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QUASI-MULTIPLIERS OF THE DUAL OF A BANACH ALGEBRA

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Abstract. In this paper we extend the notion of quasi-multipliers to the dual of a Banach algebra A whose second dual has a mixed identity. We consider algebras satisfying weaker condition than Arens regularity. Among others we prove that for an Arens regular Banach algebra which has a bounded approximate identity the space $QM_r(A^*)$ of all bilinear and separately continuous right quasi-multipliers of A^* is isometrically isomorphic to A^{**}. We discuss the strict topology on $QM_r(A^*)$ and apply our results to C^*-algebras and to the group algebra of a compact group.

1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [1] for C^*-algebras. McKennon [15] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: \ A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

Let $QM(A)$ denote the set of all separately continuous quasi-multipliers on A. It is showed in [15] that $QM(A)$ is a Banach space for the norm $\|m\| = \sup\{\|m(a, b)\|;\ a, b \in A, \|a\| = \|b\| = 1\}$. For some classical Banach algebras, the Banach space of quasi-multipliers may be identified with some other known space or algebras.

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For instance, by [15, Corollary of Theorem 22], one can identify $QM(L_1(G))$, where G is a locally compact Hausdorff group, with the measure algebra $M(G)$.

After McKennon's seminal paper the theory of quasi-multipliers on Banach algebras was developed further by Vasudevan and Goel and Takahasi[18], Vasudevan and Goel [17], Kassem and Rowlands [8], Lin [12, 13, 14], Dearden [5], Argün and Rowlands [2], Grosser [7], and Yilmaz and Rowlands [20]. Recently quasimultipliers have been studied in the context of operator spaces by Kaneda and Paulsen [10] and Kaneda [9].

In [7] and [2, p. 235] the notion of quasi-multiplier is extended to the dual of a Banach algebra and concrete representations of the space $QM(A^*)$ has been given in the case of the algebra $K_0(X)$ of all approximable operators on a Banach space X. The aim of this paper is to present a few new statements on quasi-multipliers of the dual A^* of a Banach algebra A whose second dual has a mixed identity. Before we state our main results the basic notation is introduced. We mainly adopt the notations from the monograph [4]. The reader is referred to this book for some results used in this paper, as well.

For a Banach space X, let X^* be its topological dual. The pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. We always consider X naturally embedded into X^{**} through the mapping π, which is given by $\langle \pi(x), \xi \rangle = \langle \xi, x \rangle \quad (x \in X,\ \xi \in X^*)$.

Let A be a Banach algebra. It is well known that on the second dual A^{**} there are two algebra multiplications called the first and the second Arens product, respectively. Since in the paper we use mainly the first Arens product, we recall its definition. Let $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$ be arbitrary. Then one defines $\xi \cdot a$ and $G \cdot \xi$ as $\langle \xi \cdot a, b \rangle = \langle \xi, ab \rangle$ and $\langle G \cdot \xi, b \rangle = \langle G, \xi \cdot b \rangle$, where $b \in A$ is arbitrary. Now, the first Arens product of F and G is an element $F \circ G$ in A^{**} which is given by $\langle F \circ G, \xi \rangle = \langle F, G \cdot \xi \rangle$, where $\xi \in A^*$ is arbitrary. The second Arens product, which we denote by $\circ'$, is defined in a similar way.

The space A^{**} equipped with the first (or second) Arens product is a Banach algebra and A is a subalgebra of it. It is said that A is Arens regular if the equality $F \circ G = F \circ' G$ holds for all $F,\ G \in A^{**}$. For example, every C^*-algebra is Arens regular, see [3]. Note however that $F \circ a = F \circ' a$ and $a \circ F = a \circ' F$ hold for any $a \in A$ and $F \in A^{**}$.

By A^*A we denote the subspace $\{\xi \cdot a;\ \xi \in A^*,\ a \in A\}$ of A^*. Similarly, $AA^* = \{a \cdot \xi;\ a \in A,\ \xi \in A^*\}$. If $A^*A = A^*$, then we say that A^* factors on the left. Similarly, A^* factors on the right if $AA^* = A^*$. Ülger [16] has proved that if A is Arens regular and has a b.a.i., then A^* factors on both sides.

