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ON INEQUALITIES OF HARDY–SOBOLEV TYPE

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This paper is dedicated to Professor Josip E. Pečarić

Submitted by T. Riedel

ABSTRACT. Hardy–Sobolev–type inequalities associated with the operator $L := \mathbf{x} \cdot \nabla$ are established, using an improvement to the Sobolev embedding theorem obtained by M. Ledoux. The analysis involves the determination of the operator semigroup $\{e^{-tL^*L}\}_{t>0}$.

1. INTRODUCTION

The following inequalities of Hardy and Sobolev are well-known to play a fundamental role in Analysis:

Hardy's inequality

$$\int_{\mathbb{R}^n} |\nabla f|^p d\mathbf{x} \geq C_H(n,p) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^p}{|\mathbf{x}|^p} d\mathbf{x}, \quad f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \tag{1.1}$$

with best possible constant $C_H(n,p) = \{(n-p)/p\}^p$;

Sobolev's inequality for $1 \leq p < n$ and $p^* := np/(n-p)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_S(n,p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

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with best possible constant

$$C_S(n,p) = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{(p-1)/p} \left\{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{1/n},$$

for $1 < p < n$, and

$$C_S(n,1) = \pi^{-1/2} n^{-1} \left(\Gamma(1+n/2)\right)^{1/n}.$$

From (1.1) and (1.2) it follows that for $0 < \delta < C_H(n,p), 1 \leq p < n$,

$$\begin{aligned} \|\nabla f\|^p_{L^p(\mathbb{R}^n)} &- \delta \|f/|\cdot|\|^p_{L^p(\mathbb{R}^n)} \\ &\geq \{1 - \delta/C_H(n,p)\} \|\nabla f\|^p_{L^p(\mathbb{R}^n)} \\ &\geq [\{1-\delta/C_H(n,p)\}/C_S^p(n,p)] \, \|f\|^p_{L^{p^*}(\mathbb{R}^n)}, \end{aligned}$$

and so

$$\|f\|^p_{L^{p^*}(\mathbb{R}^n)} \leq C \left\{ \|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \delta \|f/|\cdot|\|^p_{L^p(\mathbb{R}^n)} \right\}, \tag{1.3}$$

where $C \geq C_S^p(n,p)\{1-\delta/C_H(n,p)\}^{-1}$. In the case $p=2$, Stubbe [8] shows that the optimal value of the constant C is

$$C_S^2(n,2)[1-\delta/C_H(n,2)]^{-(n-1)/n}.$$

In Theorem 1 below we prove the inequality

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla) f(\mathbf{x})|^p d\mathbf{x} \geq (n/p)^p \int_{\mathbb{R}^n} |f(\mathbf{x})|^p d\mathbf{x}, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.4}$$

which is satisfied (and non-trivial) for all values of n, including $n=p$, and show that this implies Hardy's inequality for $1 \leq p \leq n$. The above argument leading to (1.3) does not work with the right-hand side $\|\nabla f\|^p_{L^p(\mathbb{R}^n)} - \delta \|f/|\cdot|\|^p_{L^p(\mathbb{R}^n)}$ replaced by $\|(\mathbf{x}\cdot\nabla) f\|^p_{L^p(\mathbb{R}^n)} - \delta\|f\|^p_{L^p(\mathbb{R}^n)}$ since, by scaling considerations, we don't have a Sobolev–type inequality

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|(\mathbf{x}\cdot\nabla) f\|_{L^p(\mathbb{R}^n)}$$

for $q \neq p$. It is natural to ask if there is some analogue of Stubbe's inequality, and indeed of the L^p version (1.3), when $\|\nabla f\|$ is replaced by $\|(\mathbf{x}\cdot\nabla) f\|$. This was the question which initiated this research. Our investigation makes use of the following result of Ledoux in [7] which, *inter alia*, improves on the standard Sobolev inequality: for every $1 \leq p < q < \infty$ and every function f in the Sobolev space $W^{1,p}(\mathbb{R}^n)$,

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla f\|^\theta_{L^p(\mathbb{R}^n)} \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}, \tag{1.5}$$

where $\theta = p/q, C$ is a positive constant which depends only on p, q and n, and $B^\alpha_{\infty,\infty}$ is the homogenous Besov space of indices (α, ∞, ∞); see [9]. The latter is the space of tempered distributions for which the norm

$$\|f\|_{B^\alpha_{\infty,\infty}} := \sup_{t>0} \{t^{-\alpha/2} \|P_t f\|_{L^\infty(\mathbb{R}^n)}\}$$

is finite, where $P_t = e^{t\Delta}, t \geq 0$, is the heat semigroup on $\mathbb{R}^n$: recall that $\{P_t\}_{t\geq 0}$ is defined by $P_0 f = f$ and

$$P_t f(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{y}) \mathbf{e}^{-|\mathbf{x}-\mathbf{y}|^2/4\mathbf{t}} \mathbf{dy}$$

