Banach J. Math. Anal. 5 (2011), no. 2, 25–32

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

# EP ELEMENTS IN BANACH ALGEBRAS

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Communicated by D. R. Larson

ABSTRACT. An element of a Banach algebra is EP, if it commutes with its Moore-Penrose inverse. We present a number of new characterizations of EP elements in Banach algebra.

## 1. INTRODUCTION AND PRELIMINARIES

Generalized inverses of matrices have important roles in theoretical and numerical methods of linear algebra. The most significant fact is that we can use generalized inverses of matrices, in the case when ordinary inverses do not exist, in order to solve some matrix equations. Similar reasoning can be applied to linear (bounded or unbounded) operators on Banach and Hilbert spaces. Then, it is interesting to consider generalized elements of Banach and $C^*$-algebras, or more general, in rings with or without involution. For general theory of generalized inverses see [2] and [11].

Let $\mathcal{A}$ be a complex unital Banach algebra. An element $a \in \mathcal{A}$ is generalized (or inner) invertible, if there exists some $b \in \mathcal{A}$ such that $aba = a$ holds. In this case $b$ is a generalized (or inner) inverse of $a$. If $aba = a$, then take $c = bab$ to obtain the following: $aca = a$ and $cac = c$. Such $c$ is called a reflexive (or normalized) generalized inverse of $a$. Finally, if $aba = a$, then $ab$ and $ba$ are idempotents. In the case of the $C^*$-algebra, we can require that $ab$ and $ba$ are Hermitian. We arrive to the definition of the Moore-Penrose inverse in $C^*$-algebras (see [13]).

*Date*: Received: 15 November 2010; Accepted: 19 November 2010.

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2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 47A05.

*Key words and phrases.* EP elements, Moore–Penrose inverse, group inverse, Banach algebra.

**Definition 1.1.** Let $\mathcal{A}$ be a unital $C^*$-algebra. An element $a \in \mathcal{A}$ is Moore-Penrose invertible, if there exists some $b \in \mathcal{A}$ such that

$$aba = a, \ bab = b, \ (ab)^* = ab, \ (ba)^* = ba$$

hold. Such $b$ is uniquely determined if it exists [13] and known as the *Moore-Penrose inverse* of $a$, writtern $a^\dagger$.

More generally, an extension [21, 22] of the Moore-Penrose inverse to more general unital Banach algebra flows from the following definition of *Hermitian elements*:

**Definition 1.2.** An element $a \in \mathcal{A}$ is said to be Hemitian if $\|\exp(ita)\| = 1$ for all $t \in R$.

**Definition 1.3.** Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. If there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad ab \text{ and } ba \text{ are Hermitian,}$$

then the element $b$ is the Moore-Penrose inverse of $a$, and it will be denoted by $a^\dagger$.

For an element $a$ in a Banach algebra, if $a^\dagger$ exists, it is uniquely determined (see [21]).

When the Moore-Penrose inverse $a^\dagger$ exists we shall say that $a \in \mathcal{A}$ is MP. The Moore-Penrose inverse has many nice approximation properties, but does not in general commute with the original element. An element with a commuting generalized inverse is very special:

**Definition 1.4.** Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. An element $b \in \mathcal{A}$ is the group inverse of $a$, if the following conditions are satisfied:

$$aba = a, \quad bab = b, \quad ab = ba.$$

The group inverse of $a \in \mathcal{A}$ is unique if it exists, written $a^\#$. When a Moore-Penrose inverse for $a \in \mathcal{A}$ is also a group inverse then $a \in \mathcal{A}$ is said to be EP.

Let $X$ be a Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear bounded operators on $X$. In addition, if $T \in \mathcal{L}(X)$, then $N(T)$ and $R(T)$ stand for the null space and the range of $T$, respectively. The ascent of $T$ is defined as $\text{asc}(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$, and the descent of $T$ is defined as $\text{dsc}(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\}$. In both cases the infimum of the empty set is equal to $\infty$. If $\text{asc}(T) < \infty$ and $\text{dsc}(T) < \infty$, then $\text{asc}(T) = \text{dsc}(T)$.

