ON STEINER LOOPS AND POWER ASSOCIATIVITY

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This paper is dedicated to Professor Themistocles M. Rassias.

Abstract. In this paper we investigate Steiner loops introduced by N.S. Mendelsohn [Aeq. Math. 6 (1991), 228–230] and provide six (seven) equivalent identities to characterize it. We also prove the power associativity of Bol loops by using closure (Hexagonal) conditions.

1. Steiner loops

In [9] Mendelsohn has defined the concept of a generalized triple system as follows. Let $S$ be a set of $\nu$ elements. Let $T$ be a collection of $b$ subsets of $S$, each of which contains three elements arranged cyclically, and such that any ordered pair of elements of $S$ appears in exactly a cyclic triplet (note the cyclic triplet \{a, b, c\} contains the ordered pairs ab, bc, ca but not ba, cb, ac). When such a configuration exists we will refer to it as a generalized triple system. If we ignore the cyclic order of the triples, the generalized triple system is a B.I.B.D.

There is one to one correspondence between generalized triple systems of order $\nu$ and quasigroups of order $\nu$ satisfying the identities $x^2 = e \cdot (xy)x = x(yx) = y$. The term generalized Steiner quasigroup means a quasigroup which satisfies the above identities.

Let $G$ be a generalized Steiner Quasigroup of order $\nu$. From $G$ a loop $G^*$ with operator $*$ is constructed as follows. The elements of $G^*$ are the same as those of $G$ together with an extra element $e$. Multiplication in $G^*$ is defined as follows: $a * e = e * a = a; a * a = e$ and for $a, b \in G$, with $a \neq b$ define $a * b = a \cdot b$. It
follows easily that $G^*$ is a loop satisfying the identities $x*e = e*x = x, x*x = e, x*(y*x) = (x*y)*x = y$ for $x, y \in G$. Also, the correspondence between generalized Steiner quasigroups and generalized Steiner loops is a bijection.

A loop which satisfies the identities

$$xx = e, \quad xe = x = ex, \quad x \cdot yx = y = xy \cdot x$$

for $x, y \in G$. Also, the correspondence between generalized Steiner quasigroups and generalized Steiner loops is a bijection.

A loop which satisfies the identities

$$xx = e, \quad xe = x = ex, \quad x \cdot (ca \cdot bb) = c,$$

is called a generalized Steiner loop (g.s.l.). In [9] the identity (2) characterizing g.s.l. is given. Five (six) equivalent identities were found immediately afterwards in 1970 to characterize g.s.l. Now we present them in the following theorem:

**Theorem 1.1.** A groupoid $G(\cdot)$ is a generalized Steiner loop if an only if $G$ satisfies any one of the following identities:

\begin{align*}
\text{(1.2)} & \quad a \cdot [(bb \cdot c) \cdot a] = c, \\
\text{(2a)} & \quad [a \cdot c(bb)] \cdot a = c, \\
\text{(2b)} & \quad a \cdot (ca \cdot bb) = c, \\
\text{(2c)} & \quad (a \cdot ca) \cdot bb = c, \\
\text{(2d)} & \quad bb \cdot (a \cdot ca) = c, \\
\text{(2e)} & \quad (bb \cdot a) \cdot (ca \cdot dd) = c,
\end{align*}

for $a, b, c, d \in G$.

**Proof.** First we consider (2) investigated in [9], here we present a different simpler proof to show that $G(\cdot)$ satisfying (2) is a g.s.l.

In (2) replace $c$ by $(dd \cdot k) \cdot bb$ and use (2) to get

$$a \cdot ka = (dd \cdot k) \cdot bb \quad \text{(1.3)}$$

and

$$bb \cdot (a \cdot ka) = k, \quad \text{for } a, b, k, \in G. \quad \text{(3a)}$$

Suppose $\nu a = ua$. Then (3a) shows that $\nu = u$, that is, $(\cdot)$ is right cancellative (r.c.). Apply r.c. in (3a) to obtain $bb = \text{constant} = e$ (say). Then (2) becomes

$$a \cdot (ee \cdot a) = c.$$

Put $c = e$ to obtain $a \cdot ea = e = ea \cdot ea$ implying $ea = a$. So $a \cdot ca = c$.

First $a = e$ in (2) yields $ce = c$ showing thereby that $e$ is an identity and then replacing $a$ by $ac$ gives $ac \cdot (c \cdot ac) = c$, that is $ac \cdot a = c$. This proves g.s.l.

Second we prove the implication of the identities in the order written above; that is, we prove that

$$\text{(2)} \Rightarrow (\text{2a}) \Rightarrow (\text{2b}) \Rightarrow (\text{2c}) ((\text{2c}')) \Rightarrow (\text{2d}) \Rightarrow (\text{2e}) \text{ finally } \Rightarrow (\text{2)}$$

to complete the cycle.

