DETERMINANTAL REPRESENTATION OF TRIGONOMETRIC POLYNOMIAL CURVES VIA SYLVESTER METHOD

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ABSTRACT. For any trigonometric polynomial $\phi(\theta)$, we give a constructive algorithm by Sylvester elimination which produces matrices $C_1, C_2, C_3$ such that $\det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0$. For a typical trigonometric polynomial, we assert that $C_1$ is positive definite, and thus the typical polynomial curve admits a determinantal representation.

1. Introduction and preliminaries

Let $A$ be an $n \times n$ matrix. The real ternary form $F_A(t, x, y)$ associated to $A$ is defined as

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. Kippenhahn [8] characterized the numerical range of $A$, $W(A) = \{\xi^*A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}$, as the convex hull of the real affine part of the dual curve of the curve $F_A(t, x, y) = 0$. The form $F_A(t, x, y)$ is hyperbolic with respect to $(1,0,0)$, i.e., $F_A(1,0,0) \neq 0$, and for any real pair $x, y$, $F_A(t, x, y)$ has only real roots in $t$. The converse part was conjectured by Fiedler [5] and Lax [9], namely, for any real ternary hyperbolic form $f(t, x, y)$, there exist Hermitian(or real symmetric) matrices $S_1$ and $S_2$ such that

$$f(t, x, y) = \det(tI_n + xS_1 + yS_2) = F_S(t, x, y),$$

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where \( S = S_1 + iS_2 \). Helton and Vinnikov [6] gave an affirmative answer to the conjecture (see also [10, 12]). In this case, we call that the form \( f(t, x, y) \) admits a determinantal representation by the matrix \( S \).

In [2], the authors of this paper study a typical roulette curve given by

\[
\phi(\theta) = \exp(in\theta) + a \exp(-i(n-1)\theta),
\]

\(0 \leq \theta \leq 2\pi, n = 2, 3, \ldots, \) and \(0 < a < 1\). In particular, they obtain that there exists a \(2n \times 2n\) matrix \( A \) so that the roulette (1.1) is exactly the algebraic curve defined by \( F_A(t, x, y) \). In other words,

\[
F_A(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0, \quad 0 \leq \theta \leq 2\pi.
\]

A more general form of the roulette curve (1.1) is a class of trigonometric polynomials given by

\[
\phi(\theta) = \sum_{j=-n}^{n} c_j \exp(ij\theta).
\]

The curve \( C_\phi \) in the Gaussian plane associated to the trigonometric polynomial \( \phi \) is defined as

\[
C_\phi = \{(\Re(\phi(\theta)), \Im(\phi(\theta))): 0 \leq \theta \leq 2\pi\}.
\]

By using Henrion method [7] based on Bezoutian resultant, it is shown in [3] that there exist \(2n \times 2n\) real symmetric matrices \( A_1, A_2, A_3 \) so that the curve \( C_\phi \) lies in the curve

\[
\det(A_1 + xA_2 + yA_3) = 0.
\]

Sufficient conditions are given in [3] that guarantee the matrix \( A_1 \) being positive definite. In this case, the curve \( C_\phi \) admits a determinantal representation by the matrix

\[
A_0 = A_1^{-1/2}(A_2 + iA_3)A_1^{-1/2},
\]

that is \( F_{A_0}(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0 \).

We continue our study to construct another algorithm, based on Sylvester matrix, that produces matrices \( C_1, C_2, C_3 \) for trigonometric polynomial \( \phi(\theta) \) in (1.3) satisfying

\[
\det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0.
\]

For a typical trigonometric polynomial \( \phi(\theta) \), we assert that \( C_1 \) is positive definite, and thus the corresponding curve \( C_\phi \) admits a determinantal representation.