If $T \in \mathcal{L}(X)$ a closed range operator, then it is not difficult to prove that the following statements are equivalent (see [23] where they were proved for linear transformations):

(i) $T^\#$ exists,

(ii) $X = N(T) \oplus R(T)$,

(iii) $R(T^2) = R(T)$ and $N(T^2) = N(T)$

(iv) $T|_{R(T)} : R(T) \to R(T)$ is a Banach space isomorphism,

In addition, if $T^\#$ exists, then, according to [23, Theorem 1],

(v) $N(T^\#) = N(T)$ and $R(T^\#) = R(T)$,

(vi) $R(TT^{\#}) = R(T)$ and $N(TT^{\#}) = N(T)$.

Note that, according to (iii), given $T \in \mathcal{L}(X)$, necessary and sufficient for $T^{\#}$ to exists is the fact that $\operatorname{asc}(T) = \operatorname{dsc}(T) \leq 1$. Obviously, $R((T^{\#})^n) = R(T^{\#}) = R(T) = R(T^n)$ and $N((T^{\#})^n) = N(T^{\#}) = N(T) = N(T^n)$ for every non-negative integer $n$.

Finally, if $a \in \mathcal{A}$ is an EP element then the group inverse $a^{\#}$ clearly exists, and coincides with the Moore-Penrose inverse $a^{\dagger}$. On the other hand if $a^{\#}$ exists then necessary and sufficient for $a$ to be EP is that $aa^{\#}(= a^{\#}a)$ is Hermitian, so that also $a$ is MP, with $a^{\dagger} = a^{\#}$.

Hence, if $H$ is a Hilbert space and $T \in \mathcal{L}(H)$ is EP, then $N(T^n) = N(T)$ and $R(T^n) = R(T)$ for all positive integers $n$. Moreover, the decomposition $H = R(T) \oplus N(T)$ is orthogonal. It follows that the orthogonal projection on every $R(T^n)$ is unique. This is the derivation of the term EP, which stands for "equal projections". The interest for EP matrices can be found in [1, 5, 6, 14]. EP operators on Hilbert spaces are considered in [7, 8, 9, 12, 15, 17]. Results on EP elements in $C^*$-algebras and rings with involution can be found in [10, 16, 19, 20]. Finally, and the most interesting results related to our work, EP elements in Banach algebras are investigated in [3, 4, 18].

In the following Banach space theorem the range and null space of the Hilbert adjoint are replaced by those of the Moore-Penrose inverse:

**Theorem 1.5.** [3] *Let $X$ be a Banach space, and consider $T \in \mathcal{L}(X)$ such that $T^{\dagger}$ exists. Then, the following statements are equivalent:*

(i) *$T$ is an EP operator;*
(ii) *$N(T) = N(T^{\dagger})$;*
(iii) *$R(T) = R(T^{\dagger})$.*

Finally, we formulate the essential corollary for the proof of our main theorem.

**Corollary 1.6.** *Let $X$ be a Banach space, and let $T \in B(X)$ such that $T^{\dagger}$ and $T^{\#}$ exist. Then the following statements are equivalent:*

(i) *$T$ is EP;*
(ii) *$R(T^{\dagger}) \subset R(T)$;*
(iii) *$R(T) \subset R(T^{\dagger})$;*
(iv) *$N(T) \subset N(T^{\dagger})$;*
(v) *$N(T^{\dagger}) \subset N(T)$.*

*Proof.* According to [3, Theorem 6], the Banach space $X$ can be decomposed using the idempotents $TT^{\#}$, $T^{\dagger}T$, and $TT^{\dagger}$, that is respectively:

$$X = N(T) \oplus R(T) = N(T) \oplus R(T^{\dagger}) = N(T^{\dagger}) \oplus R(T).$$

(i) $\Rightarrow$ (ii): Obviously, from Theorem 1.5.