An element E in the second dual A^{**} is said to be a mixed identity if it is a right identity for the first and a left identity for the second Arens product. By [4, Proposition 2.6.21], an element $E \in A^{**}$ is a mixed identity if and only if $E \cdot \xi = \xi = \xi \cdot E$, for every $\xi \in A^*$. Note that A^{**} has a mixed identity if and only if A has a b.a.i.

2. Main results

Let A be a complex Banach algebra. Assume that A^{**} is endowed with the first Arens product and A^* is a Banach A^{**}-bimodule in the natural way. The following is an extension of a definition given in [7].

Definition 2.1. A bilinear mapping $m:\ A^*\times A^{**}\to A^*$ is a *right quasi-multiplier of* A^* if

$$m(F\cdot\xi,G)=F\cdot m(\xi,G)\qquad\text{and}\qquad m(\xi,G\circ F)=m(\xi,G)\cdot F \tag{2.1}$$

hold for arbitrary $\xi\in A^*$ and $F,\ G\in A^{**}$.

Similarly, a bilinear mapping $m':\ A^{**}\times A^*\to A^*$ is a *left quasi-multiplier of* A^* if

$$m'(F\circ G,\xi)=F\cdot m'(G,\xi)\qquad\text{and}\qquad m'(G,\xi\cdot F)=m'(G,\xi)\cdot F$$

hold for arbitrary $\xi\in A^*$ and $F,\ G\in A^{**}$.

Although in our investigation we do not assume Arens regularity, we usually have to assume that the given algebra satisfies the following weaker condition. We say that a Banach algebra A satisfies condition (K) if

$$(F\cdot\xi)\cdot G=F\cdot(\xi\cdot G)\qquad(F,\ G\in A^{**},\ \xi\in A^*).$$

Of course, every Arens regular Banach algebra satisfies condition (K). However, the class of Banach algebras satisfying (K) is larger. It contains, for instance, every Banach algebra A which is an ideal in its second dual. Namely, for arbitrary $F,\ G\in A^{**}$ and $\xi\in A^*$, we have

$$\begin{aligned}\langle(F\cdot\xi)\cdot G,a\rangle&=\langle\pi(a),(F\cdot\xi)\cdot G\rangle=\langle G\circ'\pi(a),F\cdot\xi\rangle=\langle(G\circ'\pi(a))\circ F,\xi\rangle\\&=\langle G\circ'(\pi(a)\circ F),\xi\rangle=\langle\pi(a)\circ F,\xi\cdot G\rangle=\langle F\cdot(\xi\cdot G),a\rangle\qquad(a\in A).\end{aligned}$$

Thus, the class of algebras satisfying the condition (K) is strictly larger than the class of Arens regular algebras. Note however that a unital Banach algebra satisfies condition (K) if and only if it is Arens regular. Indeed, if 1 is the identity for A, then $\pi(1)$ is the identity for $(A^{**},\circ)$ and $(A^{**},\circ\prime)$. Assume that A satisfies the condition (K). For arbitrary $F,G\in A^{**}$ and $\xi\in A^*$, one has

$$\begin{aligned}\langle F\circ G,\xi\rangle&=\langle F,G\cdot\xi\rangle=\langle F\circ'\pi(1),G\cdot\xi\rangle=\langle\pi(1),(G\cdot\xi)\cdot F\rangle\\&=\langle\pi(1),G\cdot(\xi\cdot F)\rangle=\langle\pi(1)\circ G,\xi\cdot F\rangle=\langle G,\xi\cdot F\rangle=\langle F\circ' G,\xi\rangle,\end{aligned}$$

which means that the condition (K) implies Arens regularity.