for $t > 0, \mathbf{x} \in \mathbb{R}^n$. Cases of (1.5) were earlier established in [2], [3] and [4]. The inequality (1.5) is easily seen to include the classical Sobolev inequality (1.2). Ledoux's technique requires specific information on the heat semi-group $e^{t\Delta}$ in $L^2(\mathbb{R}^n)$. Our first task therefore was to determine the operator semi-group associated with the inequality (1.4), namely e^{-tL^*L}, where $L = \mathbf{x} \cdot \nabla$. This is done in section 3. We show that the analogue of (1.5) is in fact a consequence of Ledoux's result. Corollaries of this analogue in the case $p = 2$, contain the following inequalities:

$$\begin{aligned} \|rf(r\omega)\|^2_{L^{2^*}(\mathbb{R}^n)} &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \sup_{\omega \in \mathbb{S}^{n-1}} \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu))}, \end{aligned}$$

$$\begin{aligned} \|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu))} &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}, \end{aligned} \qquad (1.6)$$

where $2^* = 2n/(n-2), d\mu(r) = r^{n-1}dr, C$ is a positive constant depending only on n and, in polar co-ordinates $\mathbf{x} = r\omega, F(r)$ is the integral mean of f over the unit sphere $\mathbb{S}^{n-1}$, that is,

$$F(r) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(r\omega) d\omega.$$

These have a number of consequences. One is a Hardy–Sobolev type inequality (Corollary 4) which is an analogue of the type we set out to establish of Stubbe's inequality: that if $f, Lf \in L^2(\mathbb{R}^n), n \geq 3$, then, for $\delta \in [0, n^2/4)$,

$$\|rF\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C[\frac{n^2}{4} - \delta]^{-\frac{(n-1)}{n}} \left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \delta\|f\|^2_{L^2(\mathbb{R}^n)}\right\}.$$

It also follows from (1.6) that, for $\delta \in [0, (n-2)^2/4)$,

$$\|F\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C[\frac{(n-2)^2}{4} - \delta]^{-\frac{(n-1)}{n}} \left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \delta\|f/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\right\}. \qquad (1.7)$$

Since $\|F\|_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq |\mathbb{S}^{n-1}|^{-1/2^*}\|f\|_{L^{2^*}(\mathbb{R}^n)}$, by Hölder's inequality, (1.7) is implied by the case $p = 2$ of (1.3).

We also establish the following local Hardy–Sobolev type inequalities (see Corollaries 6 and 7): if f is supported in the annulus $A_R := \{\mathbf{x} \in \mathbb{R}^n : 1/R \leq |\mathbf{x}| \leq R\}$, then

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{2(n-1)/n}\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - (n^2/4)\|f\|^2_{L^2(\mathbb{R}^n)}\right\};$$

$$\|F\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C(\ln R)^{2(n-1)/n}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2\left\|\frac{f}{|\cdot|}\right\|^2_{L^2(\mathbb{R}^n)}\right\}. \quad (1.8)$$

The inequality (1.8) is reminiscent of the case $s = 1$ of (2.6) in [6] (proved in section 6.4); this is also proved in [1]. To be specific, it is that if $f \in C_0^\infty(\Omega)$ and $2 \le q < 2^*$,

$$\|f\|^2_{L^q(\mathbb{R}^n)} \le C|\Omega|^{2(1/q-1/2^*)}\left\{\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - \left[\frac{n-2}{2}\right]^2\left\|\frac{f}{|\cdot|}\right\|^2_{L^2(\mathbb{R}^n)}\right\}, \quad (1.9)$$

where $|\Omega|$ denotes the volume of Ω. It is noted in [6], Remark 2.4, that, in contrast to (1.8), the q in (1.9) must be strictly less than the critical Sobolev exponent $2^* = 2n/(n-2)$ if Ω includes the origin.

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2. The Hardy-type inequality (1.4)

Theorem 2.1. *Let $n \ge 1$ and $1 \le p < \infty$. Then for all $f \in C_0^\infty(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f|^p d\mathbf{x} \ge \left(\frac{n}{p}\right)^p \int_{\mathbb{R}^n} |f|^p d\mathbf{x}. \quad (2.1)$$

Proof. On integration by parts and the application of Hölder's inequality we have

$$\begin{aligned} n\int_{\mathbb{R}^n} |f(\mathbf{x})|^p d\mathbf{x} &= \int_{\mathbb{R}^n} \mathrm{div}(\mathbf{x})|\mathrm{f}(\mathbf{x})|^{\mathrm{p}} \mathrm{d}\mathbf{x} \\ &= -p \,\mathrm{Re}\int_{\mathbb{R}^n} (\mathbf{x}\cdot\nabla)\mathrm{f}(\mathbf{x})|\mathrm{f}(\mathbf{x})|^{\mathrm{p}-2}\overline{\mathrm{f}}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &\le p\left(\int_{\mathbb{R}^n} |(\mathbf{x}\cdot\nabla)f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}\left(\int_{\mathbb{R}^n} |f(\mathbf{x})|^p d\mathbf{x}\right)^{(p-1)/p} \end{aligned}$$