(ii) $\Rightarrow$ (i): Using the hypothesis $R(T^{\dagger}) \subset R(T)$ and the decomposition of $X$ derived from the idempotents $TT^{\#}$ and $T^{\dagger}T$, we have $R(T^{\dagger}) = R(T)$. By Theorem 1.5, we deduce that $T$ is EP.

The rest of the proof follows in a similar way. $\square$

## 2. EP elements in Banach algebras

There are many characterizations of EP elements in $C^*$-algebras, or rings, as well as EP operators on Hilbert spaces. The following result relies on characterization of the Moore-Penrose inverse in Banach algebras. We use $N$ to denote the set of positive integers.

**Theorem 2.1.** *Let $X$ be a Banach space, and consider $T \in \mathcal{L}(X)$ such that $T^\dagger$ and $T^\#$ exist. Then, $T$ is EP if and only if for some $m, n \in N$ one of the following equivalent conditions holds:*

(i) $(T^\#)^{n+m-1} = (T^\dagger)^m (T^\#)^{n-1}$;
(ii) $(T^\#)^{n+m-1} = (T^\#)^{n-1} (T^\dagger)^m$;
(iii) $TT^\dagger (T^\#)^{n+m-1} = (T^\dagger)^n (T^\#)^m T$;
(iv) $TT^\dagger (T^\#)^{n+m-1} = (T^\#)^m T (T^\dagger)^n$;
(v) $T(T^\#)^n (T^\dagger)^m = T^\dagger T (T^\#)^{n+m-1}$;
(vi) $T(T^\#)^n (T^\dagger)^m = (T^\#)^{n+m-1} T^\dagger T$;
(vii) $T^\dagger T (T^\#)^n = (T^\#)^n T^\dagger T$;
(viii) $(T^\dagger)^2 (T^\#)^n = T^\dagger (T^\#)^n T^\dagger$;
(ix) $T^\dagger (T^\#)^n T^\dagger = (T^\#)^n (T^\dagger)^2$;
(x) $T^\dagger (T^\#)^n = T^\# T^\dagger (T^\#)^{n-1}$;
(xi) $T^\dagger (T^\#)^n = (T^\#)^n T^\dagger$;
(xii) $(T^\#)^n T^\dagger = (T^\#)^{n-1} T^\dagger T^\#$;
(xiii) $T(T^\dagger)^{n+1} = (T^\#)^n$;
(xiv) $(T^\dagger)^{n+1} = (T^\#)^n T^\dagger$;
(xv) $(T^\dagger)^{n+1} = T^\dagger (T^\#)^n$;
(xvi) $(T^\dagger)^n = (T^\#)^n$;
(xvii) $T(T^\#)^n T^\dagger = (T^\#)^n$;
(xviii) $T^\dagger T (T^\#)^n = (T^\dagger)^n$;
(xix) $(T^\#)^n T^\dagger T = (T^\dagger)^n$;
(xx) $T^\dagger (T^\#)^n T + TT^\# (T^\dagger)^n = 2(T^\dagger)^n$;
(xxi) $T^n T T^\dagger + T^\dagger T T^n = 2T^n$;
(xxii) $T^n = T^n T T^\dagger$;
(xxiii) $T^n = T^\dagger T T^n$;
(xxiv) $T^n T^\dagger = T^\dagger T^n$.

*Proof.* If $a$ is EP, then it is easy to check that all statements hold.

On the other hand, to prove that $T$ is EP, we show that one of the conditions of Theorem 1.5 or Corollary 1.6 is satisfied.

(i) From the assumption $(T^\#)^{n+m-1} = (T^\dagger)^m (T^\#)^{n-1}$, we obtain

$$R(T) = R((T^\#)^{n+m-1}) = R((T^\dagger)^m (T^\#)^{n-1}) \subseteq R(T^\dagger).$$

Hence, by Corollary 1.6, $T$ is EP.

(ii) Suppose that $(T^\#)^{n+m-1} = (T^\#)^{n-1} (T^\dagger)^m$. If $x \in N(T^\dagger)$, then $(T^\#)^{n+m-1} x = 0$, i.e. $x \in N((T^\#)^{n+m-1}) = N(T)$. Thus, $N(T^\dagger) \subseteq N(T)$ and, by Corollary 1.6, $T$ is EP.