If A is a Banach algebra satisfying condition (K) and A^{**} has a mixed identity, then a map $m:\ A^*\times A^{**}\to A^*$ is a quasi-multiplier of A^* if and only if

$$m(F\cdot\xi,G\circ H)=F\cdot m(f,G)\cdot H \tag{2.2}$$

holds for arbitrary $F,\ G,\ H\in A^{**}$ and $\xi\in A^*$. Indeed, it is obvious that every bilinear mapping satisfying (2.1) satisfies (2.2) as well. On the other hand, if m satisfies (2.2) and E is a mixed identity for A^{**}, then one has

$$m(F\cdot\xi,G)=m(F\cdot\xi,G\circ E)=F\cdot m(\xi,G)\cdot E=F\cdot m(\xi,G).$$

Similarly, $m(\xi,G\circ H)=m(\xi,G)\cdot H$.

Let $QM_r(A^*)$ be the set of all bilinear and separately continuous right quasi-multipliers of A^*. It is obvious that $QM_r(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\xi, F)||; \quad \xi \in A^*,\ F \in A^{**},\ ||\xi|| \le 1,\ ||F|| \le 1\}.$$

Of course, the same holds for $QM_l(A^*)$, the set of all bilinear and separately continuous left quasi-multipliers of A^*.

Proposition 2.2. *Let A be a Banach algebra satisfying condition (K). Then $QM_r(A^*)$ is a Banach A^{**}-bimodule in a natural way.*

Proof. Let $m \in QM_r(A^*)$ and $H \in A^{**}$ be arbitrary. Define $H * m$ and $m * H$ as $H * m(\xi, G) = m(\xi \cdot H, G)$ and $m * H(\xi, G) = m(\xi, H \circ G)$, where $\xi \in A^*$ and $G \in A^{**}$ are arbitrary. Since equalities

$$\begin{aligned} H * m(F \cdot \xi, G) &= m\big((F \cdot \xi) \cdot H, G\big) = m\big(F \cdot (\xi \cdot H), G\big) \\ &= F \cdot m(\xi \cdot H, G) = F \cdot \big(H * m(\xi, G)\big) \end{aligned}$$

and

$$H * m(\xi, G \circ F) = m(\xi \cdot H, G \circ F) = \big(H * m(\xi, G)\big) \cdot F$$

hold for all $\xi \in A^*$ and $F,\ G \in A^{**}$ we conclude that $H * m$ is a quasi-multiplier. The boundedness of $H * m$ follows from $\|m(\xi \cdot H, G)\| \le \|m\|\|\xi\|\|H\|\|G\|$. Thus, $H * m \in QM_r(A^*)$. A similar reasoning gives $m * H \in QM_r(A^*)$.

It is easily seen that equalities $(H_1 \circ H_2) * m = H_1 * (H_2 * m)$, $m * (H_1 \circ H_2) = (m * H_1) * H_2$, and $(H_1 * m) * H_2 = H_1 * (m * H_2)$ hold for arbitrary $m \in QM_r(A^*)$ and $H_1,\ H_2 \in A^{**}$. □

For some Banach algebras A, there is a natural multiplication on the dual A^*. The following observation is related to Proposition 2.2. If A^* is a Banach algebra with multiplication $\diamond$ which is compatible with the A^{**}-bimodule structure of A^* in the sense that $F \cdot (\xi \diamond \eta) = (F \cdot \xi) \diamond \eta$ holds for arbitrary $\xi,\ \eta \in A^*$ and $F \in A^{**}$. Then $QM_r(A^*)$ has a natural structure of a left Banach A^*-module. Namely, the product $\eta \star m$ of $\eta \in A^*$ and $m \in QM_r(A^*)$ is given by $\eta \star m(\xi, F) = m(\xi \diamond \eta, F)$, where $\eta,\ \xi \in A^*$ and $F \in A^{**}$ are arbitrary.