2. Sylvester method

Consider a complex trigonometric polynomial \( \phi(\theta) \) as in (1.3). The conjugate of \( \phi(\theta) \) is denoted by

\[
\psi(\theta) = \sum_{j=-n}^{n} \overline{c}_j \exp(-ij\theta) = \sum_{j=-n}^{n} \overline{c}_{-j} \exp(ij\theta).
\]

We substitute the variable \( u = \exp(i\theta) \). Then (1.3) and (2.1) respectively become

\[
\sum_{j=-n}^{n} c_j u^{n+j} - \phi(\theta) u^n = 0,
\]
\[
\sum_{j=-n}^{n} c_{-j} u^{n+j} - \psi(\theta) u^n = 0.
\] (2.3)

Recall that the \(2\ell \times 2\ell\) Sylvester matrix \(H\) of two polynomials
\[
p(u) = \sum_{j=0}^{\ell} \gamma_{\ell-j} u^j \quad \text{and} \quad q(u) = \sum_{j=0}^{\ell} \delta_{\ell-j} u^j
\]
is defined as
\[
H = H_{p,q} = \begin{pmatrix}
\gamma_0 & \gamma_1 & \ldots & \gamma_{\ell} & 0 & 0 & \ldots & 0 \\
0 & \gamma_0 & \gamma_1 & \ldots & \gamma_{\ell} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \ldots & \ldots & \gamma_0 & \gamma_1 & \ldots & \gamma_{\ell} \\
\delta_0 & \delta_1 & \ldots & \ldots & \delta_{\ell} & 0 & \ldots & 0 \\
0 & \delta_0 & \delta_1 & \ldots & \ldots & \delta_{\ell} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \ldots & \ldots & \delta_0 & \delta_1 & \ldots & \ldots & \delta_{\ell}
\end{pmatrix}.
\]

The determinant of the matrix \(H\) is called the resultant of \(p(u)\) and \(q(u)\) with respect to \(u\). It is well known that \(p(u)\) and \(q(u)\) have a common non-constant factor if and only if \(\det(H) = 0\) (cf. \([4, 13]\)).

To construct matrices \(C_1, C_2, C_3\) satisfying (1.4), we introduce a new parameter \(t\) in (2.2) and (2.3), and write
\[
t \sum_{j=-n}^{n} c_{j} u^{n+j} - \phi(\theta) u^n = \sum_{j=0}^{2n} \gamma_{2n-j}(t, z) u^j,
\]
\[
t \sum_{j=-n}^{n} \bar{c}_{-j} u^{n+j} - \psi(\theta) u^n = \sum_{j=0}^{2n} \delta_{2n-j}(t, w) u^j.
\]

Now, let \(H\) be the \(4n \times 4n\) Sylvester matrix of polynomials
\[
p(u : t, z) = \sum_{j=0}^{2n} \gamma_{2n-j}(t, z) u^j \quad \text{and} \quad q(u : t, z) = \sum_{j=0}^{2n} \delta_{2n-j}(t, z) u^j.
\]

Denote the matrix \(H\) with rows \(r_1, r_2, \ldots, r_{4n}\) as
\[
H = H(r_1, r_2, \ldots, r_{4n}).
\] (2.4)

More precisely, the \(j\)-th row of the matrix \(H\) is
\[
r_j = (0_{j-1}, c_{n} t, c_{n-1} t, \ldots, c_0 t - \phi, \ldots, c_{-n} t, 0_{2n-j})
\]
for \(1 \leq j \leq 2n\), and
\[
r_j = (0_{j-2n-1}, c_{-n} t, c_{-n+1} t, \ldots, c_0 t - \psi, \ldots, c_{n} t, 0_{4n-j})
\]
for \(2n + 1 \leq j \leq 4n\), where \(0_k\) stands for \(k\)-dimensional zero vector. We will produce a \(2n \times 2n\) matrix associated to \(\phi(\theta)\) by modifying the matrix \(H\). At first, we define the matrix
\[
\bar{H} = \bar{H}(r_1, \ldots, r_n, \bar{r}_{n+1}, \ldots, \bar{r}_{3n}, r_{3n+1}, \ldots, r_{4n})
\] (2.5)
which is obtained from $H$ (2.4) by replacing the $n+1, n+2, \ldots, 3n$ rows with the following new rows