(iii) Assume that $TT^\dagger (T^\#)^{n+m-1} = (T^\dagger)^n (T^\#)^m T$. Since

$$R(TT^\dagger (T^\#)^{n+m-1}) = TT^\dagger R((T^\#)^{n+m-1}) = TT^\dagger R(T) = R(TT^\dagger T) = R(T),$$

and $R((T^\dagger)^n(T^\#)^mT)\subseteq R(T^\dagger)$, we conclude $R(T)\subseteq R(T^\dagger)$. So, $T$ is EP.

(iv) Let $TT^\dagger(T^\#)^{n+m-1}=(T^\#)^mT(T^\dagger)^n$ and $x\in N(T^\dagger)$. Then $TT^\dagger(T^\#)^{n+m-1}x$ $=0$, and so $(T^\#)^{n+m-1}x\in N(TT^\dagger)\cap R((T^\#)^{n+m-1})=N(T^\dagger)\cap R(T)=\{0\}$. Hence, $x\in N((T^\#)^{n+m-1})=N(T)$. Therefore, $N(T^\dagger)\subseteq N(T)$ and, $T$ is EP.

(v) If the statement $T(T^\#)^n(T^\dagger)^m=T^\dagger T(T^\#)^{n+m-1}$ holds and $x\in N(T^\dagger)$, then $T^\dagger T(T^\#)^{n+m-1}x=0$, i.e. $T(T^\#)^{n+m-1}x\in N(T^\dagger)\cap R(T)=\{0\}$. Thus, $N(T^\dagger)\subseteq N(T)$, and $T$ is EP.

(vi) The assumptions $T(T^\#)^n(T^\dagger)^m=(T^\#)^{n+m-1}T^\dagger T$ and $x\in N(T^\dagger)$ imply $(T^\#)^{n+m-1}T^\dagger Tx=0$, that is $T^\dagger Tx\in N(T)\cap R(T^\dagger)=\{0\}$. So, $x\in N(T^\dagger T)=N(T)$ and then $N(T^\dagger)\subseteq N(T)$. Therefore, $T$ is EP.

(vii) Using the hypothesis $T^\dagger T(T^\#)^n=(T^\#)^nT^\dagger T$ and the equalities

$$\begin{aligned} R(T^\dagger T(T^\#)^n) &= T^\dagger TR((T^\#)^n)=T^\dagger TR(T)\\ &= T^\dagger R(T^2)=T^\dagger R(T)=R(T^\dagger T)=R(T^\dagger),\end{aligned}$$

$$R((T^\#)^nT^\dagger T)=(T^\#)^nR(T^\dagger)=(T^\#)^n(R(T^\dagger)\oplus N(T))=R((T^\#)^n)=R(T),$$

we deduce $R(T^\dagger)=R(T)$. By Theorem 1.5, T is EP.

(viii) Let we suppose that $(T^\dagger)^2(T^\#)^n=T^\dagger(T^\#)^nT^\dagger$, and let $x\in N(T^\#)=N(T)$. Now, $T^\dagger(T^\#)^nT^\dagger x=0$ gives $(T^\#)^nT^\dagger x\in N(T^\dagger)\cap R(T)=\{0\}$. Hence, $x\in N((T^\#)^nT^\dagger)\subseteq N(T^\dagger T^{n+1}(T^\#)^nT^\dagger)=N(T^\dagger TT^\dagger)=N(T^\dagger)$. Thus, $N(T)\subseteq N(T^\dagger)$ and, by Corollary 1.6, $T$ is EP.

(ix) Assume that $T^\dagger(T^\#)^nT^\dagger=(T^\#)^n(T^\dagger)^2$. From

$$\begin{aligned} R(T^\dagger(T^\#)^nT^\dagger) &= T^\dagger(T^\#)^nR(T^\dagger)=T^\dagger(T^\#)^n(R(T^\dagger)\oplus N(T))\\ &= T^\dagger R((T^\#)^n)=T^\dagger R(T)=R(T^\dagger T)=R(T^\dagger),\end{aligned}$$

and $R((T^\#)^n(T^\dagger)^2)\subseteq R((T^\#)^n)=R(T)$, we deduce that $R(T^\dagger)\subseteq R(T)$. By Corollary 1.6, we have that $T$ is EP.

