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SOME REMARKS ON THE TRIANGLE INEQUALITY FOR NORMS

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ABSTRACT. Remarks about strengthening of the triangle inequality and its reverse inequality in normed spaces for two and more elements are collected. There is also a discussion on Fischer–Muszély equality for n-elements in a normed space. Some other estimates which follow from the triangle inequality are also presented.

1. INTRODUCTION AND PRELIMINARIES

We present three results on refinements of the triangle inequality and some related estimates of independent interest.

A) The following strengthening of the triangle inequality and its reverse inequality in normed spaces were observed already in 2003. The paper [24] was sent to AMM in 2003 and published in 2006.

Theorem 1.1. *For any nonzero elements x and y in a normed space $X = (X, \|\cdot\|)$ we have*

$$\|x + y\| \leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \tag{1.1}$$

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and

$$\|x+y\| \geq \|x\|+\|y\|-\left(2-\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|\right)\max\{\|x\|,\|y\|\}. \tag{1.2}$$

Moreover, if either $\|x\|=\|y\|$ *or* $y=cx$ *with* $c>0$, *then equality holds in both* (1.1) *and* (1.2).

Proof. ([24], p. 257) Without loss of generality we may assume that $\|x\|\leq\|y\|$. Then, by the triangle inequality

$$\begin{aligned}\|x+y\| &= \|\frac{\|x\|}{\|x\|}x+\frac{\|x\|}{\|y\|}y+(1-\frac{\|x\|}{\|y\|})y\| \\ &\leq \|x\|\ \|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|+\|y\|-\|x\| \\ &= \|y\|+(\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|-1)\|x\| \\ &= \|x\|+\|y\|+(\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|-2)\|x\|,\end{aligned}$$

which establishes the estimate (1.1). Similarly, the computation

$$\begin{aligned}\|x+y\| &= \|\frac{\|y\|}{\|y\|}y+\frac{\|y\|}{\|x\|}x+(1-\frac{\|y\|}{\|x\|})x\| \\ &\geq \|y\|\|\frac{y}{\|y\|}+\frac{x}{\|x\|}\|-|\ \|x\|-\|y\|\ | \\ &= \|y\|\|\frac{y}{\|y\|}+\frac{x}{\|x\|}\|-\|y\|+\|x\| \\ &= \|x\|+\|y\|-(2-\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|)\|y\|\end{aligned}$$

gives the inequality (1.2). □

Estimates (1.1) and (1.2) were explicitly stated and proved in [24, Theorem 1]. Estimate (1.1) appeared also in [15, Lemma 1.1] and implicitly they appeared in [17, Lemma 2]. We can put estimates (1.1) and (1.2) together as

$$\|x+y\|+\left(2-\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|\right)\min\{\|x\|,\|y\|\}\leq\|x\|+\|y\|$$

$$\leq\|x+y\|+\left(2-\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\|\right)\max\{\|x\|,\|y\|\}.$$

Moreover, we can rewrite them as the estimates for the so-called *norm-angular distance* (called also the *Clarkson distance* since he defined it in [7]) between nonzero x and y as $d[x,y]=\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\|$ (cf. [24, Remark 3]):

Corollary 1.2. *For any nonzero elements* x *and* y *in a normed space* $X=(X,\|\cdot\|)$ *we have*

$$\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\|\leq\frac{\|x-y\|+|\ \|x\|-\|y\|\ |}{\max\{\|x\|,\|y\|\}} \tag{1.3}$$

and

$$\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\| \geq \frac{\|x-y\|-|\ \|x\|-\|y\|\ |}{\min\{\|x\|,\|y\|\}}. \tag{1.4}$$

Proof. Estimate (1.3) follows directly from (1.2) and estimate (1.4) directly from (1.1). In fact, inequality (1.2) implies that

$$\begin{aligned}\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\|\max\{\|x\|,\|y\|\} &\leq \|x-y\|+2\max\{\|x\|,\|y\|\}-\|x\|-\|y\| \\ &= \|x-y\|+|\ \|x-y\|\ |,\end{aligned}$$

and inequality (1.1) gives that

$$\begin{aligned}\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\|\max\{\|x\|,\|y\|\} &\geq \|x-y\|+2\min\{\|x\|,\|y\|\}-\|x\|-\|y\| \\ &= \|x-y\|-|\ \|x-y\|\ |.\end{aligned}$$

