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POLYNOMIAL FUNCTIONS AND SPECTRAL SYNTHESIS ON ABELIAN GROUPS

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Abstract. Spectral synthesis deals with the description of translation invariant function spaces. It turns out that the basic building blocks of this description are the exponential monomials, which are built up from exponential functions and polynomial functions. The author collaborated with Laczkovich [Math. Proc. Cambridge Philos. Soc. 143 (2007), no. 1, 103–120] proved that spectral synthesis holds on an Abelian group if and only if the torsion free rank of the group is finite. The author [Aequationes Math. 70 (2005), no. 1-2, 122–130] showed that the torsion free rank of an Abelian group is strongly related to the properties of polynomial functions on the group. Here we show that spectral synthesis holds on an Abelian group if and only if the ring of polynomial functions on the group is Noetherian.

1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. We consider the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to the point-wise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. Continuous homomorphisms of G into the additive topological group of complex numbers and into the multiplicative topological group of nonzero complex numbers, respectively, are called additive and exponential functions, respectively. A function is a polynomial if

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it belongs to the algebra generated by the continuous additive functions. An exponential monomial is the product of a polynomial and an exponential.

Closed translation invariant subspaces of $\mathcal{C}(G)$ are called varieties. It turns out that exponential functions, or more generally, exponential monomials can be considered as basic building bricks of varieties. A given variety may or may not contain any exponential function or exponential monomial. If it contains an exponential function, then we say that spectral analysis holds for the variety. An exponential function in a variety can be considered as a kind of spectral value and the set of all exponential functions in a variety is called the spectrum of the variety. It follows that spectral analysis for a variety means that the spectrum of the variety is nonempty. On the other hand, the set of all exponential monomials contained in a variety is called the spectral set of the variety. It turns out that if an exponential monomial belongs to a variety, then the exponential function appearing in the representation of this exponential monomial belongs to the variety, too. Hence, if the spectral set of a variety is nonempty, then also the spectrum of the variety is nonempty and spectral analysis holds. There is, however, an even stronger property of some varieties, namely, if the spectral set of the variety spans a dense subspace of the variety. In this case we say that spectral synthesis holds for the variety. It follows, that for nonzero varieties spectral synthesis implies spectral analysis. If spectral analysis, respectively, spectral synthesis holds for every variety on an Abelian group, then we say that spectral analysis, respectively, spectral synthesis holds on the Abelian group. A famous and pioneer result of Schwartz [10] exhibits the situation by stating that if the group is the reals with the Euclidean topology, then spectral values do exist, that is, any nonzero variety contains an exponential function. In other words, in this case the spectrum is nonempty, spectral analysis holds. Furthermore, spectral synthesis also holds in this situation: there are sufficiently many exponential monomials in the variety in the sense that their linear hull is dense in the subspace.

In the investigations concerning spectral analysis and spectral synthesis a special attention is attracted by the case of discrete Abelian groups. The first relevant result is due to Lefranc ([9]) proving spectral synthesis for finite direct sums of the additive group of integers. This result has been extended to arbitrary finitely generated Abelian groups in [12].

In [5] Elliot presented a theorem stating that spectral synthesis holds on any Abelian group. Unfortunately, the proof of his theorem has a serious gap, which was pointed out by Gajda ([3]). For several years this gap has not been filled, which means, that both the spectral analysis and spectral synthesis problems have remained open. However, in [13] spectral analysis and in [1] spectral synthesis was proved for commutative torsion groups.

At the 41st International Symposium on Functional Equations in 2003, Noszvaj, Hungary a counterexample to Elliot's theorem was presented by the present author and in [14] a necessary condition was given for the presence of spectral synthesis on discrete Abelian groups: any Abelian group with spectral synthesis has finite torsion free rank. In [7] discrete Abelian groups with spectral analysis have already been characterized: they are exactly those having torsion free rank less than the continuum. Finally, a complete characterization of discrete Abelian groups is presented in [8]: spectral synthesis holds if and only if the torsion free rank of the group is finite.

All the results listed above focus on the problem of spectral analysis and spectral synthesis on the whole group, that is, on all varieties, simultaneously. Nevertheless, an equally interesting problem is to investigate the spectral properties of a single variety, only. Obviously, a possible characterization of this type may not depend merely on the algebraic structure of the group. Hence, it might be useful to reformulate the global condition on the torsion free rank transforming it into some condition which may be localized. In this paper we give an equivalent condition to the property of an Abelian group having finite torsion free rank in terms of the ring structure of polynomial functions on the group.

