+j_2}(t, \theta) = -\langle G_1 e_{j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{r+j_1, j_2}(t, \theta) = \langle G_1 e_{r+j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

\[
a_{r+j_1, r+j_2}(t, \theta) = -\langle G_1 e_{r+j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle \quad \forall j_1, j_2 \in \{1, 2, \ldots, r\}
\]

(3.15)

\[
a_{m+j_1, m+j_2}(t, \theta) = \langle G_1 e_{m+j_1}(t, \theta), e_{m+j_2}(t, \theta) \rangle \quad \forall j_1 \in \{1, 2, \ldots, m\},
\]

\[
a_{m+j_1, r+j_2}(t, \theta) = \langle G_1 e_{m+j_1}(t, \theta), e_{r+j_2}(t, \theta) \rangle \quad \forall j_1 \in \{1, 2, \ldots, m\},
\]

\[
a_{m+j_1, j_2}(t, \theta) = \langle G_1 e_{m+j_1}(t, \theta), e_{j_2}(t, \theta) \rangle \quad \forall j_1 \in \{1, 2, \ldots, m\},
\]

4. On Bertrand Supercurves in Super-Euclidean Space

In this section, we introduce Bertrand supercurve couple and give some theorems in super-Euclidean space.

Let \( M_1, M_2 \subset B^{m+n}_L \) be two supersmooth supercurves given by \((V, c)\) and \((V, c^*)\), respectively. For \((t, \theta) \in V, c^*\) is called Bertrand of the supercurve \( c \) or \( (M_1, M_2) \) is called Bertrand supercurve couple, if principal normals of body parts at the point \( c(t, \theta) \) and \( c^*(t, \theta) \) are linearly dependent where \( V \subset B^{(1,1)}_L \) is an open subset.

**Theorem 4.1.** Let \((M_1, M_2)\) be Bertrand supercurve couple which are given by coordinate neighbourhoods \((V, c)\) and \((V, c^*)\) in \((B^{m+n}_L)^0\), respectively. The distance between the points \( c(t, \theta) \in M_1 \) and \( c^*(t, \theta) \in M_2 \) is given by

\[
d(c(t, \theta), c^*(t, \theta)) = b
\]

where \( b \) is a superconstant.

**Proof.** If \((M_1, M_2)\) is Bertrand supercurve couple, we have

\[
c^*(t, \theta) = c(t, \theta) + A(t, \theta) e_2(t, \theta)
\]

(4.1)

where \( A(t, \theta) \) is supervariable. Differentiating both sides of the expression (4.1) with respect to \( t \):

\[
G_1^* c^*(t, \theta) \frac{dt^*}{dt} = G_1 c(t, \theta) + G_1 A(t, \theta) e_2(t, \theta) + (-1)^{|t||A(t, \theta)|} A(t, \theta) G_1 e_2(t, \theta)
\]

(4.2)
From the equation (3.6), we get
\[
G_1 c^* (t, \theta) \frac{dt^*}{dt} = G_1 c (t, \theta) + G_1 A (t, \theta) e_2 (t, \theta) + \left( -1 \right)^{[A (t, \theta)]} A (t, \theta) a_{23} (t, \theta) e_3 (t, \theta)
\] (4.3)

where \( t \) and \( t^* \) are arc–parameters of \( M_1 \) and \( M_2 \), respectively.

Thus we have
\[
e_1^* (t, \theta) \frac{dt^*}{dt} = \left( 1 + \left( -1 \right)^{[A (t, \theta)]} A (t, \theta) a_{21} (t, \theta) \right) e_1 (t, \theta) + G_1 A (t, \theta) e_2 (t, \theta) + A (t, \theta) a_{23} (t, \theta) e_3 (t, \theta).
\] (4.4)

Multiplying the the equation (4.4) with \( e_2 (t, \theta) \) by superscalar product, we have
\[
\langle e_1^* (t, \theta), e_2 (t, \theta) \rangle \frac{dt^*}{dt} = \left( 1 + \left( -1 \right)^{[A (t, \theta)]} A (t, \theta) a_{21} (t, \theta) \right) \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + G_1 A (t, \theta) \langle e_2 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} A (t, \theta) a_{23} (t, \theta) \langle e_3 (t, \theta), e_2 (t, \theta) \rangle.
\] (4.5)