\[
\begin{align*}
\tilde{r}_{n+1} &= r_{n+1} - \frac{c_{-n}}{c_n} r_{3n+1}, \\
\tilde{r}_{n+2} &= r_{n+2} - \frac{c_{-n}}{c_n} r_{3n+2} - \left(\frac{c_{-n+1}c_n - c_{n+1}\overline{c_n}}{c_n^2}\right) r_{3n+1} \\
\tilde{r}_{n+3} &= r_{n+3} - \frac{c_{-n}}{c_n} r_{3n+3} - \left(\frac{c_{-n+1}c_n - c_{n+1}\overline{c_n}}{c_n^2}\right) r_{3n+2} \\
&\quad - \left[\frac{c_{-n+2}c_n^2 - c_{n+1}c_n\overline{c_n} + c_{-n}(c_{n-1}^2 - c_{n-2}\overline{c_n})}{c_n^3}\right] r_{3n+1}, \\
&\quad \ldots.
\end{align*}
\]

and

\[
\begin{align*}
\tilde{r}_{3n} &= r_{3n} - \frac{c_{-n}}{c_n} r_n, \\
\tilde{r}_{3n-1} &= r_{3n-1} - \frac{c_{-n}}{c_n} r_{n-1} - \left(\frac{c_n c_{-n+1} - c_{n+1}c_n}{c_n^2}\right) r_n, \\
\tilde{r}_{3n-2} &= r_{3n-2} - \frac{c_{-n}}{c_n} r_{n-2} - \left(\frac{c_n c_{-n+1} - c_{n+1}c_n}{c_n^2}\right) r_{n-1} \\
&\quad - \left[\frac{c_n^2 c_{-n+2} - c_n c_{n-1}c_{-n+1} + c_{-n}(c_{n-1}^2 - c_{n-2}c_n)}{c_n^3}\right] r_n, \\
&\quad \ldots.
\end{align*}
\]

The general rows $\tilde{r}_{n+k}, k = 1, 2, \ldots, n$, are formulated by

\[
\tilde{r}_{n+k} = r_{n+k} + \sum_{j=1}^{k} \alpha_j r_{3n+k+1-j},
\]

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are uniquely determined so that the $(3n+1)$-th, $(3n+2)$-th, ..., $(3n+k)$-th entries of the row $\tilde{r}_{n+k}$ all equal 0, while the coefficients $\beta_1, \beta_2, \ldots, \beta_k$ of the general rows

\[
\tilde{r}_{3n+1-k} = r_{3n+1-k} + \sum_{j=1}^{k} \beta_j r_{n+j-k}, \quad k = 1, \ldots, n
\]

are uniquely determined so that the $n$-th, $(n-1)$-th, ..., $(n-k+1)$-th entries of the row $\tilde{r}_{3n+1-k}$ equal 0.

The following result is a key observation for the properties of the matrix $\tilde{H}$ in (2.5).

**Theorem 2.1.** Let $\tilde{H}$ be the matrix defined in (2.5) corresponding to the trigono-
metric polynomial $\phi(\theta)$ in (1.3). Then the following hold:

(i) The upper left $n \times n$ principal submatrix of $\tilde{H}$ is an upper triangular matrix
with diagonals $(c_n t, c_n t, \ldots, c_n t)$.

(ii) The lower right $n \times n$ principal submatrix of $\tilde{H}$ is a lower triangular matrix
with diagonals $(\overline{c_n t}, \overline{c_n t}, \ldots, \overline{c_n t})$.

(iii) The first $n$ entries and the last $n$ entries of the new rows $r_{n+1}, \ldots, r_{2n},
\tilde{r}_{2n+1}, \ldots, \tilde{r}_{3n}$ are all 0.