□

Estimates (1.1) and (1.2) mean for the norm-angular distance that

$$\begin{aligned}(2-d[x,-y])\min\{\|x\|,\|y\|\} \leq \|x\|+\|y\|-\|x+y\| \\ \leq (2-d[x,-y])\max\{\|x\|,\|y\|\}\end{aligned}$$

and they were mentioned in the book by E. Deza and M.-M. Deza [10, p. 52] as a result from [24]. Estimates (1.3) and (1.4) mean that

$$\frac{\|x-y\|-|\ \|x\|-\|y\|\ |}{\min\{\|x\|,\|y\|\}} \leq d[x,y] \leq \frac{\|x-y\|+|\ \|x\|-\|y\|\ |}{\max\{\|x\|,\|y\|\}}.$$

Estimate (1.3) is a refinement of the Massera–Schäffer inequality proved in 1958 (see [26, Lemma 5.1]; see also [14, Theorem 1] and [29, p. 516]): for nonzero vectors x and y in X we have that $d[x,y] \leq \frac{2\|x-y\|}{\max(\|x\|,\|y\|)}$, which is stronger than the Dunkl–Williams inequality $d[x,y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}$ proved in [12]. Estimates (1.2) and (1.4) can be seen as the reverse inequalities of (1.1) and (1.3), respectively. Another proof of estimate (1.4) appeared recently in [27, Theorem 1]. By the way, Mercer [27] is using the name *Maligranda inequality* for (1.3) and *reverse Maligranda inequality* for (1.4), but Pečarić-Rajić [31] called them *Maligranda–Mercer inequalities*.

Note that the Dunkl–Williams inequality holds with constant 2 if and only if X is an inner product space ([20]; cf. also [4, p. 31]) but one cannot replace the constant 2 by 1 in the Massera–Schäffer inequality even for an inner product space. Conditions for equality are proved in [20] and, consequently, we can also ask for the equality conditions in (1.3) and (1.4). The Dunkl–Williams estimate was used in the proof of the fact that the Lipschitz norm of the radial projection $k(X)$ is smaller than 2 (see [8], [33], [35], [4, p. 142]; see also [22], where the radial projection was used in the proof of the Dugunji theorem on a failure of the Brouwer fixed point theorem in arbitrary infinite dimensional Banach space, and [9, Theorem 1], where Desbiens showed that the Schäffer constant $s(X) := \limsup_{\alpha[x,y]\to 0^+} \frac{\alpha[x,y]\max\{\|x\|,\|y\|\}}{\|x-y\|}$ is equal to $k(X)$ in every finite-dimensional Banach space X). Inequality (1.1) was also used in the proof of a certain estimate of the Jordan–von Neumann constant $C_{JN}(X)$ by the James constant $J(X)$ (see [23, Lemma 1] and [25, Lemma 1]).

Recently, Betiuk–Pilarska and Prus [6] used the inequalities (1.1) and (1.2) to find estimate of the James constant of a direct sum of the spaces X_s by its James constants: $J((\sum_{s\in S} X_s)_Z) \leq 2 - \frac{1}{2}[2 - \sup_{s\in S} J(X_s)][2 - J(Z)]$. Moreover, Jiménez–Melado, Llorens–Fuster and Mazuñán–Nararro [16] introduced the notion of *Dunkl–Williams constant*

$$DW(X) = \sup\left\{\frac{\|x\|+\|y\|}{\|x-y\|} :\ x, y \in X, x \neq 0, y \neq 0, x \neq y\right\}$$

and collected its connection with some other constants.

Remark 1.3. The right-hand sides of (1.1) and (1.2) can be written also in another way since

$$\begin{aligned}\|x\|+\|y\| &- \left(2 - \|\frac{x}{\|x\|} + \frac{y}{\|y\|}\|\right)\min\{\|x\|,\|y\|\}\\ &= \min\{\|x\|,\|y\|\}\,\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| + |\ \|x\| - \|y\|\ |\end{aligned}$$

and

$$\begin{aligned}\|x\|+\|y\| &- \left(2 - \|\frac{x}{\|x\|} + \frac{y}{\|y\|}\|\right)\max\{\|x\|,\|y\|\}\\ &= \max\{\|x\|,\|y\|\}\,\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| - |\ \|x\| - \|y\|\ |\,.\end{aligned}$$