which yields (2.1). □

Remark 2.2. The inequality (2.1) implies (1.1) for $1 \le p \le n$. For we have from

$$\nabla(|\mathbf{x}|f) = \frac{\mathbf{x}}{|\mathbf{x}|}f + |\mathbf{x}|\nabla f$$

that

$$\begin{aligned} \|\nabla(|\mathbf{x}|f)\|_{L^p(\mathbb{R}^n)} &\ge \||\mathbf{x}||\nabla f|\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \\ &\ge \|(\mathbf{x}\cdot\nabla)f\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \\ &\ge \left(\frac{n-p}{p}\right)\|f\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

whence (1.1) on replacing $f(\mathbf{x})$ by $f(\mathbf{x})/|\mathbf{x}|$.

3. Calculation of the semigroup e^{-tL^*L}

Theorem 3.1. *Let $L = \mathbf{x}\cdot\nabla, \mathbf{x} = r\omega, r = |\mathbf{x}|$. Then the semigroup e^{-tL^*L} is given by*

$$(e^{-tL^*L}\psi)(\mathbf{x}) = \frac{e^{-tn^2/4}}{\sqrt{4\pi t}} r^{-n/2} \int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}} s^{-n/2}\psi(s\omega)\, s^{n-1} ds\,. \tag{3.1}$$

Proof. Before embarking on the proof, some preliminary remarks and results might be helpful. The gist of the proof is that after a change of co-ordinates, L^*L is seen to be related to the Laplacian in $\mathbb{R}$, and this then yields the result. The co-ordinate change is determined by the map $\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ defined by

$$(\Phi\psi)(s,\omega) := e^{sn/2}\psi(e^s\omega) \tag{3.2}$$

for $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$. Note that we equip $\mathbb{R}\times\mathbb{S}^{n-1}$ with the usual one dimensional Lebesgue measure on $\mathbb{R}$ and the usual surface measure on $\mathbb{S}^{n-1}$. Thus Φ preserves the L^2 norm. The inverse of Φ satisfies $\Phi^{-1} : L^2(\mathbb{R}\times\mathbb{S}^{n-1}) \to L^2(\mathbb{R}^n)$ and is given by

$$(\Phi^{-1}\varphi)(\mathbf{x}) = r^{-n/2}\varphi\big(\ln r, \omega\big). \tag{3.3}$$

The dilations $U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$U(t)\psi(\mathbf{x}) := e^{tn/2}\psi(e^t\mathbf{x})$$

form a group of unitary operators with generator $U(t) = e^{iAt}$, where A is given by

$$iA\psi = \frac{\partial}{\partial t}U(t)\psi\,|_{t=0} = (\mathbf{x}\cdot\nabla + \frac{n}{2})\psi = \frac{1}{2}(\mathbf{x}\cdot\nabla + \nabla\cdot\mathbf{x})\psi.$$

Thus

$$A = \frac{1}{i}(\mathbf{x}\cdot\nabla + \frac{n}{2}) = -iL - i\frac{n}{2}.$$

and so

$$L = iA - \frac{n}{2},$$

where A is the self-adjoint generator of dilations in $L^2(\mathbb{R}^n)$. In particular,

$$L^*L = (-iA - \frac{n}{2})(iA - \frac{n}{2}) = A^2 + \frac{n^2}{4}.$$

Since

$$(\Phi\psi)(s,\omega) = (U(s)\psi)(\omega)$$

for $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, it follows from the group property of the dilations $U(\cdot)$

that

$$(\Phi(U(t)\psi))(s,\omega) = (U(s)(U(t)\psi))(\omega) = (U(s+t)\psi)(\omega) = (\Phi\psi)(s+t,\omega).$$

In particular, in the new co-ordinates given by Φ, the dilations $U(t)$ act simply as shifts by t and should be diagonalizable with the help of a Fourier transform! We now proceed to confirm this prediction.

Define $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1})$ by

$$(M\psi)(\tau,\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} (\Phi\psi)(s,\omega)\, ds, \tag{3.4}$$

so that $M = \mathcal{F} \circ \Phi$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. Then

$$\begin{aligned}(MU(t)\psi)(\tau,\omega) &= \frac{1}{\sqrt{2\pi}} \int e^{-is\tau} (\Phi\psi)(s+t,\omega)\, ds \\ &= \frac{e^{it\tau}}{\sqrt{2\pi}} \int e^{-is\tau} (\Phi\psi)(s,\omega)\, ds = e^{it\tau}(M\psi)(\tau,\omega).\end{aligned} \tag{3.5}$$

The map $M = \mathcal{F} \circ \Phi$ is the Mellin transformation and has an explicit representation using the group structure of $\mathbb{R}^+$ under multiplication: it is the Fourier transform on this group.