(iv) The form associated to $\phi(\theta)$ in (1.3) is given by

\[
R(t, x, y) \equiv \det(H) = \det(\tilde{H}) = |c_n|^{2n} t^{2n} \times \det(H_0),
\]

where $H_0$ is the $2n \times 2n$ principal submatrix of $\tilde{H}$ by deleting the first $n$ and last $n$ rows and columns.
(v) If we denote the matrix $H_0$ by
\[ H_0 = H_0(t, \phi, \psi) = H_0(t, x + iy, x - iy) = tC_1 + xC_2 + yC_3, \]  
then we have
\[ \det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0. \]

The matrix $C_1$ obtained in Theorem 2.1 is not necessarily Hermitian and is therefore not positive definite; see, for example, the remark at the end of this section. It is shown in [2] that a special trigonometric polynomial (1.1) admits a determinantal representation. We apply Theorem 2.1 to more general typical trigonometric polynomials of the form $\phi(\theta) = \exp(in\theta) + a \exp(-im\theta)$ which guarantee the positive definiteness of $C_1$.

**Theorem 2.2.** Let $\phi(\theta)$ be a trigonometric polynomial defined by
\[ \phi(\theta) = \exp(in\theta) + a \exp(-im\theta), \]
$0 \leq \theta \leq 2\pi$, where $0 < m < n$ are positive integers and $0 < a < 1$ is a positive real number. Then the matrix $H_0 = tC_1 + xC_2 + yC_3$ in (2.7) satisfies the following conditions:

(i) The $2n \times 2n$ matrices $C_1, C_2, C_3$ are Hermitian and $C_1$ is positive definite.

(ii) The matrix $C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}$ satisfies
\[ F_{C_0}(t, x, y)\det(C_1) = \det(H_0). \]

(iii) For $0 \leq \theta \leq 2\pi$,
\[ F_{C_0}(1, \cos(n\theta) + a \cos(m\theta), \sin(n\theta) - a \sin(m\theta)) = 0. \]

**Proof.** From (2.7), the matrix $H_0(0, x, y) = xC_2 + yC_3$ is the following form
\[ \begin{pmatrix} 0 & P(x, y) \\ Q(x, y) & 0 \end{pmatrix}, \]
where $P(x, y)$ is a lower triangular Toeplitz matrix
\[ P(x, y) = \begin{pmatrix} p_1(x, y) & 0 & 0 & \ldots \\ rp_2(x, y) & p_1(x, y) & 0 & \ldots \\ rp_3(x, y) & p_2(x, y) & p_1(x, y) & \ldots \\ \vdots & \ldots & \ldots & \ldots \end{pmatrix} \in M_n, \]
with
\[ p_1(x, y) = \frac{(-c_n + c_{-n})x + i(-c_n - c_{-n})y}{c_n}, \]
\[ p_2(x, y) = \frac{(c_{n+1}c_n - c_{-n}c_{-n-1})(x - iy)}{c_n^2}, \]
\[ p_3(x, y) = \frac{(c_{n+2}c_n^2 - c_{n+1}c_{n-1}c_n + c_{-n}(c_{n-1}^2 - c_{n-2}c_n))}{c_n^3}, \]
\[ \ldots . \]
and $Q(x, y)$ is an upper triangular Toeplitz matrix

$$Q(x, y) = \begin{pmatrix} q_1(x, y) & q_2(x, y) & q_3(x, y) & \cdots \\ 0 & q_1(x, y) & q_2(x, y) & \cdots \\ 0 & 0 & q_1(x, y) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in M_n$$

with

$$q_1(x, y) = [(-c_n + \overline{c_{-n}})x + i(c_n + \overline{c_{-n}})y] / c_n,$$

$$q_2(x, y) = [(c_n \overline{c_{-n+1}} - c_{n-1} \overline{c_{-n}})(x + iy)] / c_n^2,$$