(x) Let the equality $T^\dagger(T^\#)^n=T^\#T^\dagger(T^\#)^{n-1}$ is satisfied. Because

$$R(T^\dagger(T^\#)^n)=T^\dagger R((T^\#)^n)=T^\dagger R(T)=R(T^\dagger T)=R(T^\dagger),$$

and $R(T^\#T^\dagger(T^\#)^{n-1})\subseteq R(T^\#)=R(T)$, it follows $R(T^\dagger)\subseteq R(T)$. Hence, $T$ is EP.

(xi) Suppose that $T^\dagger(T^\#)^n=(T^\#)^nT^\dagger$. In particular,

$$R(T^\dagger(T^\#)^n)=T^\dagger R((T^\#)^n)=T^\dagger R(T^\#)=T^\dagger R(T)=R(T^\dagger T)=R(T^\dagger).$$

On the other hand,

$$R((T^\#)^nT^\dagger)=(T^\#)^n(R(T^\dagger)\oplus N(T))=R((T^\#)^n)=R(T).$$

So, $R(T^\dagger)=R(T)$ and, by Theorem 1.5, T is an EP operator.

(xii) Assume that $(T^\#)^nT^\dagger=(T^\#)^{n-1}T^\dagger T^\#$. If $x\in N(T^\#)=N(T)$, then $(T^\#)^nT^\dagger x=0$ implies $T^\dagger x\in N(T)\cap R(T^\dagger)=\{0\}$. Now $x\in N(T^\dagger)$ and $N(T)\subseteq N(T^\dagger)$. Therefore, $T$ is EP.

(xiii) From $T(T^\dagger)^{n+1}=(T^\#)^n$ and $x\in N(T^\dagger)$, it follows $(T^\#)^nx=0$, that is $x\in N((T^\#)^n)=N(T)$. Thus, $N(T^\dagger)\subseteq N(T)$ and $T$ is EP.

(xiv) By the assumption $(T^\dagger)^{n+1}=(T^\#)^nT^\dagger$ and the equality

$$R((T^\#)^nT^\dagger)=(T^\#)^nR(T^\dagger)=(T^\#)^n(R(T^\dagger)\oplus N(T))=R((T^\#)^n)=R(T),$$

we deduce $R(T) = R((T^\dagger)^{n+1}) \subseteq R(T^\dagger)$. As before, T is an EP operator.

(xv) If $(T^\dagger)^{n+1} = T^\dagger(T^\#)^n$ and $x \in N(T^\dagger)$, then $T^\dagger(T^\#)^n x = 0$. So $(T^\#)^n x \in N(T^\dagger) \cap R(T) = \{0\}$. Hence, $x \in N(T)$ and $N(T^\dagger) \subseteq N(T)$, i.e. $T$ is EP.

(xvi) The condition $(T^\dagger)^n = (T^\#)^n$ gives

$$R(T) = R((T^\#)^n) = R((T^\dagger)^n) \subseteq R(T^\dagger).$$

Thus, T is an EP operator.

(xvii) Let $T(T^\#)^n T^\dagger = (T^\#)^n$ and $x \in N(T^\dagger)$. It follows that $(T^\#)^n x = 0$, i.e. $x \in N((T^\#)^n) = N(T)$. So, $N(T^\dagger) \subseteq N(T)$ and $T$ is EP.

(xviii) Suppose that $T^\dagger T(T^\#)^n = (T^\dagger)^n$. If $x \in N(T^\dagger)$, then $T^\dagger T(T^\#)^n x = 0$ implying $(T^\#)^n x \in N(T^\dagger T) \cap R((T^\#)^n) = N(T) \cap R(T) = \{0\}$. Hence, $x \in N((T^\#)^n) = N(T)$ and $N(T^\dagger) \subseteq N(T)$. Therefore, $T$ is EP.