The next step is to show that

$$(MA\psi)(\tau,\omega) = \tau(M\psi)(\tau,\omega) \tag{3.6}$$

for ψ in the domain $\mathcal{D}(A)$: it follows that $\psi \in \mathcal{D}(A)$ if and only if $(\tau,\omega) \mapsto \tau(M\psi)(\tau,\omega) \in L^2(\mathbb{R} \times \mathbb{S}^{n-1})$. To see (3.6) we note that $iAe^{itA} = \partial_t U(t)$ and so, from (3.5)

$$\begin{aligned}(MiAe^{iAt}\psi)(\tau,\omega) &= (M\partial_t U(t)\psi)(\tau,\omega) = \partial_t (MU(t)\psi)(\tau,\omega) \\ &= \partial_t e^{it\tau}(M\psi)(\tau,\omega) = i\tau e^{it\tau}(M\psi)(\tau,\omega).\end{aligned}$$

Setting $t = 0$ yields (3.6).

We are now in a position to complete the proof of the theorem. We have $e^{-tL^*L} = e^{-tn^2/4} e^{-tA^2}$ and by (3.4)

$$(Me^{-tA^2}\psi)(\tau,\omega) = e^{-t\tau^2}(M\psi)(\tau,\omega).$$

So

$$e^{-tA^2} = M^{-1} e^{-t\tau^2} M.$$

Since $M = \mathcal{F} \circ \Phi$, we see that

$$e^{-tA^2} = \Phi^{-1} \circ \mathcal{F}^{-1}\big(e^{-t\tau^2} \mathcal{F} \circ \Phi\big).$$

Of course,

$$\begin{aligned}\mathcal{F}^{-1}\big(e^{-t\tau^2} M\psi\big)(\lambda,\omega) &= \mathcal{F}^{-1}\big(e^{-t\tau^2}\mathcal{F} \circ \Phi\big)(\lambda,\omega) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda\tau} e^{-t\tau^2} e^{-is\tau} (\Phi\psi)(s,\omega) ds d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-t\tau^2 + i(\lambda - s)\tau} d\tau \Big) (\Phi\psi)(s,\omega)\, ds\end{aligned}$$

The integral in big parentheses is a Gaussian integral which gives

$$\int_{\mathbb{R}} e^{-t\tau^2 + i(\lambda-s)\tau} d\tau = \sqrt{\frac{\pi}{t}} e^{-\frac{(\lambda-s)^2}{4t}}.$$

Thus

$$\mathcal{F}^{-1}\big(e^{-t\tau^2}M\psi\big)(\lambda,\omega) = \frac{1}{\sqrt{4\pi t}}\int e^{-\frac{(\lambda-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds =: \varphi_t(\lambda,\omega)$$

and, with $\mathbf{x} = r\omega$,

$$\begin{aligned}(e^{-tA^2}\psi)(r\omega) &= (\Phi^{-1}\varphi_t)(r\omega)\\ &= r^{-n/2}\varphi_t(\ln r,\omega)\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_{\mathbb{R}} e^{-\frac{(\ln r-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds.\end{aligned}$$

Since $(\Phi\psi)(s,\omega) = e^{sn/2}\psi(e^s\omega)$, we get from the change of variables $z = e^s$,

$$\begin{aligned}(e^{-tA^2}\psi)(r\omega) &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_{\mathbb{R}} e^{-\frac{(\ln r-s)^2}{4t}}(\Phi\psi)(s,\omega)\,ds\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{\frac{n}{2}-1}\psi(z\omega)dz.\end{aligned}$$

So

$$\begin{aligned}(e^{-tL^*L}\psi)(r\omega) &= e^{-tn^2/4}(e^{-tA^2}\psi)(r\omega)\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}e^{-tn^2/4}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{\frac{n}{2}-1}\psi(z\omega)\,dz\\ &= \frac{1}{\sqrt{4\pi t}}r^{-n/2}e^{-tn^2/4}\int_0^\infty e^{-\frac{(\ln r-\ln z)^2}{4t}}z^{-\frac{n}{2}}\psi(z\omega)\,z^{n-1}dz\end{aligned}$$

which is (3.1).

Once it is realised that A is simply multiplication by τ in the sense of (3.6), it is clear that A is the momentum operator on $\mathbb{R}$, that is, $\Phi A\Phi^{-1}$ is given by

$$\Phi A\Phi^{-1} = -i\partial_s \otimes \mathbf{1}_{\mathbb{S}^{n-1}}$$

On using this and the functional calculus we get

$$\Phi L^*L\Phi^{-1} = (\Phi A\Phi^{-1})^2 + \frac{n^2}{4} = -\partial_s^2 \otimes \mathbf{1}_{S^{n-1}} + \frac{n^2}{4}.$$

Thus, $L^*L = -\Phi^{-1}\partial_s^2 \otimes \mathbf{1}_{S^{n-1}}\Phi + \frac{n^2}{4}$ and

$$e^{-tL^*L} = e^{-tn^2/4}e^{-t\Phi^{-1}\partial_s^2\otimes\mathbf{1}_{S^{n-1}}\Phi} = e^{-tn^2/4}\Phi^{-1}e^{-t\partial_s^2\otimes\mathbf{1}_{S^{n-1}}}\Phi \tag{3.7}$$

which is a convenient way of expressing (3.1). □

On substituting (3.2) and (3.3) and making an obvious change of variables, we obtain from (3.1) the following representation for e^{-tA^2}; see also (3.7).