$$q_3(x, y) = [(c_n^2 \overline{c_{-n+2}} - c_{n-1} c_n \overline{c_{-n+1}}) + \overline{c_{-n}}(c_{n-1}^2 - c_{n-2} c_n)](x + iy) / c_n^3,$$

$$\cdots.$$

Hence the matrices $C_2, C_3$ are Hermitian, and

$$\det(H_0(0, x, y)) = \det(xC_2 + yC_3)$$

$$= (-1)^n p_1(x, y)^n q_1(x, y)^n$$

$$= (-1)^n \{-c_n(x + iy) + c_{-n}(x - iy)\}^n / |c_n|^{2n},$$

Let $\ell = n - m$. Then the matrix $C_1$ is given by

$$ \begin{pmatrix} I_\ell & 0_{2n-2\ell, \ell} & aI_\ell \\ 0_{2n-2\ell, \ell} & (1 - a^2)I_{2n-2\ell} & 0_{2n-2\ell, \ell} \\ aI_\ell & 0_{2n-2\ell, \ell} & I_\ell \end{pmatrix},$$

which is a real symmetric positive definite matrix. The matrix

$$C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}$$

gives a homogeneous polynomial

$$F_{C_0}(t, x, y) = \det(tI_n + xC_1^{-1/2}C_2C_1^{-1/2} + yC_1^{-1/2}C_3C_1^{-1/2})$$

satisfying

$$F_{C_0}(t, x, y)\det(C_1) = \det(H_0) = \det(tC_1 + xC_2 + yC_3).$$

The assertion $(iii)$ follows from the Sylvester construction (2.6) and (2.7) for the trigonometric polynomial $\phi(\theta)$, i.e.,

$$F_{C_0}(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0, \ 0 \leq \theta \leq 2\pi.$$

$\Box$

**Remark 2.3.** Although the matrix $C_1$ in Theorem 2.2 is positive definite for $\phi(\theta) = \exp(i\theta) + a \exp(-i\theta)$, in general, $C_1$ is not Hermitian for an arbitrary trigonometric polynomial $\phi(\theta)$ given in (1.3). For example, let $n = 2$ and

$$\phi(\theta) = \exp(2i\theta) - \frac{1}{4} \exp(i\theta) - \frac{17}{72} + \frac{1}{36} \exp(-i\theta) + \frac{1}{72} \exp(-2i\theta).$$

Then

$$\phi(\theta) \exp(2i\theta) = (\exp(i\theta) + \frac{1}{3})(\exp(i\theta) + \frac{1}{4})(\exp(i\theta) - \frac{1}{3})(\exp(i\theta) - \frac{1}{2}).$$
The matrices constructed by Theorem 2.2 are
\[ C_1 = \begin{pmatrix} 20732 & -5192 & -4828 & 648 \\ -9 & 20714 & -5039 & -4666 \\ -4666 & -5039 & 20714 & -9 \\ r648 & -4828 & -5192 & 20732 \end{pmatrix}, \]
and
\[ xC_2 + yC_3 = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & \alpha \\ \bar{\alpha} & \bar{\beta} & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \end{pmatrix}, \]
where \( \alpha = -20448x - 21024yi, \ \beta = 648x - 648yi. \) The matrix \( C_1 \) is not Hermitian.

3. Discussion

Let \( 0 < m < n \) be two positive integers and \( 0 < a < 1 \) be a real number. Consider a trigonometric polynomial \( \phi(\theta) = \exp(i\theta) + a \exp(-i\theta), 0 \leq \theta \leq 2\pi \) which defines a real affine curve by the relation

\[ x = x(\theta) = \Re(\phi(\theta)), \ y = y(\theta) = \Im(\phi(\theta)), \]

\( 0 \leq \theta \leq 2\pi. \) Based on Bezoutian, the authors of this paper [3] gave a constructive proof by providing real symmetric matrices \( A_1, A_2, A_3 \) so that the curve \((x(\theta), y(\theta))\) lies on \( \det(A_1 + xA_2 + yA_3) = 0. \)