From the definition of Bertrand supercurve couple \( \varepsilon^{(L)} \langle e_1^* (t, \theta), e_2 (t, \theta) \rangle = 0 \). Thus we obtain
\[
G_1 \varepsilon^{(L)} (A (t, \theta)) = 0
\]
\[
\varepsilon^{(L)} (A (t, \theta)) = b
\] (4.6)

and
\[
0 = \varepsilon^{(L)} (a_{21} (t, \theta)) s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + s \langle a_{21} (t, \theta) \rangle s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle a_{21} (t, \theta) \rangle s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle a_{21} (t, \theta) \rangle s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle a_{21} (t, \theta) \rangle s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle a_{21} (t, \theta) \rangle s \langle e_1 (t, \theta), e_2 (t, \theta) \rangle + G_1 s (A (t, \theta)) s \langle e_2 (t, \theta), e_2 (t, \theta) \rangle + G_1 s (A (t, \theta)) s \langle e_2 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle + \left( -1 \right)^{[A (t, \theta)]} \varepsilon^{(L)} (A (t, \theta)) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle + A (t, \theta) a_{23} (t, \theta) s \langle e_3 (t, \theta), e_2 (t, \theta) \rangle
\] (4.7)

where \( b \) is a superconstant. From the definition of the distance on total super-Euclidean space, we can easily find
\[
d (c (t, \theta), c^* (t, \theta)) = b
\] (4.8)

where \( b \) is the superconstant.

\[\boxdot\]

**Theorem 4.2.** Let \((M_1, M_2)\) be Bertrand supercurve couple which are given by coordinate neighbourhoods \((V, c)\) and \((V, c^*)\) in \(\langle B_{L}^{m+n+1} \rangle_1\), respectively. The distance between the points \(c (t, \theta) \in M_1\) and \(c^* (t, \theta) \in M_2\) is given by
\[
d (c (t, \theta), c^* (t, \theta)) = b
\]

where \( b \) is a superconstant.

**Proof.** If \((M_1, M_2)\) is Bertrand supercurve couple, we have
\[
c^* (t, \theta) = c (t, \theta) + A (t, \theta) e_6 (t, \theta)
\] (4.9)
where $A(t, \theta)$ is supervariable. Differentiating both sides of the expression (4.9) with respect to $t$:

$$G_1^*c^*(t, \theta) \frac{dt^n}{dt} = G_1c(t, \theta) + G_1A(t, \theta)e_6(t, \theta) + (-1)^{|t||A(t, \theta)|} A(t, \theta)G_1e_6(t, \theta). \quad (4.10)$$

From the equation (3.13), we get

$$G_1^*c^*(t, \theta) \frac{dt^n}{dt} = G_1c(t, \theta) + G_1A(t, \theta)e_6(t, \theta) + (-1)^{|t||A(t, \theta)|} A(t, \theta)a_{67}(t, \theta)e_7(t, \theta) \quad (4.11)$$

where $t$ and $t^*$ are arc-parameters of $M_1$ and $M_2$, respectively.

Thus, we have

$$e_5^*(t, \theta) \frac{dt}{dt} = (1 + (-1)^{|t||A(t, \theta)|} A(t, \theta)a_{65}(t, \theta)) \cdot \langle e_5(t, \theta), e_6(t, \theta) \rangle + G_1A(t, \theta)e_6(t, \theta) + A(t, \theta)a_{67}(t, \theta)e_7(t, \theta) \quad (4.12)$$

Multiplying the the equation (4.12) with $c_6(t, \theta)$ by superscalar product, we get

$$\langle e_5^*(t, \theta), e_6(t, \theta) \rangle \frac{dt}{dt} = (1 + (-1)^{|t||A(t, \theta)|} A(t, \theta)a_{65}(t, \theta)) \cdot \langle e_5(t, \theta), e_6(t, \theta) \rangle + G_1A(t, \theta)\langle e_6(t, \theta), c_7(t, \theta) \rangle + A(t, \theta)a_{67}(t, \theta)\langle c_7(t, \theta), e_6(t, \theta) \rangle \quad (4.13)$$