Corollary 3.2. *Let P_t denote e^{-tA^2}. Then*

$$\Phi P_t\Phi^{-1}\varphi(r,\omega) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\exp\{-\frac{1}{4t}(r-s)^2\}\varphi(s\omega)ds. \tag{3.8}$$

4. The main inequalities

The fact that $\Phi e^{-tA^2}\Phi^{-1}$ in (3.8) is essentially radial means that the analogue of (1.5) derived by Ledoux's technique is a consequence of the one-dimensional case of (1.5). Defining B^α to be the space of all tempered distributions g on $\mathbb{R}\times\mathbb{S}^{n-1}$ for which the norm

$$\|g\|_{B^\alpha} := \sup_{t>0}\{t^{-\alpha/2}\|\Phi e^{-tA^2}\Phi^{-1}g\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\} < \infty, \tag{4.1}$$

one obtains from the $n=1$ case of (1.5), that for any $\omega\in\mathbb{S}^{n-1}$,

$$\begin{aligned}
&\int_{\mathbb{R}} |g(r,\omega)|^q dr \le C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \\
&\times \left(\sup_{t>0, r\in\mathbb{R}} t^{\theta/2(1-\theta)}\left|\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(r-s)^2/4t}g(s,\omega)ds\right|\right)^{q(1-\theta)} \\
&= C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \left(\sup_{t>0, r\in\mathbb{R}} t^{\theta/2(1-\theta)}\left|\Phi e^{-tA^2}\Phi^{-1}g(r,\omega)\right|\right)^{q(1-\theta)} \\
&\le C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \left(\sup_{t>0} t^{\theta/2(1-\theta)}\left\|\Phi e^{-tA^2}\Phi^{-1}g\right\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\right)^{q(1-\theta)} \\
&\le C^q \int_{\mathbb{R}} \left|\frac{\partial g(r,\omega)}{\partial r}\right|^p dr \|g\|^{q(1-\theta)}_{B^{\theta/(\theta-1)}}.
\end{aligned}$$

On integrating with respect to ω over $\mathbb{S}^{n-1}$ we obtain

Theorem 4.1. *Let $1\le p<q<\infty$ and suppose that g is such that $\Phi A\Phi^{-1}g \equiv -i(\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$ and $g\in B^{\theta/(\theta-1)}, \theta = p/q$. Then there exists a positive constant C, depending on p and q, such that*

$$\|g\|_{L^q(\mathbb{R}\times\mathbb{S}^{n-1})} \le C\|(\partial/\partial r)g\|^\theta_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{1-\theta}_{B^{\theta/(\theta-1)}}. \tag{4.2}$$

The theorem has two natural corollaries featuring the Hardy-type inequality (2.1), the first an inequality of Sobolev type , and the second of Gagliardo-Nirenberg type.

Corollary 4.2. *(i) Let $p^* := np/(n-p), 1\le p\le n-1$, and suppose $(\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$ and $\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot,\omega)\|_{L^p(\mathbb{R})}<\infty$. Then*

$$\|g\|_{L^{p^*}(\mathbb{R}\times\mathbb{S}^{n-1})} \le C\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot,\omega)\|^{(n-1)/n}_{L^p(\mathbb{R})}. \tag{4.3}$$

(ii) If $G=\mathcal{M}(g)$ denotes the integral mean of g, namely,

$$G(r) = \mathcal{M}(g)(r) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} g(r,\omega)d\omega,$$

then if $g, (\partial/\partial r)g \in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$,

$$\|G\|_{L^{p^*}(\mathbb{R})} \le C\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{(n-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.4}$$

If g is supported in $[-\Lambda,\Lambda]\times\mathbb{S}^{n-1}$, then

$$\|g\|_{L^{p^*}(\mathbb{R}\times\mathbb{S}^{n-1})}\le C\Lambda^{(n-1)/n^2}\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot,\omega)\|^{(n-1)/n}_{L^{p^*}(\mathbb{R})}; \tag{4.5}$$

also

$$\|G\|_{L^{p^*}(\mathbb{R})}\le C\Lambda^{(n-1)/n}\|(\partial/\partial r)g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.6}$$

Proof. From (3.8), it follows that, for any $s\in[1,\infty)$,

$$t^{-\theta/2(\theta-1)}\|\Phi P_t\Phi^{-1}g\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})}\le Ct^{-\theta/2(\theta-1)-1/2s}\sup_{\omega\in\mathbb{S}^{n-1}}\|g\|_{L^s(\mathbb{R})}.$$