Kato, Saito and Tamura ([18], Theorem 1) generalized inequalities (1.1) and (1.2) to n-elements with $n \geq 2$: *For any nonzero elements $x_1, x_2, \ldots, x_n$ in a normed space $X = (X, \|\cdot\|)$ we have*

$$\|\sum_{k=1}^{n} x_k\| \leq \sum_{k=1}^{n}\|x_k\| - \left(n - \|\sum_{k=1}^{n}\frac{x_k}{\|x_k\|}\|\right)\min_{k=1,2,\ldots,n}\|x_k\| \tag{1.5}$$

and

$$\|\sum_{k=1}^{n} x_k\| \geq \sum_{k=1}^{n}\|x_k\| - \left(n - \|\sum_{k=1}^{n}\frac{x_k}{\|x_k\|}\|\right)\max_{k=1,2,\ldots,n}\|x_k\|. \tag{1.6}$$

Remark 1.4. If either $\|x_1\| = \|x_2\| = \ldots = \|x_n\|$ or for a fixed $i \in I = \{1, 2, \ldots, n\}$ we have that $0 \neq x_i \in X$ and $x_k = c_k x_i$ with $c_k > 0$ for $k \in I \setminus \{i\}$, then equality holds in both (1.5) and (1.6).

Immediately from inequalities (1.5) and (1.6) we have the following equivalence.

Corollary 1.5. *For nonzero vectors $x_1, x_2, \ldots, x_n$ in a normed space $X = (X, \|\cdot\|)$ we have equality $\|\sum_{k=1}^{n} x_k\| = \sum_{k=1}^{n}\|x_k\|$ if and only if $\|\sum_{k=1}^{n}\frac{x_k}{\|x_k\|}\| = n$.*

Let us compare the above considerations with the Fischer–Muszély equality for n-elements in a normed space. The statement was proved for two-elements ($n = 2$) by Fischer–Muszély [13] (see also [5, Lemma 1], [3, Lemma 2.1] and [2, Lemma 11.4]) and for three-elements ($n = 3$) by Lin [21, Lemma 1]. An induction proof for all $n \geq 2$ can be found in [1, pp. 335-336] and [2, p. 463]. We present a direct proof following the ideas of Baker [5] and Lin [21]. Some results, without knowledge of the Fisher–Muszély theorem, were presented also by Kato, Saito and Tamura (cf. [18], Lemma 1 and Theorems 3, 4).

Theorem 1.6. (a). *If $x_1, x_2, \ldots, x_n$ are elements in a normed space $X = (X, \|\cdot\|)$, then the equality*

$$\|\sum_{k=1}^{n} x_k\| = \sum_{k=1}^{n} \|x_k\| \tag{1.7}$$

holds if and only we have equality

$$\|\sum_{k=1}^{n} a_k x_k\| = \sum_{k=1}^{n} a_k \|x_k\| \tag{1.8}$$

for any positive numbers $a_1, a_2, \ldots, a_n$.

(b). *If, in addition, X is a strictly convex normed space, that is, its sphere does not contain any segment, then the equalities* (1.7) *and* (1.8) *for nonzero $x_1, x_2, \ldots, x_n \in X$ are equivalent to the equalities*

$$\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \ldots = \frac{x_n}{\|x_n\|}. \tag{1.9}$$

Proof. (a) Of course, it is sufficient to prove the implication (1.7)$\Longrightarrow$ (1.8). Without loss of generality we can assume that $a_1 = \max_{k=1,2,\ldots,n} a_k$. Then, by (1.7), we obtain

$$\begin{aligned}
\|\sum_{k=1}^{n} a_k x_k\| &= \|a_1 \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} (a_1 - a_k) x_k\| \\
&\geq a_1 \|\sum_{k=1}^{n} x_k\| - \|\sum_{k=1}^{n} (a_1 - a_k) x_k\| \\
&\geq a_1 \|\sum_{k=1}^{n} x_k\| - \sum_{k=1}^{n} (a_1 - a_k) \|x_k\| \\
&= \sum_{k=1}^{n} a_k \|x_k\|.
\end{aligned}$$

The reverse inequality follows from the triangle inequality and thus we obtain the equality (1.8).

(b) It is well-known that a normed space X is strictly convex if and only if the equality $\|x+y\| = \|x\| + \|y\|$ for nonzero x, y implies that $x = cy$ for some $c > 0$.