From the definition of Bertrand supercurve couple $\varepsilon^{(L)} \langle e_5^*(t, \theta), e_6(t, \theta) \rangle = 0$. Thus we obtain

$$G_1\varepsilon^{(L)}(A(t, \theta)) = 0 \quad (4.14)$$

and

$$0 = \varepsilon^{(L)}(a_{65}(t, \theta)) s \langle e_1(t, \theta), e_6(t, \theta) \rangle + s \langle a_{65}(t, \theta) \rangle s \langle e_5(t, \theta), e_6(t, \theta) \rangle + G_1s(A(t, \theta)) \langle e_6(t, \theta), c_7(t, \theta) \rangle + A(t, \theta)a_{67}(t, \theta)\langle c_7(t, \theta), e_6(t, \theta) \rangle \quad (4.15)$$

where $b$ is a superconstant. From the definition of the distance on total super-Euclidean space, we can easily find

$$d(c(t, \theta), c^*(t, \theta)) = b \quad (4.16)$$

where $b$ is the superconstant.

\textbf{Theorem 4.3.} Let $M_1, M_2$ be supersmooth supercurves which are given by coordinate neighbourhoods $(V, c)$ and $(V, c^*)$ in $(B^{m+n}_L)_0$, respectively. Then, $M_1, M_2$ are Bertrand supercurves if and only if

$$\lambda\varepsilon^{(L)}(a_{21}(t, \theta)) + \mu\varepsilon^{(L)}(a_{23}(t, \theta)) = 1$$

where $\lambda, \mu$ are superconstants and $a_{21}(t, \theta), a_{23}(t, \theta)$ are supercurvatures in $M_1$. \hfill \Box
Proof: If \((M_1, M_2)\) is Bertrand supercurve couple from Theorem 4.1, we have

\[
e(t, \theta) \frac{dt}{ds} = (1 - (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)e_1(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta)c_3(t, \theta) \tag{4.17}
\]

where \(s(A(t, \theta))\) is a odd part of supervariable \(A(t, \theta)\) and \(b\) is a superconstant. Differentiating both sides of the expression (4.17) with respect to \(t\) and from the equation (4.6), an equation

\[
a_{12}(t, \theta)e_2^*(t, \theta) = G_1A_1(t, \theta)e_1(t, \theta) + G_1B_1(t, \theta)e_3(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)e_1(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta)c_3(t, \theta) \tag{4.18}
\]

is obtained that \(A_1(t, \theta) = (1 - (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)\) and \(B_1(t, \theta) = (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta)\). Since \((M_1, M_2)\) is Bertrand supercurve couple, we have

\[
e_1(t, \theta) = A_1(t, \theta)e_1(t, \theta) + B_1(t, \theta)e_3(t, \theta) \tag{4.19}
\]

where \(A_1(t, \theta)\) and \(B_1(t, \theta)\) are supervariables. Let us differentiate the equation (4.19) with respect to \(t\) and use the equation (3.3)

\[
a_{12}(t, \theta)e_2^*(t, \theta) = G_1A_1(t, \theta)e_1(t, \theta) + G_1B_1(t, \theta)e_3(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)e_1(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta)c_3(t, \theta) \tag{4.20}
\]

is found. Since \(\{e_2(t, \theta), e_2^*(t, \theta)\}\) is a linearly dependent set and using the equation (4.18), we get

\[
\varepsilon^{(L)}(G_1A_1(t, \theta)) = 0 \text{ and } \varepsilon^{(L)}(G_1B_1(t, \theta)) = 0. \tag{4.21}
\]