If $1\le p<n-1$ set $\theta=p/q, q=p(p+1)$ and $s=p$. Then, from Theorem 4.1

$$\|g\|_{L^{p(p+1)}(\mathbb{R}\times\mathbb{S}^{n-1})}\le C\|(\partial/\partial r)g\|^{1/(p+1)}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\sup_{\omega\in\mathbb{S}^{n-1}}\|g\|^{p/(p+1)}_{L^p(\mathbb{R})}. \tag{4.7}$$

Thus $g\in L^{p(p+1)}(\mathbb{R}\times\mathbb{S}^{n-1})\cap L^p(\mathbb{R}\times\mathbb{S}^{n-1})$, and since

$$\frac{np}{(n-p)}=\frac{p(p+1)}{(n-p)}+\frac{p(n-p-1)}{(n-p)}$$

we have by Hölder's inequality,

$$\int_{\mathbb{R}\times\mathbb{S}^{n-1}}|g|^{p^*}d\lambda\le\left(\int_{\mathbb{R}\times\mathbb{S}^{n-1}}|g|^{p(p+1)}d\lambda\right)^{1/(n-p)}\left(\int_{\mathbb{R}\times\mathbb{S}^{n-1}}|g|^p d\lambda\right)^{(n-p-1)/(n-p)}.$$

Hence, from (4.7),

$$\begin{aligned}\|g\|_{L^{p^*}(\mathbb{R}\times\mathbb{S}^{n-1})} &\le \|g\|^{(p+1)/n}_{L^{p(p+1)}(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{(n-p-1)/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\\ &\le C\|(\partial/\partial r)g\|^{1/n}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot,\omega)\|^{(n-1)/n}_{L^p(\mathbb{R})}.\end{aligned}$$

If $p=n-1$, we choose $s=n-1, q=p^*=n(n-1)$ and $\theta=1/n$. Then Theorem 3 gives (4.3) immediately. The inequality (4.5) follows on applying Hölder's inequality to $\|g(\cdot,\omega)\|_{L^p(\mathbb{R})}$. The inequalities (4.4) and (4.6) follow from (4.3) and (4.5) respectively, on substituting G for g and noting that

$$\begin{aligned}\|G'\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} &\le \|(\partial/\partial r)g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\\ \|G\|_{L^p(\mathbb{R})} &\le |\mathbb{S}^{n-1}|^{-1/p}\|g\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}.\end{aligned}$$

□

Corollary 4.3. *(i) Let $1\le p<q<\infty, m=(q/p)-1$, and suppose that $(\partial/\partial r)g\in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$ and $\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot\omega)\|_{L^m(\mathbb{R})}<\infty$. Then*

$$\|g\|_{L^q(\mathbb{R}\times\mathbb{S}^{n-1})}\le C\|(\partial/\partial r)g\|^{p/q}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\sup_{\omega\in\mathbb{S}^{n-1}}\|g(\cdot,\omega)\|^{1-p/q}_{L^m(\mathbb{R})}. \tag{4.8}$$

(ii) If $(\partial/\partial r)g\in L^p(\mathbb{R}\times\mathbb{S}^{n-1})$ and $g\in L^m(\mathbb{R}\times\mathbb{S}^{n-1})$, then, with $G=\mathcal{M}(g)$,

$$\|G\|_{L^q(\mathbb{R})}\le C\|(\partial/\partial r)g\|^{p/q}_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})}\|g\|^{1-p/q}_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})}. \tag{4.9}$$

Proof. From (3.8), with $\theta = p/q$ and $m = q/p - 1$, we deduce that

$$\begin{aligned} t^{-\theta/2(\theta-1)}\|\Phi P_t \Phi^{-1} g\|_{L^\infty(\mathbb{R}\times\mathbb{S}^{n-1})} &\le Ct^{-\theta/2(\theta-1)-1/2m} \sup_{\omega\in\mathbb{S}^{n-1}} \|g(\cdot,\omega)\|_{L^m(\mathbb{R})} \\ &\le C \sup_{\omega\in\mathbb{S}^{n-1}} \|g(\cdot,\omega)\|_{L^m(\mathbb{R})} \end{aligned}$$

and this yields (4.8). The inequality (4.9) follows from (4.8) on substituting G for g. □

The case $p = 2$ of Corollary 4.2 is of special interest.

