Harmonic Finsler manifolds of (α, β)-type

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Abstract. In this paper we construct a new class of harmonic and asymptotically harmonic Finsler manifolds of (α, β)-type. This class is defined by a Riemannian metric α and a special 1-form β.

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1 Introduction

A complete Riemannian manifold (M, α) is said to be harmonic if for any $x \in M$, its volume density function $\sqrt{\det(\alpha_{ij}(x))}$ is a radial function (i.e., it depends only on the distance to the origin in the polar coordinates centered at x). All Riemannian harmonic manifolds (M, α) are Einstein manifolds, that is they have constant Ricci curvature [2]. Further, both 2-dimensional and 3-dimensional harmonic Riemannian manifolds have constant sectional curvature. Known examples of harmonic Riemannian manifolds are the Euclidean space, rank one symmetric spaces and Damek–Ricci spaces [1, Chapter 5] and [2, 4].

It is known that, the density function of a Riemannian harmonic manifold plays a vital role in studying many problems such as the classification of such manifolds [2, 4]. Finsler metrics are natural generalization of Riemannian metrics. In [6], harmonic manifolds have been extended to Finsler geometry. There are significant differences between the two geometries which are reflected in the study of harmonic Finsler spaces, such as the dependence of the metric on fiber coordinates, nonsymmetry of the Finsler distance, nonexistence of canonical volume form, nonlinearity of the Finsler Laplacian, and appearance of non-Riemannian quantities.

Matsumoto in [5] gave an elegant method to build Finsler metrics from the Riemannian ones using a 1-form. These metrics are known as (α, β)-metrics and include Randers, Kropina, Matsumoto and square metrics for which many interesting results have been obtained cf. [3, 6, 7, 8].

In the present paper, we use the known machinery of harmonic Riemannian manifolds to construct a new class of harmonic Finsler manifolds $(M, F, d\mu)$, namely

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harmonic Finsler manifolds of (α, β)*-type*, using either Busemann-Hausdorff volume measure or Holmes-Thompson volume measure on M. The main results of the present work are as follows:
(a) Theorem 3.1 in which we construct a harmonic Kropina Finsler manifold $(M, F = \frac{\alpha^2}{\beta}, d\mu)$ from a harmonic Riemannian manifold (M, α) and a constant Killing 1-form β with respect to α. Also, we prove that $(M, F := \frac{\alpha^2}{\beta}, d\mu)$ is of Einstein type.
(b) Theorem 3.4 in which we provide, from a harmonic Riemannian manifold (M, α) and a 1-form β whose length $||\beta||_\alpha$ is a radial function, a harmonic (α, β)-Finsler manifold $(M, F = \alpha\, \phi(\frac{\beta}{\alpha}), d\mu)$.
(c) It is known that, if (M, α) is noncompact harmonic Riemannian then it is asymptotically harmonic, that is, M has no conjugate points and the mean curvature of its horospheres is constant. We make use of this observation and our above mentioned results to provide examples of asymptotically harmonic Finsler manifolds of (α, β)-type in Theorem 3.5.

2 A Finsler geometric setting

Let M be a smooth connected n-dimensional manifold of dimension $n \geq 2$.

Let (M, F) be a Finsler manifold. The Finsler distance d_F induced by the Finsler structure F [9] is defined on M by

$$d_F(p, q) := \inf \left\{ \int_0^1 F(\dot{\xi}(t))\, dt \ \mid \xi : [0, 1] \to M,\ C^1 \text{ curve joining } p \text{ to } q \right\}.$$

It should be noted that the *Finsler distance* is nonsymmetric, that is, $d_F(p, q) \neq d_F(q, p)$. In other words, the distance depends on the direction of the curve. The non-reversibility property is also reflected in the notions of Cauchy sequence and completeness. Thus, being different from the Riemannian case, a positively (or forward) complete Finsler manifold (M, F) is not necessarily negatively (or backward) complete. For example, a Randers metric is forward complete solely. A Finsler metric is said to be *complete* if it is both forward and backward complete.

It is worth mentioning that every function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}$ generates a *radial function* η_x around a point $x \in M$ defined by $\eta_x(x_o) = \eta(r_x(x, x_o))$, where $r_x(x, x_o)$ is the geodesic distance between $x, x_o \in M$ induced by the Finsler function [6].