Then, using \(\varepsilon^{(L)}(A_1(t, \theta)) = \text{superconstant and } \varepsilon^{(L)}(B_1(t, \theta)) = \text{superconstant}

\[
\varepsilon^{(L)} \left( \frac{A_1(t, \theta)}{B_1(t, \theta)} \right) = a \tag{4.22}
\]

is written where \(a\) is a superconstant. From the equation (4.18) and (4.19),

\[
B_1(t, \theta) = (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)B_1(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta). \tag{4.23}
\]

If we divide the equation (4.23) with \(B_1(t, \theta)\) and separate into the even and odd parts, then we get

\[
(-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{21}(t, \theta)B_1(t, \theta) + (-1)^{|t|b + s(A(t, \theta))}| (b + s(A(t, \theta)))a_{23}(t, \theta). \tag{4.24}
\]

From even and odd parts of the equation (4.24), we get

\[
\lambda \varepsilon^{(L)} (a_{21}(t, \theta)) + \mu \varepsilon^{(L)} (a_{23}(t, \theta)) = 1 \tag{4.25}
\]

and

\[
0 = b \cdot s (a_{21}(t, \theta)) + s (A(t, \theta)[\varepsilon^{(L)} (a_{21}(t, \theta)) + s (a_{21}(t, \theta))]
+ \varepsilon^{(L)} \left( \frac{A_1(t, \theta)}{B_1(t, \theta)} \right) b \cdot s (a_{23}(t, \theta))
+ s \left( \frac{A_1(t, \theta)}{B_1(t, \theta)} \right) b \cdot \{ \varepsilon^{(L)} (a_{23}(t, \theta)) + s (a_{23}(t, \theta)) \}
+ s (A(t, \theta) \{ s (a_{23}(t, \theta)) + \varepsilon^{(L)} (a_{23}(t, \theta)) \}). \tag{4.26}
\]
Theorem 4.4. Let $M_1, M_2$ be supersmooth supercurves which are given by coordinate neighbourhoods $(V, c)$ and $(V, c^*)$ in $(B_{L}^{m+n})_1$, respectively. Then $(M_1, M_2)$ is Bertrand supercurve couple if and only if

$$\lambda e^{(L)}(a_{65}(t, \theta)) + \mu e^{(L)}(a_{67}(t, \theta)) = 1$$

where $\lambda, \mu$ are superconstants and $a_{65}(t, \theta)$, $a_{67}(t, \theta)$ are supercurvatures in $M_1$.

Proof. If $(M_1, M_2)$ is Bertrand supercurve couple from Theorem 4.2, we have

$$e_5^e(t, \theta) \frac{d e_5^e(t, \theta)}{d t} = (1 - (-1)^{|t|} b + s(A(t, \theta))) a_{65}(t, \theta) e_5^e(t, \theta) + (-1)^{|t|} b + s(A(t, \theta)) a_{67}(t, \theta) e_7(t, \theta)$$

where $s(A(t, \theta))$ is a odd part of supervariable $A(t, \theta)$ and where $b$ is a superconstant.

Differentiating both sides of the expression (4.27) with respect to $t$ and from the equation (3.13), an equation

$$a_{67}^e(t, \theta) e_5^e(t, \theta) \frac{d e_5^e(t, \theta)}{d t} = G_{1} A_{1}(t, \theta) e_5^e(t, \theta) + G_{1} B_{1}(t, \theta) e_3(t, \theta) + [(-1)^{|t|} A_{1}(t, \theta)] a_{56}(t, \theta) - (-1)^{|t|} B_{1}(t, \theta) a_{67}(t, \theta) e_7(t, \theta)$$

is obtained that $A_{1}(t, \theta)$ and $B_{1}(t, \theta)$ are supervariables. Differentiating the equation (4.29) with respect to $t$ gives

$$a_{65}^e(t, \theta) e_5^e(t, \theta) \frac{d e_5^e(t, \theta)}{d t} = G_{1} A_{1}(t, \theta) e_5^e(t, \theta) + G_{1} B_{1}(t, \theta) e_7(t, \theta)$$