If we assume that (1.7) holds, then for any $1 < j \le n$

$$\begin{aligned} \|x_1 + x_j\| &\ge \|\sum_{k=1}^n x_k\| - \|\sum_{k\ne 1,j} x_k\| \ge \|\sum_{k=1}^n x_k\| - \sum_{k\ne 1,j} \|x_k\| \\ &= \sum_{k=1}^n \|x_k\| - \sum_{k\ne 1,j} \|x_k\| = \|x_1\| + \|x_j\|, \end{aligned}$$

and whence $\|x_1 + x_j\| = \|x_1\| + \|x_j\|$, which by the strict convexity of X gives that $x_1 = c_j x_j$ with $c_j > 0, 1 < j \le n$. Consequently, $c_j = \frac{\|x_1\|}{\|x_j\|}$ or $\frac{x_1}{\|x_1\|} = \frac{x_j}{\|x_j\|}$ for any $1 < j \le n$, and (1.9) is proved.

If (1.9) holds, then for any positive numbers $a_1, a_2, \dots, a_n$

$$\begin{aligned} \|\sum_{k=1}^n a_k x_k\| &= \|\sum_{k=1}^n a_k \|x_k\| \frac{x_k}{\|x_k\|}\| = \|\sum_{k=1}^n a_k \|x_k\| \frac{x_i}{\|x_i\|}\| \\ &= \sum_{k=1}^n a_k \|x_k\| \ \|\frac{x_i}{\|x_i\|}\| = \sum_{k=1}^n a_k \|x_k\|, \end{aligned}$$

and the theorem is proved. □

Remark 1.7. If (1.9) holds, then we have equalities (1.7) and (1.8) without any restriction on a normed space X. It will be interesting to characterize equalities in (1.5) and (1.6).

Some other sharpenings of (1.3) and (1.4) in the case $n \ge 3$ (for $n = 2$ they are just estimates (1.1) and (1.2) was proved by Mitani, Saito, Kato and Tamura [28, Theorem 1]: *For any nonzero elements $x_1, x_2, \dots, x_n$ in a normed space $X = (X, \|\cdot\|)$ we have*

$$\|\sum_{k=1}^n x_k\| + \sum_{k=2}^n \left(k - \|\sum_{i=1}^k \frac{x_i^*}{\|x_i^*\|}\|\right) \left(\|x_k^*\| - \|x_{k+1}^*\|\right) \le \sum_{k=1}^n \|x_k\|$$

$$\le \|\sum_{k=1}^n x_k\| - \sum_{k=2}^n \left(k - \|\sum_{i=n-(k-1)}^n \frac{x_i^*}{\|x_i\|}\right) \left(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|\right),$$

where $x_1^*, x_2^*, \dots, x_n^*$ are the rearrangements of $\|x_1\|, \|x_2\|, \dots, \|x_n\|$ satisfying $\|x_1^*\| \ge \|x_2^*\| \ge \dots \ge \|x_n^*\|$ and $x_0^* = x_{n+1}^* = 0$.

Pečarić and Rajić [31] generalized inequalities (1.3) and (1.4) to n-elements with $n \ge 2$: *if nonzero $x_1, x_2, \dots, x_n \in X$, then*

$$\max_{1\le i\le n} \{\frac{S - D_i}{\|x_i\|}\} \le \|\sum_{k=1}^n \frac{x_k}{\|x_k\|}\| \le \min_{1\le i\le n} \{\frac{S + D_i}{\|x_i\|}\},$$

where $S = \|\sum_{k=1}^n x_k\|$ and $D_i = \sum_{k=1}^n |\|x_k\| - \|x_i\||, i = 1, 2, \dots, n$. They also observed (cf. [31], Corollary 2.3) that from these estimates we can obtain the estimates (1.5) and (1.6) in the form

$$\sum_{k=1}^{n} \|x_k\| \leq \|\sum_{k=1}^{n} x_k\| + \left(n - \|\sum_{k=1}^{n} \frac{x_k}{\|x_k\|}\|\right) \max_{k=1,2,\ldots,n} \|x_k\|$$

and

$$\sum_{k=1}^{n} \|x_k\| \geq \|\sum_{k=1}^{n} x_k\| + \left(n - \|\sum_{k=1}^{n} \frac{x_k}{\|x_k\|}\|\right) \min_{k=1,2,\ldots,n} \|x_k\|.$$