We compare the two construction matrices obtained in [3] and Theorem 2.2 by investigating the following example. The relation between Bezoutian and Sylvester resultants can be found in [11]. Let \( n = 2, m = 1, a = 4/5, \)

\[ \phi(\theta) = \exp(2i\theta) + \frac{4}{5} \exp(-i\theta), \]

Then the matrix \( H_0(t, x, y) = tC_1 + xC_2 + yC_3 \) in (2.7) is computed by

\[ C_1 = \begin{pmatrix} 1 & 0 & 0 & 4/5 \\ 0 & 9/25 & 0 & 0 \\ 0 & 0 & 9/25 & 0 \\ 4/5 & 0 & 0 & 1 \end{pmatrix}, \]

\[ C_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 4/5 & -1 \\ -1 & 4/5 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ C_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & -4i/5 & -i \\ i & 4i/5 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \]

We have that
\[ (C_1)^{-1/2}C_2(C_1)^{-1/2} = \frac{5}{9} \begin{pmatrix} 0 & \sqrt{5} & -2\sqrt{5} & 0 \\ \sqrt{5} & 0 & 4 & -2\sqrt{5} \\ -2\sqrt{5} & 4 & 0 & \sqrt{5} \\ 0 & -2\sqrt{5} & \sqrt{5} & 0 \end{pmatrix} \]
and
\[(C_1)^{-1/2}C_3(C_1)^{-1/2} = \frac{5}{9} \begin{pmatrix} 0 & -i\sqrt{5} & -2i\sqrt{5} & 0 \\ i\sqrt{5} & 0 & -4i & -2i\sqrt{5} \\ 2i\sqrt{5} & 4i & 0 & -i\sqrt{5} \\ 0 & 2i\sqrt{5} & i\sqrt{5} & 0 \end{pmatrix}.\]

Thus the matrix \(C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}\) in Theorem 2.2 is given by
\[C_0 = \frac{10}{9} \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -2\sqrt{5} & 0 & 0 & \sqrt{5} \\ 0 & -2\sqrt{5} & 0 & 0 \end{pmatrix}.\] (3.1)

On the other hand, the matrices constructed by Bezoutian in [3] satisfying
\[6250000 \det(tC_1 + xC_2 + yC_3) = \det(tA_1 + xA_2 + yA_3)\]
are given by
\[A_1 = \begin{pmatrix} 27 & 0 & -63 & 0 \\ 0 & 27 & 0 & -3 \\ -63 & 0 & 207 & 0 \\ 0 & -3 & 0 & 7 \end{pmatrix},\]
and
\[A_2 = \begin{pmatrix} -15 & 0 & 35 & 0 \\ 0 & 65 & 0 & 15 \\ 35 & 0 & 85 & 0 \\ 0 & 15 & 0 & -35 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -60 & 0 & -10 \\ -60 & 0 & -10 & 0 \\ 0 & -10 & 0 & 40 \\ -10 & 0 & 40 & 0 \end{pmatrix},\]

The matrix \(A_1^{-1/2}\) is a scalar multiple of the matrix
\[S = \begin{pmatrix} p & 0 & q & 0 \\ 0 & u & 0 & v \\ q & 0 & r & 0 \\ 0 & v & 0 & w \end{pmatrix},\]

where
\[p = \sqrt{218(6217 + 98\sqrt{5})}, \quad q = 7\sqrt{218(13 - 2\sqrt{5})}, \quad r = \sqrt{218(257 + 98\sqrt{5})}, \quad u = \sqrt{298(1373 + 54\sqrt{5})}, \quad v = 3\sqrt{298(17 - 6\sqrt{5})}, \quad w = 3\sqrt{298(637 + 6\sqrt{5})}.\]