(xix) By the hypothesis $(T^\#)^n T^\dagger T = (T^\dagger)^n$, if $x \in N(T^\dagger)$ then $(T^\#)^n T^\dagger T x = 0$, giving $T^\dagger T x \in N(T) \cap R(T^\dagger) = \{0\}$. So, $x \in N(T^\dagger T) = N(T)$. Thus, $N(T^\dagger) \subseteq N(T)$ and $T$ is EP.

(xx) Assume that $T^\dagger(T^\#)^n T + TT^\#(T^\dagger)^n = 2(T^\dagger)^n$, and let $x \in N(T^\dagger)$. Now, $T^\dagger T(T^\#)^n x = T^\dagger(T^\#)^n T x = 0$. So, $(T^\#)^n x \in N(T) \cap R(T) = \{0\}$. Hence, $N(T^\dagger) \subseteq N(T)$ and $T$ is EP.

(xxi) If $T^n TT^\dagger + T^\dagger TT^n = 2T^n$ and $x \in N(T^\#) = N(T)$, we have $T^n TT^\dagger x = 0$. Hence, $TT^\dagger x \in N(T) \cap R(T) = \{0\}$. Then we conclude $x \in N(TT^\dagger) = N(T^\dagger)$. So, $N(T) \subseteq N(T^\dagger)$ and $T$ is EP.

(xxii) Let $T^n = T^n TT^\dagger$, and let $x \in N(T^\dagger)$. Now $T^n x = 0$, that is $x \in N(T^n) = N(T)$. Therefore, $N(T^\dagger) \subseteq N(T)$ and T is EP.

(xxiii) If we assume that $T^n = T^\dagger TT^n$, then we have

$$R(T) = R(T^n) = R(T^\dagger TT^n) = T^\dagger R(T) = R(T^\dagger T) = R(T^\dagger).$$

Using Theorem 1.5, we get that $T$ is EP.

(xxiv) Let $T^n T^\dagger = T^\dagger T^n$. From

$$R(T^n T^\dagger) = T^{n-1}R(TT^\dagger) = T^{n-1}R(T) = R(T^n) = R(T)$$

and

$$R(T^\dagger T^n) = T^\dagger R(T^n) = T^\dagger R(T) = R(T^\dagger T) = R(T^\dagger),$$

we have $R(T^\dagger) = R(T)$. By Theorem 1.5, T is an EP operator. □

Now, we return to a general case, i.e. $\mathcal{A}$ is a complex unital Banach algebra, and $a \in \mathcal{A}$ is both Moore-Penrose and group invertible.

**Corollary 2.2.** *Theorem 2.1 holds if we change $\mathcal{L}(X)$ by an arbitrary complex Banach algebra $\mathcal{A}$, and if we change $T$ by an $a \in \mathcal{A}$ such that $a^\dagger$ and $a^\#$ exist.*

*Proof.* The left multiplication by $a$ is the mapping $L_a : \mathcal{A} \to \mathcal{A}$, which is defined as $L_a(x) = ax$ for all $x \in \mathcal{A}$. Then it is easy to see that $L_{a^\dagger} = (L_a)^\dagger$ and $L_{a^\#} = (L_a)^\#$ in the Banach algebra $\mathcal{L}(\mathcal{A})$. Now, if any one of statements (i)-(xxiv) holds for $a$, then the same statement holds for $L_a$ in $\mathcal{L}(\mathcal{A})$. Hence, by Theorem 2.1, $L_a$ is EP in $\mathcal{L}(\mathcal{A})$. From [3, Remark 12], it follows that $a$ is EP in $\mathcal{A}$. □

Notice that the right multiplication by $a$, which is considered as the mapping $R_a : \mathcal{A} \to \mathcal{A}$, and defined as $R_a(x) = xa$ for all $x \in \mathcal{A}$, achieves the same effect as $L_a$ in proof of the preceding result.

**Acknowledgement.** We are grateful to the referee for his/her very valuable comments concerning the paper. The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 174007.

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