Let A be a general Banach algebra. Then a map $T : A^* \to A^*$ is called a right multiplier of A^* if

$$T(F \cdot \xi) = F \cdot T(\xi),$$

for all $\xi \in A^*, F \in A^{**}$. With $M_r(A^*)$ we denote the space of all bounded linear right multipliers on A^*. It is obvious that for each $F \in A^{**}$ the right multiplication operator $R_F\xi = \xi \cdot F$ is a right multiplier on A^*. If A^{**} has a mixed identity, then each bounded linear right multiplier on A^* is a right multiplication operator. Indeed, let E be a mixed identity for A^{**} and $T \in M_r(A^*)$ be arbitrary. Then equalities

$$\langle T\xi, a\rangle = \langle E \circ a, T\xi\rangle = \langle E, T(a \cdot \xi)\rangle = \langle R_{T^*(E)}\xi, a\rangle$$

hold for all $a \in A$ and $\xi \in A^*$, which means $T = R_{T^*(E)}$.

Theorem 2.3. *If A^{**} has a mixed identity, then*

$$\rho_T(\xi, F) = (T\xi) \cdot F \qquad (T \in M_r(A^*),\ \xi \in A^*,\ F \in A^{**})$$

defines an injective linear map $\rho: M_r(A^*) \to QM_r(A^*)$ with norm $\|\rho\| \le 1$. Moreover, ρ is onto if A^{**} has an identity. If A^{**} has a mixed identity with norm one, then ρ is an isometry.

Proof. Let $T \in M_r(A^*)$ be arbitrary. It is obvious that ρ_T is a bilinear map from $A^* \times A^{**}$ to A^* and that it is bounded with $\|T\|$. For $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$, we have

$$\rho_T(F \cdot \xi, G) = T(F \cdot \xi) \cdot G = (F \cdot T\xi) \cdot G = F \cdot (T\xi \cdot G) = F \cdot \rho_T(\xi, G)$$

and

$$\rho_T(\xi, G \circ F) = (T\xi) \cdot (G \circ F) = (T\xi \cdot G) \cdot F = \rho_T(\xi, G) \cdot F.$$

Thus, $\rho_T \in QM_r(A^*)$. It follows from the definition that $\rho: M_r(A^*) \to QM_r(A^*)$ is linear. Obviously, $\|\rho_T\| \le \|T\|$, which gives $\|\rho\| \le 1$. Let $E \in A^{**}$ be a mixed identity. If $\rho_T = 0$, then we have $(T\xi) \cdot E = 0$ for every $\xi \in A^*$ and consequently $T = 0$. Assume that E is an identity for A^{**}. Let $m \in QM_r(A^*)$ be arbitrary. It is easily seen that $T\xi = m(\xi, E)$ $(\xi \in A^*)$ defines a bounded right multiplier of A^*. Since equalities $\rho_T(\xi, F) = (T\xi) \cdot F = m(\xi, E) \cdot F = m(\xi, E \circ F) = m(\xi, F)$ hold for all $\xi \in A^*$ and $F \in A^{**}$ we conclude that ρ is onto.

At the end assume that E is mixed identity for A^{**} of norm one. Let $T \in M_r(A^*)$ and $\varepsilon > 0$ be arbitrary. If $\xi \in A^*$ is such that $\|\xi\| \le 1$ and $\|T\| - \varepsilon < \|T\xi\|$, then

$$\|\rho_T\| \ge \|\rho_T(\xi, E)\| = \|T\xi\| > \|T\| - \varepsilon.$$

Thus, ρ is an isometry. □

Corollary 2.4. *If A is a C^*-algebra, then ρ is an isometrical isomorphism from $M_r(A^*)$ onto $QM_r(A^*)$.*

Proof. It is well known that every C^*-algebra is Arens regular and has b.a.i. Thus, A satisfies condition (K) and its second dual A^{**} is unital. □

If A is a Banach algebra satisfying condition (K) and A^{**} has an identity, then Theorem 2.3 allows a natural definition of multiplication in $QM_r(A^*)$. Namely, for arbitrary $m_1,\ m_2 \in QM_r(A^*)$, let $T_1,\ T_2 \in M_r(A^*)$ be uniquely determined multipliers satisfying $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. Then

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2 T_1}$$

gives a well defined multiplication. It is easy to see that $QM_r(A^*)$ is a unital Banach algebra.