From the equation (4.28) and $\{e_6(t, \theta), e_7^e(t, \theta)\}$ is a linearly dependent set, we get

$$\varepsilon^{(L)}(G_{1} A_{1}(t, \theta)) = 0$$

Then, using $\varepsilon^{(L)}(A_{1}(t, \theta)) = \text{superconstant}$ and $\varepsilon^{(L)}(B_{1}(t, \theta)) = \text{superconstant}$

$$\varepsilon^{(L)} \left( \frac{A_{1}(t, \theta)}{B_{1}(t, \theta)} \right) = a$$

is written where $a$ is a supernumber. From the equation (4.28) and (4.29), we have

$$B_{1}(t, \theta) = \frac{(-1)^{|t|} A_{1}(t, \theta)}{B_{1}(t, \theta)} a_{65}(t, \theta) B_{1}(t, \theta) + (-1)^{|t|} A_{1}(t, \theta) a_{67}(t, \theta).$$

If we divide the equation (4.33) with $B_{1}(t, \theta)$ and separate into the even and odd parts, then we get

$$(-1)^{|t|} A_{1}(t, \theta) = (b + s(A(t, \theta))) \left[ \varepsilon^{(L)}(a_{65}(t, \theta)) + s(a_{65}(t, \theta)) \right]$$

$$+ \left[ \varepsilon^{(L)} \left( \frac{A_{1}(t, \theta)}{B_{1}(t, \theta)} \right) + s \left( \frac{A_{1}(t, \theta)}{B_{1}(t, \theta)} \right) \right] 
\cdot (b + s(A(t, \theta)))$$

From even and odd parts of the equation (4.34), we get

$$\lambda e^{(L)}(a_{65}(t, \theta)) + \mu e^{(L)}(a_{67}(t, \theta)) = 1$$

(4.35)
and
\[ 0 = b \cdot s(a_{65}(t, \theta)) + s(A(t, \theta)[\varepsilon^{(L)}(a_{65}(t, \theta)) + s(a_{65}(t, \theta))] \\
+ \varepsilon^{(L)} \left( \frac{A_{1}(t, \theta)}{B_{1}(t, \theta)} \right) b \cdot s(a_{67}(t, \theta)) \\
+s \left( \frac{A_{1}(t, \theta)}{B_{1}(t, \theta)} \right) \left[ b \cdot \{ \varepsilon^{(L)}(a_{67}(t, \theta)) + s(a_{67}(t, \theta)) \} \\
+ s(A(t, \theta) \{ s(a_{67}(t, \theta)) + \varepsilon^{(L)}(a_{67}(t, \theta)) \} \right]. \] (4.36)

\[ \square \]

**Example** Let \( B_{L}^{2+2} \) be a \((2, 2)\) dimensional total super-Euclidean space, \( V \subset B_{L}^{1,1} \) be an open subset,

\[ c: \quad V \subset B_{L}^{1,1} \quad \rightarrow \quad B_{L}^{2,2} \]

\[ (t, \theta) \quad \rightarrow \quad c(t, \theta) = (t^{2} + 2, \theta \beta^{2}, \theta + 2 \beta^{1} t - 3, \theta t^{2}) \] (4.37)

be a supercurve. Supercurve \( c(t, \theta) \) is supersmooth because the functions
\[ c^{1}(t, \theta) = t^{2} + 2, \quad c^{2}(t, \theta) = \theta \beta^{2}, \quad c^{3}(t, \theta) = \theta + 2 \beta^{1} t - 3, \quad c^{4}(t, \theta) = \theta t^{2} \] (4.38)

are supersmooth. If we compute \( G_{1}c(t, \theta), G_{2}c(t, \theta) \) and \( G_{1}G_{2}c(t, \theta) \), then we have
\[
\begin{align*}
G_{1}c(t, \theta) & = (2t, 0, 2\beta^{1}, 2\theta \cdot t) \\
G_{2}c(t, \theta) & = (0, \beta^{2}, 1, t^{2}) \\
G_{1}G_{2}c(t, \theta) & = (0, 0, 0, 2t)
\end{align*}
\]