2. Polynomial functions on Abelian groups

Although the theory of polynomial functions on Abelian groups can be presented in a more general setting (see e.g. [6], [11]), for our purposes it is enough to consider complex valued polynomial functions. The additive group of integers, respectively complex numbers will be denoted by $\mathbb{Z}$, respectively $\mathbb{C}$.

Let G be an Abelian group. For any function $f : G \to \mathbb{C}$ and for any y in G one introduces the *difference of f with increment y* or shortly *the difference of f* as the function $\Delta_y f : G \to \mathbb{C}$ defined by

$$\Delta_y f(x) = f(x+y) - f(x)$$

for any x in G. For any positive integer n and for any $y_1, y_2, \dots, y_n$ in G we use the notation $\Delta_{y_1,y_2,\dots,y_n}$ for the product

$$\Delta_{y_1} \circ \Delta_{y_2} \circ \dots \circ \Delta_{y_n} .$$

In particular, if $y_1 = y_2 = \dots = y_n$, then we write Δ_y^n for $\Delta_{y_1,y_2,\dots,y_n}$.

The functional equation

$$\Delta_{y_1,y_2,\dots,y_{n+1}} f(x) = 0 \tag{2.1}$$

will be referred to as *Fréchet's equation*. Another basic functional equation is

$$\Delta_y^n f(x) = n! f(y) , \tag{2.2}$$

which is called the *monomial equation*. In these equations we suppose that $f : G \to \mathbb{C}$ is a function and the equations hold for all $x, y, y_1, y_2, \dots, y_{n+1}$, respectively. Solutions of the Fréchet's equation (2.1) are called *polynomial functions of degree at most n* and nonzero solutions of the monomial equation (2.2) are called *monomial functions of degree n*. The zero function can be considered as a monomial function of degree -1.

It is known (see e.g. [2], [11]) that any solution $f : G \to \mathbb{C}$ of (2.1) has a unique representation in the form

$$f(x) = \sum_{j=0}^{n} a_j(x)$$

for all x in G, where $a_j : G \to \mathbb{C}$ is a solution of (2.2) with j in place of n. In other words, any nonzero polynomial function of degree at most n has a unique representation as a sum of nonzero monomial functions of degree not higher than n.

Let n be a positive integer. If G, H are any sets and an arbitrary function $F : G^n \to H$ is given, then the function $x \mapsto F(x, x, \ldots, x)$ is called the *diagonalization of* F. The function $F : G^n \to H$ is called *symmetric* if

$$F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = F(x_1, x_2, \ldots, x_n)$$

holds for any $x_1, x_2, \ldots, x_n$ in G and for any permutation σ of the set $\{1, 2, \ldots, n\}$.

If G is an Abelian group and n is a positive integer, then the function $F : G^n \to \mathbb{C}$ is n-*additive* if the function $t \mapsto F(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is a homomorphism of G into $\mathbb{C}$ for each $i = 1, 2, \ldots, n$ and for any elements $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ in G. We call 1-additive functions simply *additive*. Sometimes this terminology is extended for $n = 0$ by considering any constant function to be 0-additive. It is clear that if σ is any permutation of the set $\{1, 2, \ldots, n\}$, then the function $symF : G^n \to \mathbb{C}$ defined by

$$symF(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma} F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

(the summation extends for all permutations σ of the set $\{1, 2, \ldots, n\}$) is obviously symmetric and has the same diagonalization as F. Moreover, if F is n-additive, then $symF$ is n-additive, too.

In [2] (see also [11]) it is proved that if n is a positive integer and G is an Abelian group, then the diagonalization of any nonzero n-additive symmetric function $F : G^n \to \mathbb{C}$ is a monomial function of degree n. Taking $symF$ instead of F we see that this holds for the diagonalization of any nonzero n-additive function. We note that from the results of [2] it follows that the converse of this statement is also true: any monomial function of degree n is the diagonalization of some n-additive symmetric function.

In the class of complex polynomial functions there is a special subclass. Let G be an Abelian group, n a positive integer, P a complex polynomial of degree N in n variables and $a_1, a_2, \ldots, a_n$ (complex) additive functions. Then it is easy to see that the function $x \mapsto P\big(a_1(x), a_2(x), \cdots, a_n(x)\big)$ is a polynomial function of degree at most N. We call polynomial functions of this kind simply *polynomials*. We note that in [11] the terminology is different: these types of polynomial functions are called *normal polynomials* and polynomial functions are called simply polynomials. Roughly speaking, a polynomial on an Abelian group is a polynomial of complex homomorphisms. Polynomials are the elements of the complex algebra generated by all complex homomorphisms.