More precisely \(S = 2\sqrt{1089149}A_1^{-1/2}\). The matrices \(A_1^{-1/2}A_2A_1^{-1/2}\) and \(A_1^{-1/2}A_3A_1^{-1/2}\) are respectively real symmetric matrices of the form
\[
\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{13} & 0 & a_{33} & 0 \\ 0 & a_{24} & 0 & a_{44} \end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix} 0 & a_{13} & 0 & a_{14} \\ a_{13} & 0 & a_{23} & 0 \\ 0 & a_{23} & 0 & a_{34} \\ a_{14} & 0 & a_{34} & 0 \end{pmatrix},
\]

where \(a_{ij}\)'s are distinct non-zero real numbers. Therefore none of entries of the matrix \(A_0 = A_1^{-1/2}(A_2 + iA_3)A_1^{-1/2}\) is 0, while the matrix \(C_0\) in (3.1) obtained by
Theorem 2.2 is rather sparse. The sparsity of $A_0$ and $C_0$, obtained by the two methods, is an interesting subject for further study.

We have proposed two constructive algorithms for determinantal representations of the trigonometric polynomial $\phi(\theta) = \exp(i n \theta) + a \exp(-i m \theta)$ by matrices $A_0 = A_1^{-1/2}(A_2 + i A_3)A_1^{-1/2}$ and $C_0 = C_1^{-1/2}(C_2 + i C_3)C_1^{-1/2}$ satisfying (1.2). It is interesting to ask whether the two matrices $A_0$ and $C_0$ are unitarily similar. At this time, we cannot answer this question. Nevertheless, we give a positive answer for the case when

$$\phi(\theta) = \exp(2i \theta) + 4/5 \exp(-i \theta).$$

According to [2], there constructs a matrix

$$B = \frac{10}{9}\begin{pmatrix}
0 & -4 & 0 & 0 \\
0 & 0 & -4 & -3 \\
-5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

satisfying

$$729\det(tI_4 + x\Re(B) + y\Im(B)) = 15625\det(tC_1 + xC_2 + yC_3).$$

At first, we show that the matrices $A_0$ and $B$ are unitarily similar by a unitary intertwining matrix $W$:

$$WA_1^{-1/2}(A_2 + i A_3)A_1^{-1/2} = BW.$$  \hspace{1cm} (3.2)

Setting $WA_1^{1/2} = V$, the matrix $V$ satisfies

$$VA_1^{-1}(A_2 + i A_3) = WA_1^{1/2}A_1^{-1}(A_2 + i A_3) = WA_1^{-1/2}(A_2 + i A_3) = BW A_1^{1/2} = BV,$$

and

$$VA_1^{-1}V^* = WA_1^{1/2}A_1^{-1}A_1^{1/2}W^* = WW^* = I_4. \hspace{1cm} (3.3)$$

Conversely, if $V$ satisfies (3.2) and (3.3) then the unitary matrix $W = VA_1^{-1/2}$ satisfies $WA_1^{-1/2}(A_2 + i A_3)A_1^{-1/2}W^* = B$. Such a matrix $V$ is given by

$$V = \begin{pmatrix}
-3i/2 & 3/2 & -3i/2 & 3/2 \\
3i/2 & 3/2 & 3i/2 & 3/2 \\
-3i/2 & -9/2 & 9i/2 & 3/2 \\
9i/2 & -3/2 & -27i/2 & 1/2
\end{pmatrix}.$$  \hspace{1cm} (3.4)

This shows that $A_0$ and $B$ are unitarily similar.

On the other hand, the matrix $C_0$ is unitarily similar to $B$, and $UC_0U^* = B$ for the unitary matrix

$$U = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1/\sqrt{5} & 0 & 0 & -2/\sqrt{5} \\
2/\sqrt{5} & 0 & 0 & 1/\sqrt{5}
\end{pmatrix}.$$  \hspace{1cm} (3.5)

Thus, both $A_0$ and $C_0$ are unitarily similar to $B$. 
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