Because of satisfying the equation (3.5), we get
\[ \varepsilon^{(L)} \left( \langle G_{2}c(t, \theta), G_{1}G_{2}c(t, \theta) \rangle \right) = 2t > 0 \]
\[ \varepsilon^{(L)} \left( \langle G_{2}c(t, \theta), G_{2}c(t, \theta) \rangle \right) = 0 \]
\[ \varepsilon^{(L)} \left( \langle G_{1}G_{2}c(t, \theta), G_{1}G_{2}c(t, \theta) \rangle \right) = 0 \]

\( e_{1}(t, \theta), e_{3}(t, \theta) \) and \( e_{4}(t, \theta) \) are obtained by
\[
\begin{align*}
e_{1}(t, \theta) & = \left( 1, 0, 2\beta^{1} \cdot (2t)^{-1}, \theta \right) \\
e_{3}(t, \theta) & = \left( 0, \beta^{2} \cdot (2t)^{-1}, (2t)^{-1}, 2^{-1}t \right) \\
e_{4}(t, \theta) & = (0, 0, 0, 2t).
\end{align*}
\] (4.40)

Computing \( G_{1}c(t, \theta), G_{2}c(t, \theta) \) and \( G_{1}G_{2}c(t, \theta) \), the supervectors
\[ \{G_{1}c(t, \theta), G_{2}c(t, \theta), G_{1}G_{2}c(t, \theta)\} \]

are linearly independent and then, system of the supervectors
\[ \{e_{1}(t, \theta), e_{2}(t, \theta), e_{3}(t, \theta), e_{4}(t, \theta)\} \]
is Frenet frame of supercurve \( c \). Let the matrix \( M(t, \theta) \) be
\[
M(t, \theta) = \begin{bmatrix}
1 & 0 & \theta & -2\beta^{1}(2t)^{-1} \\
0 & \beta^{2}(2t)^{-1} & 2^{-1}t & -(2t)^{-1} \\
0 & 0 & 2t & 0
\end{bmatrix} \] (4.41)

and \( e_{2}(t, \theta) = \left( -2\beta^{2} \beta^{1} (2t)^{-1}, -1, 0, -\beta^{2} \right) \) is computed. \( a_{12}(t, \theta) \) and \( a_{33}(t, \theta) \),
\[ a_{12}(t, \theta) = \beta^{1} \beta^{2} t^{-2} \] (4.42)
and
\[ a_{33}(t, \theta) = -t^{-1} \] (4.43)
are obtained. Finally, the matrix
\[
A = \begin{pmatrix}
0 & \beta^1 \beta^2 t^{-2} & 0 & 0 \\
-\beta^1 \beta^2 t^{-2} & 0 & 0 & 0 \\
0 & 0 & -t^{-1} & 0 \\
0 & 0 & 0 & -t^{-1}
\end{pmatrix}
\] (4.44)
can be obtained.