Definition 2.1. [5] Let $\alpha = \sqrt{\alpha_{ij}(x)\, y^i\, y^j}$ be a Riemannian metric defined on M and $\beta = b_i(x)\, dx^i$ be a 1-form on M with $b := ||\beta||_\alpha < b_o$, where b_o is a real positive number. The Finsler metric $F = \alpha\, \phi(s)$, $s := \frac{\beta}{\alpha}$, is said to be an (α, β)-metric if ϕ is a positive smooth function defined on the interval $(-b_o, b_o)$ such that

$$\phi(s) - s\, \phi'(s) + (b^2 - s^2)\, \phi''(s) > 0, \quad |s| \leq b < b_o.$$

Unlike the Riemannian case, there are several volume forms used in Finsler geometry. The most important ones are *Busemann-Hausdorff and Holmes-Thompson volume forms* [7].

Definition 2.2. The Busemann-Hausdorff volume form $d\mu_{BH} := \sigma_{BH}(x)\,dx$ is defined at a point $x \in M$ as follows

$$\sigma_{BH}(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}(B^n_F(1))}, \qquad \text{where } \mathrm{Vol}(B^n_F(1)) := \mathrm{Vol}\left(\{(y^i) \in \mathbb{R}^n \mid F(x, y^i\partial_i) < 1\}\right)$$

and $\mathrm{Vol}(\mathbb{B}^n(1))$ denotes the Euclidean volume of a unit Euclidean ball:

$$\mathrm{Vol}(\mathbb{B}^n(1)) = \frac{1}{n}\,\mathrm{Vol}(\mathbb{S}^{n-1}) = \frac{1}{n}\,\mathrm{Vol}(\mathbb{S}^{n-2}) \int_0^\pi \sin^{n-2} t\,dt.$$

Definition 2.3. The Holmes-Thompson volume form is defined at a point $x \in M$ by $d\mu_{HT} := \sigma_{HT}(x)\,dx$, where

$$\sigma_{HT}(x) := \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{S_xM} \det(g_{ij}(x,y))\,dy,$$

where $S_xM = \{y \in T_xM \mid F(y) = 1\}$.

Definition 2.4. [6] *A forward complete Finsler manifold $(M, F, d\mu)$ endowed with a smooth volume measure $d\mu$ is (globally) harmonic if in polar coordinates the volume density function $\overline{\sigma}_p(r,y)$ is a radial function around (each) $p \in M$, where $\overline{\sigma}_p(r,y) := \frac{\sigma_p(r,y)}{\sqrt{\det(\dot{g}_p(p,y))}}$ and $\dot{g}_p$ is the restriction of g on the indicatrix I_pM. That is, $\overline{\sigma}_p(r,y)$ is independent of $y \in I_pM$; so it can be written briefly as $\overline{\sigma}_p(r)$.*

There are some equivalent definitions to Definition 2.4 that can be found in [6].

Definition 2.5. [6] Suppose that r is a Finsler distance defined on an open subset of M, that is $F(\nabla r) = 1$, where ∇r is the gradient of r. The Finsler mean curvature of the level hypersurface $r^{-1}(t)$ at $x \in M$ with respect to ∇r_x is defined by

$$\Pi_{\nabla r}(x) := \frac{d}{dt}\,\log(\sigma_x(t, x^a))|_{t=t_o}, \tag{2.1}$$

for some $t_o \in Im(r)$.

Definition 2.6. [6] A forward complete, simply connected Finsler manifold $(M, F, d\mu)$ without conjugate points is called an *asymptotically harmonic Finsler manifold (or simply, AHF-manifold)* if the mean curvature of Finslerian horospheres (Finsler geodesic sphere with infinite radius) $\Pi_\infty := \lim_{r\to\infty} \Pi_{\nabla r}$ is a real constant.

Consequently, a simply connected, forward complete, noncompact harmonic Finsler manifold with constant Finsler mean curvature of horospheres is an AHF-manifold cf. [1, Chapter 5] and [6].

For general Finsler metrics, the Busemann-Hausdorff and Holmes-Thompson volume forms may be expressed hardly by elementary functions. However, it is done for some particular Finsler metrics, namely (α,β)-Finsler metrics.