**To prove (2) \Rightarrow (2a)**

Suppose (2) $a \cdot [(bb \cdot c) \cdot a] = c$ holds.

From the above prove we see that

$$bb = e, \quad ae = a, \quad ac \cdot a = e.$$

Now $[a \cdot c(bb)] \cdot a = (a \cdot ce) \cdot a = ac \cdot a = c$ which is (2a).

**To prove (2a) \Rightarrow (2b)**
Now \([a \cdot c(bb)] \cdot a = c\) holds.
Replace \(c\) by \(bb \cdot (c \cdot dd)\) in (2b) and use (2b) to get
\[
[a \cdot \{(bb \cdot (c \cdot dd)) \cdot bb\}] \cdot a = bb \cdot (c \cdot dd)
\]
that is
\[
ac \cdot a = bb \cdot (c \cdot dd) \quad (1.4)
\]
Let \(ac = au\). (4a) yields \(c = u\) implying l.c. (left cancellative). (4) is \(ac \cdot a = bb \cdot (c \cdot dd) = bb \cdot (c \cdot ww)\) giving \(dd = \text{constant} = e\) (say) by l.c. (4) becomes
\[
ac \cdot a = e \cdot (ce).
\]
Let \(ac = uc\). Then (4b) shows that \(u = a\), that is, \(\cdot\) is r.c. \(c = e\) in (4b) gives \(ae \cdot a = e = a \cdot a\), that is, \(ae = a\) (using r.c.). Then (4a) gives \(ac \cdot a = c\). Now
\[
a \cdot (ca \cdot bb) = a \cdot (ca \cdot e) = a \cdot ca = c
\]
which is (2b). This proves that (2a) \(\Rightarrow\) (2b).

Next we prove (2b) \(\Rightarrow\) (2c)

(2b) \(a \cdot (ca \cdot bb) = c\) holds.
Let \(ca = da\). This in (2b) shows \(c = d\), that is, r.c. Let \(ca = cd\). Then
\[
a \cdot (ca \cdot bb) = d \cdot (cd \cdot bb)\]
implies \(a = d\), (use r.c.), that is, l.c. l.c. in (2b) gives
\(bb = e\) (say).
Thus, \(a \cdot (ca \cdot e) = c\).
\[
c = a \quad \text{gives} \quad ae = a \quad \text{and} \quad a \cdot ca = c.
\]
(1.5)
Now from (5) results \((a \cdot ca) \cdot bb = (a \cdot ca) \cdot e = a \cdot ca = e\) which is (2c).

Remark 1.2. Instead of (2c) we consider
\[
(ac \cdot a) \cdot bb = c.
\]
(2c')

From (5) \(a \cdot ca = c\), replacing \(c\) by \(ac\) we get \(a \cdot (ac \cdot a) = ac\), that is \(ac \cdot a = c\)
(use l.c.. Then
\[
(ac \cdot a) \cdot bb = (ac \cdot a) \cdot e = ac \cdot a = c
\]
which is (2c'). That is (2b) \(\Rightarrow\) (2c').

Next we take (2c) \(\Rightarrow\) (2d)

(2c) \((a \cdot ca) \cdot bb = c\) holds.
Set \(ca = da\) in (2c) to obtain \(c = d\), that is, r.c. Let \(ca = cd\). Then (2c) and
r.c. yields l.c. l.c. in (2c) gives \(bb = \text{constant} = e\) (say). Hence (2c) becomes
\[
(a \cdot ca) \cdot e = c
\]
(1.6)
\(c = e\) in (6) gives
\[
(a \cdot ea) \cdot e = e \Rightarrow a \cdot ea = e \Rightarrow ea = a.
\]
(6a)
With \(a = e\), (6) shows that \(ce \cdot e = c\) and
\[
a \cdot ca = ce.
\]
Now \(c\) replaced by \(ac\) gives
\[
a \cdot (ac \cdot a) = ac \cdot e,
\]
that is, \(a \cdot ce = ac \cdot e\).
Then \(c = e\) gives \(ae = ae \cdot e = a\) by (6).
Now, \( bb \cdot (a \cdot ca) = e \cdot (a \cdot ca) = a \cdot ca = ce = e \) which is (2d).

Remark 1.3. Suppose (2c′) holds.

With \( ac = au \), (2c′) gives l.c. Then changing \( b \) in (2c′) yields \( bb = constant = e \).