**Example** Let \( B_L^{2+2} \) be a \((2, 2)\) dimensional total super-Euclidean space, \( V \subset B_L^{1,1} \) be an open subset and
\[
c^* : V \subset B_L^{1,1} \rightarrow B_L^{2,2}
\]
\[
(t, \theta) \mapsto c^*(t, \theta) = (t^2 + 1, \theta \beta^2 - \frac{t}{\beta^2 \beta^1}, 2 \beta^1 t + \theta - 3, \theta t^2 - \frac{t}{\beta^1})
\] (4.45)
be a supercurve. \( c^*(t, \theta) \) supercurve is supersmooth because the functions
\[
c^1(t, \theta) = t^2 + 1, \quad c^2(t, \theta) = \theta \beta^2 - (\beta^2 \beta^1)^{-1} t,
\]
\[
c^3(t, \theta) = 2 \beta^1 t + \theta - 3, \quad c^4(t, \theta) = \theta t^2 - (\beta^1)^{-1} t
\]
are supersmooth. We compute \( G_1 c^*(t, \theta), G_2 c^*(t, \theta) \) and \( G_1 G_2 c^*(t, \theta) \) as
\[
G_1 c^*(t, \theta) = (2t, - (\beta^2 \beta^1)^{-1}, 2 \beta^1, 2\theta - (\beta^1)^{-1})
\]
\[
G_2 c^*(t, \theta) = (0, \beta^2, 1, t^2)
\]
\[
G_1 G_2 c^*(t, \theta) = (0, 0, 0, 2t).
\]
Because of satisfying the equation (3.5),
\[
e^{(l)} \langle (G_2 c^*(t, \theta), G_1 G_2 c^*(t, \theta)) \rangle = 2t > 0
\]
\[
e^{(l)} \langle (G_2 c^*(t, \theta), G_2 c^*(t, \theta)) \rangle = 0
\]
\[
e^{(l)} \langle (G_1 G_2 c^*(t, \theta), G_1 G_2 c^*(t, \theta)) \rangle = 0
\]
e_1^*(t, \theta), e_3^*(t, \theta) \) and \( e_3^*(t, \theta) \) are obtained as
\[
e_1^*(t, \theta) = \left(1, - (\beta^2 \beta^1)^{-1}(2t)^{-1}, 2 \beta^1 (2t)^{-1}, \theta - (\beta^1)^{-1}(2t)^{-1}\right)
\]
\[
e_3^*(t, \theta) = \left(0, \beta^2 (2t)^{-1}, (2t)^{-1}, 2^{-1} t\right)
\]
\[
e_3^*(t, \theta) = (0, 0, 0, 2t).
\]
Computing \( G_1 c^*(t, \theta), G_2 c^*(t, \theta) \) and \( G_1 G_2 c^*(t, \theta) \), the supervectors \{\( G_1 c^*(t, \theta), G_2 c^*(t, \theta), G_1 G_2 c^*(t, \theta) \)\} are linearly independent and then system of the supervectors \{\( e_1^*(t, \theta), e_2^*(t, \theta), e_3^*(t, \theta), e_4^*(t, \theta) \)\} is Frenet frame of supercurve \( c \).
Let the matrix \( M(t, \theta) \) be
\[
M(t, \theta) = \begin{bmatrix}
1 - (\beta^2 \beta^1)^{-1}(2t)^{-1} & \theta - (\beta^1)^{-1}(2t)^{-1} & -2 \beta^1 (2t)^{-1} \\
0 & \beta^2 (2t)^{-1} & -2^{-1}t \\
0 & 2t & 0
\end{bmatrix}
\] (4.46)
and \( e_5^*(t, \theta) = -\left( (\beta^2 \beta^1)^{-1} + 2 \beta^2 \beta^1 \right)(2t)^{-1}, -1, 0, -\beta^2 \) is computed. \( a_{12}^*(t, \theta) \) and \( a_{33}^*(t, \theta) \),
\[
a_{12}^*(t, \theta) = \frac{2 (\beta^2 \beta^1)^2 - 1}{2 \beta^2 \beta^1 t^2}
\] (4.47)
and
\[
a_{33}^*(t, \theta) = -t^{-1}
\] (4.48)
are obtained. Finally, the matrix

\[
A = \begin{pmatrix}
0 & \frac{2 (\beta^2 \beta^1)^2 - 1}{2 \beta^2 \beta^1 t^2} & 0 & 0 \\
1 - 2 (\beta^2 \beta^1)^2 & 0 & 0 & 0 \\
\frac{2 \beta^2 \beta^1 t^2}{2 \beta^2 \beta^1} & 0 & -t^{-1} & 0 \\
0 & 0 & 0 & -t^{-1}
\end{pmatrix}
\]  

(4.49)

can be obtained. Finally, since \( \varepsilon^{(L)} \{ e_2(t, \theta), e^*_2(t, \theta) \} \) is linearly dependent and

\[
\varepsilon^{(L)} \langle e^*_1(t, \theta), e_2(t, \theta) \rangle = 0,
\]

we can say that \( (c(t, \theta), c^*(t, \theta)) \) is Bertrand supercurve couple. The distance of Bertrand supercurve couple, as in equation (4.23),

\[
d(c(t, \theta), c^*(t, \theta)) = 1
\]  

(4.50)
is easily found.

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