These last inequalities were also obtained, in a more general form and even for convex functions, by Dragomir [11]:

$$\sum_{k=1}^{n} \|x_k\|^p - n^{1-p}\|\sum_{k=1}^{n} x_k\|^p \leq \left(\sum_{k=1}^{n} \|x_k\|^{p-1} - \|\sum_{k=1}^{n} \frac{x_k}{\|x_k\|}\|^p\right) \max_{k=1,2,\ldots,n} \|x_k\|$$

and

$$\sum_{k=1}^{n} \|x_k\|^p - n^{1-p}\|\sum_{k=1}^{n} x_k\|^p \geq \left(\sum_{k=1}^{n} \|x_k\|^{p-1} - \|\sum_{k=1}^{n} \frac{x_k}{\|x_k\|}\|^p\right) \min_{k=1,2,\ldots,n} \|x_k\|,$$

where $p \geq 1$ and $n \geq 2$.

B) In 1930 Alfred Tarski posed in [34] the question to prove that

$$|\ |x| - |y|\ | = |x+y| + |x-y| - |x| - |y|$$

holds for all real numbers x, y, and in 1931 appeared his nice solution [34]. Of course, this equality is not true for complex numbers (as a counterexample it is enough to take $x = 1$ and $y = i$).
Note also that since for all $x, y \in \mathbb{R}$ we have the equality $\max\{|x+y|, |x-y|\} = |x| + |y|$ and also

$$\begin{aligned} |x+y| + |x-y| - |x| - |y| &= |x+y| + |x-y| - \max\{|x+y|, |x-y|\} \\ &= \min\{|x+y|, |x-y|\}, \end{aligned}$$

it yields that for all real x, y:

$$|\ |x| - |y|\ | = |x+y| + |x-y| - |x| - |y| = \min\{|x+y|, |x-y|\}.$$

Next we state a corresponding result in normed spaces.

Theorem 1.8. *For any elements x, y in a normed space $X = (X, \|\cdot\|)$ we have that*

$$|\ \|x\| - \|y\|\ | \leq \|x+y\| + \|x-y\| - \|x\| - \|y\| \leq \min\{\|x+y\|, \|x-y\|\} \tag{1.10}$$

and

$$|\ \|x\| - \|y\|\ | \leq \|x\| + \|y\| - |\ \|x+y\| - \|x-y\|\ |\ . \tag{1.11}$$

Proof. It is clear that for every $x, y \in X$

$$\|x\| + \|y\| + |\ \|x\| - \|y\|\ | = 2\max\{\|x\|, \|y\|\}$$

and

$$\|x\| + \|y\| - |\ \|x\| - \|y\|\ | = 2\min\{\|x\|, \|y\|\}.$$

By the triangle inequality

$$\|x+y\| + \|x-y\| \geq \|x+y \pm (x-y)\| = 2\|x\| \text{ or } = 2\|y\|.$$

Thus,

$$\|x+y\| + \|x-y\| \geq 2\max\{\|x\|, \|y\|\},$$

and by combining these facts we obtain the first estimate in (1.10). The second estimate in (1.10) follows directly from the triangle inequality.
By the triangle inequality

$$|\ \|x+y\| - \|x-y\|\ | \leq \|x+y \pm (x-y)\| = 2\|x\| \text{ or } = 2\|y\|,$$

so that

$$|\ \|x+y\| - \|x-y\|\ | \leq 2\min\{\|x\|, \|y\|\}$$

and also the estimates in (1.11) are proved. □

C) In a normed space $X = (X, \|\cdot\|)$ for a fixed $u \in X$ and $p \geq 1$ we consider new norms $\|\cdot\|_{u,p}$ defined by

$$\|x\|_{u,p} = (\|x + \|x\|u\|^p + \|x - \|x\|u\|^p)^{1/p}. \tag{1.12}$$

The norm $\|\cdot\|_{u,1}$ was considered by Odell and Schlumprecht [30, p. 178] to produce a strictly convex norm in every separable Banach space (cf. also [32, p. 118]).

Theorem 1.9. *The functionals $\|\cdot\|_{u,p}$ defined by (1.12) are norms in X which are equivalent to the norm $\|\cdot\|$.*

To prove this theorem we will need the following lemma of independent interest.

Lemma 1.10. *For fixed x, y in a normed space X and $p \geq 1$ consider the function $f_p : \mathbb{R} \to [0, \infty)$ defined by*

$$f_p(t) = (\|x + ty\|^p + \|x - ty\|^p)^{1/p},\ t \in \mathbb{R}.$$

Then f_p is an even convex function and so is increasing on $[0, \infty)$.