An important property of complex polynomial functions is that they form a commutative ring.

Theorem 1. Let G be an Abelian group. Then the set of all complex polynomial functions on G is a commutative ring.

Proof. Clearly, the sum of two polynomial functions is a polynomial function. For the rest of the proof, by the results of [2], it is enough to show that the product of the diagonalizations of an m-additive and an n-additive function is the diagonalization of an $m + n$-additive function. Let $f : G^m \to \mathbb{C}$ and $g : G^n \to \mathbb{C}$ be m-additive, respectively n-additive functions, then the function $h : G^{m+n} \to \mathbb{C}$

defined by,

$$h(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = f(x_1, x_2, \dots, x_m)g(y_1, y_2, \dots, y_n)$$

for $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$ in G is clearly $m+n$-additive. It is obvious, that the diagonalization of h is the product of the diagonalizations of f and g. This implies our statement.

The commutativity of the ring of the polynomial functions is obvious. □

Clearly the polynomials form a subring in the ring of polynomial functions. The following theorem characterizes those Abelian groups on which polynomial functions and polynomials coincide (see [15]).

Theorem 2. The torsion free rank of a discrete Abelian group is finite if and only if any complex generalized polynomial on the group is a polynomial.

3. Spectral synthesis

In this section, based on the results in [14], [15] and [7], we show that the presence of spectral synthesis on an Abelian group heavily depends on the structure of the ring of polynomials on the group. Homomorphisms of an Abelian group into the multiplicative group of nonzero complex numbers will be called *exponentials*.

Let G be an Abelian group. Given a function $f : G \to \mathbb{C}$ and an element y in G the function $\tau_y f : G \to \mathbb{C}$ defined by

$$\tau_y f(x) = f(x+y)$$

is called the *translate* of f by y. A linear space of complex valued functions on G, which is closed under forming point-wise limits and containing all translates of its elements is called a *variety*. We say that *spectral synthesis holds* for a given variety if the sum of all finite dimensional varieties is dense in this variety. It is known ([17]), that this is the case if and only if the elements of the function algebra generated by polynomials and exponentials, the so-called *exponential polynomials*, form a dense subspace in the variety. We say that *spectral synthesis holds* on an Abelian group, if spectral synthesis holds for each variety on the group. For more about spectral synthesis the reader should consult [16].

Theorem 3. Let G be an Abelian group. Spectral synthesis holds on G if and only if the ring of polynomial functions over G is Noetherian.

Proof. Suppose that spectral synthesis holds on G. Then, by [14], the torsion free rank of G is finite, and, by [15], the linear space of complex additive functions on G is of finite dimension. Let $a_1, a_2, \dots, a_n$ be a basis of this space. By the linear independence of the functions $a_1, a_2, \dots, a_n$ there exist elements $x_1, x_2, \dots, x_n$ in G such that the matrix $\big(a_i(x_j)\big)_{i,j=1}^n$ is regular. For $k = 1, 2, \dots, n$ we consider the systems of linear equations

$$\delta_{i,k} = \sum_{j=1}^n \lambda_{k,j} a_j(x_i)$$

for the unknowns $\lambda_{k,j}$ $(j = 1, 2, \ldots, n)$, where $i = 1, 2, \ldots, n$. Here $\delta_{i,k} = 1$ for $i = k$ and $\delta_{i,k} = 0$ for $i \neq k$. Using the unique solution $\lambda_{k,j}$ we define the functions

$$A_k(x) = \sum_{j=1}^{n} \lambda_{k,j} a_j(x)$$

for each x in G. Then the functions A_k for $k = 1, 2, \ldots, n$ are additive and linearly independent, as $A_k(x_i) = 1$ for $i = k$ and $A_k(x_i) = 0$ for $i \neq k$. Hence the functions $A_1, A_2, \ldots, A_n$ form a basis in the space of all complex additive functions on G.

Then, again by the results in [15], every polynomial function $p : G \to \mathbb{C}$ has the form

$$p(x) = P\big(A_1(x), A_2(x), \ldots, A_n(x)\big)\,, \tag{3.1}$$

where $P : \mathbb{C}^n \to \mathbb{C}$ is a complex polynomial. We show that equation (3.1) sets up an isomorphism between the ring R of polynomial functions on G and the ring $\mathbb{C}[z_1, z_2, \ldots, z_n]$ of complex polynomials in n variables.