\( c = e \) in (2c′) gives \( ae \cdot a = e \), that is, \( ae = a \). With \( a = e \) (2c′) shows \( ec = c \), \( ac \cdot a = c \) and \( a \cdot ca = c \) (replace \( c \) by \( ca \)). Now \( bb \cdot (a \cdot ca) = a \cdot ca = c \) which is (2d). Thus (2c′) ⇒ (2d).

Next we tackle (2d) ⇒ (2e)

(2d) \( bb \cdot (a \cdot ca) = c \) holds.

\( ca = da \) in (2d) gives \( c = d \), that is, r.c. Then (2d) yields \( bb = constant = e \)

(2d) is \( e \cdot (a \cdot ca) = c \). \( c = e \) gives \( a \cdot ea = e \) or \( ea = a \) and \( a \cdot ca = c \). Then \( a = e \) results to \( ce = c \).

Now \( (bb \cdot a) \cdot (ca \cdot dd) = a \cdot ca = c \) which is (2e).

Finally, to complete the cycle, we prove that (2e) ⇒ (2)

(2e) \( (bb \cdot a) \cdot (ca \cdot dd) = c \) holds.

\( ca = ua \) in (2e) gives r.c. and \( bb = constant = e \).

(2e) is \( ea \cdot (ca \cdot e) = c \).

First \( c = e \) yields \( ea = ea \cdot e \). Second \( c = a \) gives \( ea \cdot e = a = ea \) and then \( ae = a \). Further \( a \cdot ca = c \). Now \( a \cdot [(bb \cdot c) \cdot a] = a \cdot ca = c \) which is (2).

This completes the proof of the theorem. □

2. Bol loop and power associativity

There are several closure conditions in Quasigroups and Loops theory [1, 2, 3, 4, 5, 6, 7, 8] of which \( R \)-condition (Reidemeister condition) connected to groups, \( T \)-condition (Thomsen condition) connected to Abelian groups, \( H \)-condition (Hexagonal condition) connected to power associativity are well known.

H-condition

For \( x_1, x_2, x_3, y_1, y_2, y_3 \) in \( G \), a groupoid if

\[
\begin{align*}
  x_1y_2 &= x_2y_1, \\
  x_1y_3 &= x_2y_2 = x_3y_1 \\
\end{align*}
\]

implies \( x_2y_3 = x_3y_2 \), \hspace{1cm} (2.1)

then \( G \) is said to satisfy the closure condition known as Hexagonal condition.

Geometrically, \( H \)-condition means the following:

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]

\[
\begin{array}{c}
  y_3 \\
  y_2 \\
  y_1 \\
\end{array}
\begin{array}{c}
  x_3 \\
  x_2 \\
  x_1 \\
\end{array}
\]
$H$-condition implies power associativity, that is,
\[ x \cdot y^{n+m} = xy^n \cdot y^m \]
for all $x, y \in G$ and all $m, n \in \mathbb{Z}$, integers.

(Left) Bol Loop

A loop $G(\cdot)$ is said to be a (left) Bol loop provided
\[ x \cdot (y \cdot xz) = (x \cdot yx) \cdot z, \quad \text{for } x, y, z \in G \quad (2.2) \]
holds.

It is well known that left (right) Bol loop is power associative. Here we prove it by using $H$-condition.

**Theorem 2.1.** The left Bol loop is power associative (by using the hexagonal closure condition).

**Proof.** Suppose (8) holds.

Set $y = x^{-1}$ (inverse of $x$) in (9) to obtain $x \cdot (x^{-1} \cdot xz) = xz \Rightarrow x^{-1} \cdot xz = z,$
that is $G(\cdot)$ satisfies l.i.p. (left inverse property) or $G(\cdot)$ is a left inverse property loop. Suppose
\[ x_1, y_2 = x_2 y_1, \quad x_1 y_3 = x_2 y_2 = x_3 y_1 \quad \text{holds in } G. \quad (7') \]

First use (7') to get
\[ x_3^{-1} x_2 = y_1^{-1} y_2, \quad x_1 = x_2 y_1^{-1} y_2 = x_2 (x_3^{-1} x_2). \]

Now
\[ x_1 y_3 = (x_2 (x_3^{-1} x_2)) \cdot y_3 \overset{\text{by } (8)}{=} x_2 (x_3^{-1} \cdot x_2 y_3) \overset{\text{also}}{=} x_2 y_2. \]

Thus
\[ x_3^{-1} \cdot x_2 y_3 = y_2 \quad \text{or} \quad x_2 y_3 = x_3 y_2 \quad (\text{using l.i.p.}). \]

Thus $H$-condition holds. Hence $G$ is power associative. 

A loop in which $xy \cdot x = x \cdot yx$ holds is said to satisfy elasticity law. In passing, we mention that a loop satisfying elasticity law is power associative.

**References**


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