Lemma 2.1. *[3] The Busemann-Hausdorff volume form of the (α,β)-Finsler metric $F = \alpha\,\phi(s)$, with $||\beta||_\alpha = b$, is given by*

$$d\mu_{BH} = f_{BH}(b)\;d\mu_\alpha, \qquad f_{BH}(b) = \frac{\int_0^\pi \sin^{n-2} t\,dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi^n(b\,\cos t)}\,dt}.$$

The Holmes-Thompson volume measure is given by

$$d\mu_{HT} = f_{HT}(b)\ d\mu_\alpha, \qquad f_{HT}(b) = \frac{\int_0^\pi T(b\,\cos t)\,\sin^{n-2} t\,dt}{\int_0^\pi \sin^{n-2} t\,dt},$$

where $d\mu_\alpha = \sqrt{\det(\alpha_{ij}(x))}\,dx$ *denotes the Riemannian volume form of* α *and* T *is defined by*

$$T(s) = \phi(\phi - s\phi')^{n-2}\left[(\phi - s\phi') + (b^2 - s^2)\,\phi''\right]. \tag{2.2}$$

3 Harmonic and asymptotically harmonic Finsler manifolds of (α, β)-type

Definition 3.1. [7] A 1-form β is called a constant Killing 1-form with respect to α if $||\beta||_\alpha$ is constant and $r_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}) = 0$, where $b_{i|j}$ denotes covariant derivative of β with respect to α.

Hereafter, we assume that M is a smooth connected forward complete manifold and $d\mu$ is either Busemann-Hausdorff or Holmes-Thompson volume measures on M.

Theorem 3.1. *Let* (M, α) *be a harmonic Riemannian manifold of dimension* $n \geq 2$. *Let* β *be a constant Killing* 1*-form with respect to* α *such that* $||\beta||_\alpha < b_o$. *Then* $(M, F = \frac{\alpha^2}{\beta},\ d\mu)$ *is harmonic non-Riemannian Finsler manifold of Einstein type.*

Proof. As (M, α) is a harmonic Riemannian manifold, the volume density function $\sqrt{\det(\alpha_{ij})}$ of $d\mu_\alpha$ is a radial function, say $l(r)$. In other words, $d\mu_\alpha = l(r)\,dr \wedge d\Theta$. Since β is a constant Killing 1-form with respect to α, thereby $||\beta||_\alpha$ is constant, say A. Then, in view of Lemma 2.1, we have for $\phi(s) = \frac{1}{s}$, $s > 0$,

$$d\mu_{BH} = \frac{\int_0^\pi \sin^{n-2} t\,dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi^n(A\,\cos t)}\,dt}\ l(r)\,dr \wedge d\Theta$$

and

$$d\mu_{HT} = \frac{\int_0^\pi T(A\,\cos t)\,\sin^{n-2} t\,dt}{\int_0^\pi \sin^{n-2} t\,dt}\ l(r)\,dr \wedge d\Theta.$$

Hence, the corresponding volume density functions are radial functions. The Kropina Finsler manifold $(M, F = \frac{\alpha^2}{\beta}, d\mu)$ is a harmonic since both $d\mu_{BH}$ and $d\mu_{HT}$ are constant multiples of $d\mu_\alpha$, which is radial. Moreover, it is of Einstein type in view of [7, Theorem 10.8] (α is Einstein; being harmonic Riemannian, and β is constant killing). □

Remark 3.2. The Einstein harmonic Kropina Finsler manifold $(M, F = \frac{\alpha^2}{\beta},\ d\mu)$ appearing in Theorem 3.1 has the properties that:
(1) it has constant Einstein factor of F. This follows from [7, Theorem 10.8].
(2) it has constant flag curvature for $n = 2$, $n = 3$. This follows from the fact that all harmonic Riemannian spaces of dimension 2 and 3 have constant sectional curvature [2] together with [7, Corollary 10.3].

Now, we provide a constructive way to find a new class of harmonic Finsler manifolds.