Note that $QM_l(A^*)$ as well has a natural multiplication if A is a Banach algebra satisfying condition (K) and A^{**} has a mixed identity. Indeed, let $M_l(A^*)$ be the space of all bounded left multipliers on A^*, i.e., bounded linear operators T on A^* satisfying $T(\xi \cdot F) = T\xi \cdot F$, for all $\xi \in A^*$ and $F \in A^{**}$. A similar reasoning as in Theorem 2.3 shows that the mapping $\lambda: M_l(A^*) \to QM_l(A^*)$, which is defined by

$$\lambda_S(F, \xi) = F \cdot S\xi \qquad (S \in M_l(A^*),\ \xi \in A^*,\ F \in A^{**}),$$

is a linear bijection. Thus, a natural multiplication on $QM_l(A^*)$ is given by $\lambda_{S_1} \circ_\lambda \lambda_{S_2} := \lambda_{S_1 S_2}$.

If A is a Banach algebra such that A^{**} has an identity, say E, of norm one, then one can identify $QM_r(A^*)$ by $M_r(A^*)$ and $QM_l(A^*)$ by $M_l(A^*)$. Since right multipliers on A^* are precisely right multiplication operators with elements in A^{**} and left multipliers are left multiplication operators with same elements we conclude that if A^{**} has an identity of norm one, then Banach algebras $QM_r(A^*)$ and $QM_l(A^*)$ are isomorphic.

Theorem 2.5. *Let A be a Banach algebra satisfying condition (K) and A^{**} has an identity E. Assume A^* factors on the right. Then there exists an isomorphism of A^{**} onto $QM_r(A^*)$.*

Proof. Define a map $\psi : A^{**} \to QM_r(A^*)$ by $\psi(H) = \rho_{R_H}$, where R_H is the right multiplication operator on A^* determined by $H \in A^{**}$. Then, for arbitrary $\xi \in A^*, F \in A^{**}$,

$$\psi(H)(\xi, F) = (\xi \cdot H) \cdot F.$$

We check only the multiplicativity of ψ since the linearity and continuity are evident. Let $H_1, H_2 \in A^{**}$. By Theorem 2.3, there exist $T_1, T_2 \in M_r(A^*)$ such that $\psi(H_1) = \rho_{T_1}$ and $\psi(H_2) = \rho_{T_2}$. Hence, for arbitrary $\xi \in A^*, F \in A^{**}$, we have

$$T_1(\xi) \cdot F = (\xi \cdot H_1) \cdot F \qquad \text{and} \qquad T_2(\xi) \cdot F = (\xi \cdot H_2) \cdot F.$$

It follows

$$\begin{aligned}(\psi(H_1) \circ_\rho \psi(H_2))(\xi, F) &= \rho_{T_2 T_1}(\xi, F) = T_2(T_1(\xi)) \circ F = T_1\xi \cdot (H_2 \circ F) \\ &= \xi \cdot (H_1 \circ H_2 \circ F) = \psi(H_1 \circ H_2)(\xi, F),\end{aligned}$$

which means ψ is a homomorphism.

Assume that $\psi(H) = 0$ for $H \in A^{**}$. Since the mapping ρ is one to one $R_H = 0$. Hence, for each $\xi \in A^*$, one has $\xi \circ H = 0$. Since, by the assumption, A^* factors on the right, we conclude $H = 0$. Thus, ψ is one to one. Homomorphism ψ is onto, as well. Namely, if $m \in QM_r(A^*)$, then there exist $T \in M_r(A^*)$ such that $m = \rho_T = \rho_{R_{T^*(E)}} = \psi(T^*(E))$. □

The previous theorem holds, for instance, for every Arens regular Banach algebra with a b.a.i., in particular for every C^*-algebra.