Let H denote the subgroup H of G generated by the elements $x_1, x_2, \ldots, x_n$. Then every element y in H can be written in the form

$$y = m_1x_1 + m_2x_2 + \cdots + m_nx_n \tag{3.2}$$

with some integers $m_1, m_2, \ldots, m_n$. This representation is unique, which follows from the fact that the equation

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0 \tag{3.3}$$

with some integers $m_1, m_2, \ldots, m_n$ implies $m_1 = m_2 = \cdots = m_n = 0$. Indeed, by equation (3.3) we have for $k = 1, 2, \ldots, n$

$$m_k = m_1A_k(x_1) + m_2A_k(x_2) + \cdots + m_nA_k(x_n) = 0\,.$$

Using the representation (3.2) for y we define

$$\Phi(y) = (m_1, m_2, \ldots, m_n)\,.$$

Then clearly $\Phi : H \to \mathbb{Z}^n$ is an isomorphism.

Returning back to the mapping $p \mapsto P$ of the ring R into the polynomial ring $\mathbb{C}[z_1, z_2, \ldots, z_n]$ defined in (3.1), first we show that P is uniquely defined by p. Supposing the contrary, we have that there is a polynomial function $p \neq 0$ in R such that the corresponding P in $\mathbb{C}[z_1, z_2, \ldots, z_n]$ satisfies

$$P\big(A_1(x), A_2(x), \ldots, A_n(x)\big) = 0$$

for each x in G. If $(m_1, m_2, \ldots, m_n)$ is arbitrary in $\mathbb{Z}^n$, then for the element $y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$ in H we have $A_k(y) = m_k$, which means

$$P(m_1, m_2, \ldots, m_n) = P\big(A_1(y), A_2(y), \ldots, A_n(y)\big) = 0\,,$$

that is, $P = 0$ on $\mathbb{Z}^n$. It follows that $P = 0$, hence $p = 0$. It is then clear, that the mapping $p \mapsto P$ is a ring isomorphism, hence, as $\mathbb{C}[z_1, z_2, \ldots, z_n]$ is Noetherian, the necessity part of our theorem is proved.

Suppose now that the ring R of polynomial functions on G is Noetherian and spectral synthesis does not hold on G. Then, by the results in [7], the torsion free rank of G is infinite. It follows that G has a subgroup H which is isomorphic to the direct product of $\aleph_0$ copies of $\mathbb{Z}$, say $H = \prod_{n\in\mathbb{N}} Z_n$, where $Z_n = \mathbb{Z}$ for each $n = 0, 1, \ldots$. Let p_n denote the projection of H onto $\mathbb{Z}_n$, that is

$$p_n(x) = x(n)$$

for each $x : \mathbb{N} \to \mathbb{Z}$ in H and n in N. Then $p_n : H \to \mathbb{C}$ is a homomorphism of H into the additive group of $\mathbb{C}$, that is

$$p_n(x + y) = p_n(x) + p_n(y)$$

holds for all x, y in H and n in $\mathbb{N}$. It is well known that any homomorphism of a subgroup of an Abelian group into a divisible Abelian group can be extended to a homomorphism of the whole group. As the additive group of complex numbers is obviously divisible, the homomorphisms p_n of H can be extended to complex homomorphisms of the whole group G (see e.g. [4], Vol.I., (A.7) Theorem, p.441.). We shall denote the extensions by p_n, too. The functions p_n for $n = 0, 1, \ldots$ belong to the ring R. Let I_n denote the ideal in R generated by the polynomial functions $p_0, p_1, \ldots, p_n$ for $n = 0, 1, \ldots$. Suppose that p_{n+1} belongs to I_n for some n in $\mathbb{N}$. This means that there are polynomial functions $q_0, q_1, \ldots, q_n$ such that

$$p_{n+1}(x) = \sum_{j=0}^{n} q_j(x) p_j(x)$$

holds for each x in H. Let $x^{(n+1)}$ be the element of H whose $n+1$-st component is 1, the others being zero. Putting $x = x^{(n+1)}$ in the above equation we have $1 = 0$, a contradiction.

Hence the ideals $I_0 \subset I_1 \subset \ldots$ form a ascending chain, which contradicts the assumption that R is Noetherian. Our theorem is proved. □

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