Corollary 4.4. *(i) Let f be such that $Lf \in L^2(\mathbb{R}^n), L = \mathbf{x}\cdot\nabla$, and*

$$\sup_{\omega\in\mathbb{S}^{n-1}} \|f(\cdot,\omega)\|_{L^2(\mathbb{R}^+;d\mu)} < \infty.$$

Then, for $n \ge 3$,

$$\begin{aligned} \|rf(r\omega)\|^2_{L^{2^*}(\mathbb{R}^n)} &\le C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \sup_{\omega\in\mathbb{S}^{n-1}} \|f(\cdot,\omega)\|^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu))}, \end{aligned} \tag{4.10}$$

where $2^* = 2n/(n-2)$ and $d\mu = r^{n-1}dr$.
(ii) If $f, Lf \in L^2(\mathbb{R}^n)$, then, with $F := \mathcal{M}(f)$,

$$\begin{aligned} \|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\le C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \|f\|^{2(1-1/n)}_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{4.11}$$

For $0 \le \delta < n^2/4$, we have

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C\left(n^2/4 - \delta\right)^{-(n-1)/n}\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \delta\|f\|^2_{L^2(\mathbb{R}^n)}\right\}. \tag{4.12}$$

Proof. On using the facts that $\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}\times\mathbb{S}^{n-1})$ is an isometry and, with $g := \Phi f$,

$$\begin{aligned} \|(\partial/\partial r)g\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{n-1})} &= \|\Phi A\Phi^{-1} g\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{n-1})} \\ &= \|Af\|^2_{L^2(\mathbb{R}^n)} \\ &= \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)} \end{aligned}$$

since $A^2 = L^*L - (n^2/4)$ from (3.6), it follows from (4.3) that

$$\begin{aligned} \|\Phi f\|^2_{L^{2^*}(\mathbb{R}\times\mathbb{S}^{n-1})} &\le C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{1/n} \\ &\times \sup_{\omega\in\mathbb{S}^{n-1}} \|f(\cdot,\omega)\|^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu)}. \end{aligned}$$

Then (4.10) follows since

$$\|\Phi f\|_{L^{2^*}(\mathbb{R}\times\mathbb{S}^{n-1})} = \|rf(r,\omega)\|_{L^{2^*}(\mathbb{R}^n)}.$$

The inequality (4.11) follows in a similar way from (4.4) since

$$\|\mathcal{M}(\Phi f)\|_{L^{2^*}(\mathbb{R})} = \|rF(r)\|_{L^{2^*}(\mathbb{R}^+;d\mu)}.$$

From Young's inequality we have for any $\varepsilon > 0$ that

$$n[\varepsilon/(n-1)]^{1-1/n} ab \le a^n + \varepsilon b^{n/(n-1)}.$$

On applying this to (4.11) we get

$$\varepsilon^{1-1/n}\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - [\big(\frac{n}{2}\big)^2 - \varepsilon]\|f\|^2_{L^2(\mathbb{R}^n)}\}.$$

This yields (4.12) on setting $\varepsilon = n^2/4 - \delta$. □

Corollary 4.5. *(i) Let* $\nabla h \in L^2(\mathbb{R}^n), n \ge 3,$ *and*

$$\sup_{\omega\in\mathbb{S}^{n-1}} \|h(\cdot,\omega)/|\cdot|\|^2_{L^2(\mathbb{R}^+;d\mu)} < \infty.$$

Then

$$\begin{aligned}\|h\|^2_{L^{2^*}(\mathbb{R}^n)} &\le C\big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \big(\frac{n-2}{2}\big)^2\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\big\}^{1/n} \\ &\times \sup_{\omega\in\mathbb{S}^{n-1}}\big\{\|h(\cdot,\omega)/|\cdot|\|^2_{L^2(\mathbb{R}^+;d\mu)}\big\}^{1-1/n}. \end{aligned} \tag{4.13}$$

(ii) If $h, \nabla h \in L^2(\mathbb{R}^n)$ *then, with* $H := \mathcal{M}(h)$,

$$\begin{aligned}\|H\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} &\le C\big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \big(\frac{n-2}{2}\big)^2\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\big\}^{1/n} \\ &\times \big\{\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\big\}^{1-1/n}. \end{aligned} \tag{4.14}$$

For $0 \le \delta < (n-2)^2/4$, *we have*

$$\begin{aligned}\|H\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \le C\left((n-2)^2/4-\delta\right)^{-(n-1)/n}\Big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} \\ - \delta\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\Big\}. \end{aligned} \tag{4.15}$$

Proof. Since $n \ge 3$, we have that $f := h/|\cdot| \in L^2(\mathbb{R}^n)$. We claim that $Lf \in L^2(\mathbb{R}^n)$. For

$$\begin{aligned}|\nabla(|\mathbf{x}|f)|^2 &= \left|\frac{\mathbf{x}}{|\mathbf{x}|}f + |\mathbf{x}|\nabla f\right|^2 \\ &= |f|^2 + (|\mathbf{x}||\nabla f|)^2 + 2\mathrm{Re}[\bar{\mathrm{f}}(\mathbf{x}\cdot\nabla)\mathrm{f}]\end{aligned}$$

and, on integration by parts, initially for $f \in C_0^\infty(\mathbb{R}^n)$ and then by the usual continuity argument,

$$\begin{aligned}\int_{\mathbb{R}^n}\overline{f}(\mathbf{x}\cdot\nabla)f d\mathbf{x} &= \sum_{j=1}^n\int_{\mathbb{R}^n} x_j\overline{f}\frac{\partial f}{\partial x_j}d\mathbf{x} \\ &= -\sum_{j=1}^n\int_{\mathbb{R}^n} f\left\{\overline{f} + x_j\frac{\partial\overline{f}}{\partial x_j}\right\}d\mathbf{x} \\ &= -\int_{\mathbb{R}^n}\left\{n|f|^2 + f(\mathbf{x}\cdot\nabla)\overline{f}\right\}d\mathbf{x}.\end{aligned}$$