Let H be a Hilbert space and let $A = K(H)$, the algebra of all compact operators on H. The dual of the space of compact operators is the space of all trace-class operators, $C_1(H)$. The second dual of A is $B(H)$. Since $K(H)$ is a C^*-algebra we have $QM_r(C_1(H)) \cong B(H)$.

Theorem 2.6. *Let A be a Banach algebra satisfying condition (K) and assume that A^{**} has an identity E. If A^{**} is Arens regular then $QM_r(A^*)$ is Arens regular.*

Proof. Let ψ be as in the proof of Theorem 2.5. Thus, it is an onto homomorphism. Of course, $\psi^{**} : (A^{**})^{**} \to (QM_r(A^*))^{**}$ has the same property, as well. Let $\tilde{F},\ \tilde{G} \in (QM_r(A^*))^{**}$. Then there exist $F, G \in (A^{**})^{**}$ such that $\psi^{**}(F) = \tilde{F},\ \psi^{**}(G) = \tilde{G}$. Thus,

$$\tilde{F} \circ \tilde{G} = \psi^{**}(F) \circ \psi^{**}(G) = \psi^{**}(F \circ G) = \psi^{**}(F \circ' G) = \tilde{F} \circ' \tilde{G}. \qquad \square$$

Beside the norm topology there are two other useful topologies on $QM_r(A^*)$. The first is the strict topology β which is given by seminorms

$$m \to \|m * F\| \qquad (F \in A^{**},\ m \in QM_r(A^*)).$$

The second is the quasi-strict topology γ. It is given by seminorms

$$m \to \|m(\xi, F)\| \qquad (\xi \in A^*,\ F \in A^{**},\ m \in QM_r(A^*)).$$

Let τ denote the topology on $QM_r(A^*)$ generated by the norm.

If A^{**} has a mixed identity, then $\gamma \subseteq \beta \subseteq \tau$. Indeed, let a net $\{m_\alpha\}_{\alpha\in I} \subseteq QM_r(A^*)$ converge to $m \in QM_r(A^*)$ in the topology β and let $\xi \in A^*$ be arbitrary. Since A^{**} has a mixed identity the second dual A^{**} is factorable. For arbitrary $F \in A^{**}$, there exist $G, H \in A^{**}$ such that $F = G \circ H$. It follows, by the definition of the topology β, that $||m_\alpha * G - m * G|| \to 0$. Thus

$$\begin{aligned} ||m_\alpha(\xi, F) - m(\xi, F)|| &= ||m_\alpha(\xi, G \circ H) - m(\xi, G \circ H)|| \\ &= ||(m_\alpha * G)(\xi, H) - (m * G)(\xi, H)|| \to 0, \end{aligned}$$

which means that $\{m_\alpha\}_{\alpha\in I}$ converges to m in the topology γ. It is obvious that $\beta \subseteq \tau$.

Theorem 2.7. *Let A be a Banach algebra satisfying condition (K).*
(i) The space $(QM_r(A^*), \gamma)$ is complete.
(ii) If A^{**} has a mixed identity of norm one, then $(QM_r(A^*), \beta)$ is complete.

Proof. (i) Let $\{m_\alpha\}_{\alpha\in I}$ be a γ-Cauchy net in $QM_r(A^*)$. Then, for arbitrary $\xi \in A^*$ and $F \in A^{**}$, we have a Cauchy net $\{m_\alpha(\xi, F)\}_{\alpha\in I}$ in the norm topology of A^*. Let $m(\xi, F) = \lim_\alpha m_\alpha(\xi, F)$. It is obvious that in this way we have defined a bilinear mapping m on $A^* \times A^{**}$ satisfying condition (2.1). Also by uniform boundedness principle ([11], p. 172 and [6], p. 489), m is separately continuous and therefore $m \in QM_r(A^*)$.