Proof. Let $0 \leq \alpha, \beta \leq 1$ be such that $\alpha + \beta = 1$ and $s, t \in \mathbb{R}$. Then, by the triangle inequality, homogeneouity of the norm $\|\cdot\|$ and the Minkowski inequality for two-dimensional l_2^p-norm, that is, for $a, b, c, d \geq 0$ it yields that

$$\|(a+c, b+d)\|_p \leq \|(a,b)\|_p +, \|(c,d)\|_p$$

or, equivalently,

$$[(a+c)^p + (b+d)^p]^{1/p} \leq (a^p + b^p)^{1/p} + (c^p + d^p)^{1/p},$$

we obtain the convexity of f_p:

$$\begin{aligned} f_p(\alpha s+\beta t) &= (\|x+(\alpha s+\beta t)y)\|^p+\|x-(\alpha s+\beta t)y)\|^p)^{1/p} \\ &= (\|\alpha(x+sy)+\beta(x+ty)\|^p+\|\alpha(x-sy)+\beta(x-ty)\|^p)^{1/p} \\ &\le ([\alpha\|x+sy\|+\beta\|x+ty\|]^p+[\alpha\|x-sy\|+\beta\|x-ty\|]^p)^{1/p} \\ &\le ([\alpha\|x+sy\|]^p+[\alpha\|x-sy\|]^p)^{1/p} \\ &+ ([\alpha\|x+ty\|]^p+[\alpha\|x-ty\|]^p)^{1/p}=\alpha f_p(s)+\beta f_p(t). \end{aligned}$$

Moreover, by the triangle inequality and the concavity of $u^{1/p}$, we get, for $t\in\mathbb{R}$, that

$$\begin{aligned} f_p(0) &= 2^{1/p}\|x\|\le 2^{1/p}\frac{\|x+ty\|+\|x-ty\|}{2} \\ &\le 2^{1/p}(\frac{\|x+ty\|^p+\|x-ty\|^p}{2})^{1/p}=f_p(t). \end{aligned}$$

In particular, if $0\le s<t$, then

$$\begin{aligned} f_p(s) &= f_p\left(\frac{s}{t}t+(1-\frac{s}{t})0\right)\le\frac{s}{t}f_p(t)+(1-\frac{s}{t})f_p(0) \\ &\le \frac{s}{t}f_p(t)+(1-\frac{s}{t})f_p(t)=f_p(t). \end{aligned}$$

Note also that $|f_p(s)-f_p(t)|\le 2^{1/p}|s-t|\|y\|$ for all $s,t\in\mathbb{R}$. □

We are now ready to prove Theorem 1.9.

Proof. We need to show the triangle inequality for $\|\cdot\|_{u,p}$. For any $x,y\in X$, by using the monotonicity from Lemma 1.10, twice the triangle inequality of the norm $\|\cdot\|$ and the Minkowski inequality for the two-dimensional l_2^p-norm, we obtain that

$$\begin{aligned} \|x+y\|_{u,p} &= (\|x+y+\|x+y\|u\|^p+\|x+y-\|x+y\|u\|^p)^{1/p} \\ &\le (\|x+y+(\|x\|+\|y\|)u\|^p+\|x+y-(\|x\|+\|y\|)u\|^p)^{1/p} \\ &= (\|(x+\|x\|u)+(y+\|y\|u)\|^p+\|(x-\|x\|u)+(y-\|y\|u)\|^p)^{1/p} \\ &\le ([\|x+\|x\|u\|+\|y+\|y\|u\|]^p+[\|x-\|x\|u\|+\|y-\|y\|u\|]^p)^{1/p} \\ &\le (\|x+\|x\|u\|^p+\|x-\|x\|u\|^p)^{1/p} \\ &+ (\|y+\|y\|u\|^p+\|y-\|y\|u\|^p)^{1/p}=\|x\|_{u,p}+\|y\|_{u,p}. \end{aligned}$$

Moreover, by the convexity of u^p and the triangle inequality,

$$\|x\|_{u,p}\ge 2^{1/p-1}(\|x+\|x\|u\|+\|x-\|x\|u\|)\ge 2^{1/p}\|x\|,$$

and also, by the triangle inequality,

$$\|x\|_{u,p}\le 2^{1/p}(\|x\|+\|x\|\|u\|).$$

Thus

$$2^{1/p}\|x\|\le\|x\|_{u,p}\le 2^{1/p}(1+\|u\|)\|x\|.$$

□

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