This gives

$$2\mathrm{Re}\int_{\mathbb{R}^n}[\bar{\mathrm{f}}(\mathbf{x}\cdot\nabla)\mathrm{f}]d\mathbf{x} = -\mathrm{n}\int_{\mathbb{R}^n}|\mathrm{f}|^2 d\mathbf{x}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla(|\mathbf{x}|f)|^2 d\mathbf{x} &= \int_{\mathbb{R}^n}(|\mathbf{x}||\nabla f|)^2\, d\mathbf{x} - (n-1)\int_{\mathbb{R}^n}|f|^2 d\mathbf{x}\\ &\geq \int_{\mathbb{R}^n}|Lf|^2 d\mathbf{x} - (n-1)\int_{\mathbb{R}^n}|f|^2 d\mathbf{x} \qquad (4.16)\end{aligned}$$

which confirms our claim. On substituting (4.16) and $f = h/|\cdot|$ in (4.10), we get

$$\begin{aligned}\|h\|^2_{L^{2^*}(\mathbb{R}^n)} &\leq C\Big\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} + (n-1)\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\\ &- (n^2/4)\|h/|\cdot|\|^2_{L^2(\mathbb{R}^n)}\Big\}^{1/n}\sup_{\omega\in\mathbb{S}^{n-1}}\|h/|\cdot|\|^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu)}\end{aligned}$$

which yields (4.13); (4.14) follows similarly from (4.11) and (4.14) yields (4.15). □

If in (4.6) $g = \Phi f$, where f is supported in the annulus $A_R := \{\mathbf{x}\in\mathbb{R}^n : 1/R \leq |\mathbf{x}| \leq R\}$, then G is supported in the interval $[-\ln R, \ln R]$ and we have as in the proof of Corollary 4

Corollary 4.6. *Let $f \in C_0^\infty(A_R)$. Then, with $F := \mathcal{M}(f)$,*

$$\|rF(r)\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{\frac{2(n-1)}{n}}\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}. \qquad (4.17)$$

On putting $f = h/|\cdot|$ in (4.17) and using (4.16), we have

Corollary 4.7. *Let $h \in C_0^\infty(A_R)$. Then, with $H := \mathcal{M}(h)$,*

$$\|H\|^2_{L^{2^*}(\mathbb{R}^+;d\mu)} \leq C(\ln R)^{\frac{2(n-1)}{n}}\left\{\|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \frac{(n-2)^2}{4}\|\frac{h}{|\cdot|}\|^2_{L^2(\mathbb{R}^n)}\right\}.$$

Finally we have the following $p = 2$ case of Corollary 3(ii).

Corollary 4.8. *Let $2 < q < \infty$ and $m = q/2 - 1$. Then, if f is such that $f, Lf \in L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}}|f(s,\omega)|^m s^{(\frac{nm}{2}-1)}dsd\omega < \infty$, we have that $\int_{\mathbb{R}^+}|F(s)|^q s^{(\frac{nq}{2}-1)}ds < \infty$ and*

$$\begin{aligned}\int_{\mathbb{R}^+}|F(s)|^q s^{(\frac{nq}{2}-1)}ds &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^2\\ &\times \left\{\int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}}|f(s,\omega)|^m s^{(\frac{nm}{2}-1)}dsd\omega\right\}^2\end{aligned}$$

Proof. Corollary 4.3(ii) with $p = 2$ yields

$$\begin{aligned} \|\mathcal{M}(\Phi f)\|_{L^q(\mathbb{R})} &\leq C\left\{\|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4}\|f\|^2_{L^2(\mathbb{R}^n)}\right\}^{2/q} \\ &\times \|\Phi f\|^{1-2/q}_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})}. \end{aligned}$$

Since

$$\|\mathcal{M}(\Phi f)\|^q_{L^q(\mathbb{R})} = \int_{\mathbb{R}^+} |F(s)|^q s^{(\frac{nq}{2}-1)} ds$$

and

$$\|\Phi f\|^m_{L^m(\mathbb{R}\times\mathbb{S}^{n-1})} = \int_{\mathbb{R}^+}\int_{\mathbb{S}^{n-1}} |f(s,\omega)|^m s^{(\frac{nm}{2}-1)} ds d\omega$$

the corollary follows. □

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