(ii) Let $\{m_\alpha\}_{\alpha\in I}$ be a $\beta-$Cauchy net in $QM_r(A^*)$. For each $F \in A^{**}$, the mapping $T_F^\alpha : A^* \to A^*$ which is given by $T_F^\alpha(\xi) = m_\alpha(\xi, F)$ defines elements in $M_r(A^*)$. It is easy to show that $\rho_{T_F^\alpha} = m_\alpha \circ F$. It follows from the definition of the $\beta-$topology that $\{\rho_{T_F^\alpha}\}_{\alpha\in I}$ is a Cauchy net in the norm of $QM_r(A^*)$. By Theorem 2.3, ρ is isometry and therefore $\{T_F^\alpha\}$ is a Cauchy net in the norm of $M_r(A^*)$. By the completeness of $M_r(A^*)$, there exists $T_F \in M_r(A^*)$ such that $||T_F^\alpha - T_F|| \to 0$. Since $\gamma \subseteq \beta$ the net $\{m_\alpha\}_{\alpha\in I}$ is a Cauchy net in γ topology. By the first part of this theorem, $(QM_r(A^*), \gamma)$ is complete. Hence there exist $m \in QM_r(A^*)$ such that

$$\lim_\alpha m_\alpha(\xi, F) = m(\xi, F) \quad \text{for all} \quad \xi \in A^* \quad \text{and} \quad F \in A^{**}.$$

For each $G \in A^{**}$,

$$\begin{aligned} \rho_{T_F}(\xi, G) &= \lim_\alpha \rho_{T_F^\alpha}(\xi, G) = \lim_\alpha (m_\alpha \circ F)(\xi, G) = \lim_\alpha m_\alpha(\xi, F \circ G) \\ &= m(\xi, F \circ G) = m \circ F(\xi, G). \end{aligned}$$

It follows that

$$||m_\alpha \circ F - m \circ F|| = ||\rho_{T_F^\alpha} - \rho_{T_F}|| = ||T_F^\alpha - T_F|| \to 0,$$

which implies that m is the $\beta-$limit of the net $\{m_\alpha\}_{\alpha\in I}$, i.e., $QM_r(A^*)$ is complete in β topology. □

At the end we consider the group algebra of a compact group G. By [21], $L_1(G)$ is Arens regular if and only if G is finite. However, since $L_1(G)$ is a two-sided ideal in its second dual ([19]), it satisfies condition (K). Note that the dual $L_1(G)^*$ can be identified with $L_\infty(G)$.

Let $M(G)$ be the convolution algebra of all bounded regular measures on G. Recall that the convolution product of $f \in L_1(G)$ and $\mu \in M(G)$ is given by

$$f * \mu(x) = \int_G f(xy^{-1})\, d\mu(y).$$

Of course, $L_\infty(G)$ is a Banach $L_1(G)^{**}$-bimodule. However, the space $L_\infty(G)$ has also a natural structure of a Banach $M(G)$-bimodule. The same holds for $L_\infty(G)^* = L_1(G)^{**}$. We will denote all these module multiplications by $*$.

Proposition 2.8. *Let G be a compact group and $A = L_1(G)$. Then the equation*

$$(\theta_\mu(\xi, F) := (\xi * \mu) * F \qquad (\mu \in M(G), \xi \in L_\infty(G),\ F \in L_1(G)^{**})$$

defines a linear isomorphism between $M(G)$ and a subspace of $QM_r(A^*)$.

Proof. Note that by the definition of module action $(\xi * \mu) * F = \xi * (\mu * F)$. From this and condition (K) we conclude that $\theta_\mu \in QM_r(L_1(G)^*)$. Of course, $\theta :\ M(G) \to QM_r(L_1(G)^*)$ is a bounded linear map. We claim that θ is injective. Indeed, suppose that $\theta_\mu = 0$. Then $(\xi * \mu) * F = 0$ for all $\xi \in L_\infty(G)$ and $F \in (L_\infty(G))^*$. Since $L_1(G)$ has a b.a.i. it follows $\xi \circ \mu = 0$. In particular, for each $\xi \in C_0(G)$, $\xi \circ \mu = 0$. Since the measure algebra $M(G)$ is the dual of $C_0(G)$ and it has a b.a.i., $\mu = 0$, as required. □

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