Theorem 3.2. *Let (M, α) be a harmonic Riemannian manifold. Let $T(s)$ be defined by (2.2). If $T(s) - 1$ is an odd function of s, then $(M, F = \alpha\,\phi(s),\, d\mu_{HT})$ is a harmonic Finsler manifold.*

Proof. As (M, α) is a harmonic Riemannian manifold, the volume density function $\sqrt{\det(\alpha_{ij})}$ of $d\mu_\alpha$ is a radial function, say $l(r)$, that is, $d\mu_\alpha = l(r)\, dr \wedge d\Theta$. Now, if $T(s) - 1$ is an odd function of s, then, according to [3, Corollary 2.2], the Holmes-Thompson volume measure coincides with $d\mu_\alpha$. That is,

$$d\mu_{HT} = d\mu_\alpha = l(r)\, dr \wedge d\Theta.$$

Hence, $(M, F = \alpha\,\phi(s),\, d\mu_{HT})$ is a harmonic Finsler manifold of (α, β)-type. □

It is worth mentioning that $T(s) - 1$ is an odd function of s for some significant particular ϕ and thereby Theorem 3.2 can be applied. For example, in the case of Randers metrics, $\phi(s) = 1 + s$ and by (2.2), $T(s) = 1 + s$, which implies that $T(s) - 1 = s$ is an odd function of s. Hence, we have the following result.

Corollary 3.3. *All Randers manifolds $(M, F = \alpha + \beta, d\mu_{HT})$ with harmonic α are harmonic.*

Theorem 3.4. *Let (M, α) be a harmonic Riemannian manifold. Let β be a 1-form such that its length $||\beta||_\alpha$ is a radial function and $||\beta||_\alpha < b_o$. Then, $(M, F = \alpha\,\phi(s),\, d\mu)$ is a harmonic Finsler manifold of (α, β)-type.*

Proof. As (M, α) is a harmonic Riemannian manifold, the volume density function $\sqrt{\det(\alpha_{ij})}$ of (M, α) is a radial function, say $l(r)$. In other words, $d\mu_\alpha = l(r)\, dr \wedge d\Theta$. Since, $||\beta||_\alpha$ is a radial function. Then, in view of Lemma 2.1, we obtain

$$\sigma_{BH} = f_{BH}(r)\, l(r) \qquad \text{and} \qquad \sigma_{HT} = f_{HT}(r)\, l(r).$$

Hence, the corresponding volume density functions are radial functions. Consequently, $(M, F = \alpha\phi(s), d\mu)$ is harmonic Finsler manifold of (α, β)-type. □

We end this work by applying our aforementioned results to asymptotically harmonic Finsler manifolds of (α, β)-type. The mean curvature of a Finslerian geodesic sphere is denoted by $\Pi^F_{\nabla r}$ and the mean curvature of a Riemannian geodesic sphere is denoted by $\Pi^\alpha_{\nabla r}$. Consequently, the mean curvature of Finslerian horospheres and Riemannian horospheres are denoted by Π^F_∞, Π^α_∞, respectively.

Theorem 3.5. *Let (M, α) be a complete, simply connected noncompact harmonic Riemannian space with horospheres of constant mean curvature. The following assertions hold:*
(1) *If β is a constant Killing 1-form with respect to α, then $(M, F = \frac{\alpha^2}{\beta},\, d\mu)$ is an AHF-manifold.*
(2) *If $T(s) - 1$ is an odd function of s, where $T(s)$ is defined by (2.2), then $(M, F = \alpha\,\phi(s),\, d\mu_{HT})$ is an AHF-manifold.*
(3) *If $||\beta||_\alpha$ is radial, then $(M, F = \alpha\,\phi(s),\, d\mu)$ is an AHF-manifold.*

Proof. Suppose (M, α) is a harmonic Riemannian manifold.
(1) By Theorem 3.1, $(M, F = \alpha^2/\beta,\, d\mu)$ is a harmonic Finsler manifold. Moreover, each of $d\mu_{HT}$ and $d\mu_{BH}$ is a constant multiple of $d\mu_\alpha$. Hence, the mean curvature of geodesic sphere $\Pi^F_{\nabla r}$ is equal to $\Pi^\alpha_{\nabla r}$, by (2.1). Now, since (M, α) is a complete, simply connected noncompact harmonic Riemannian manifold with horospheres of constant mean curvature, that is, Π^α_∞ exists. Consequently, Π^F_∞ is a real constant. Then it is an AHF-manifold.
(2) Assume $T(s) - 1$ is an odd function of s. By Theorem 3.2, $(M, F = \alpha\, \phi(s),\, d\mu_{HT})$ is a harmonic Finsler manifold and $d\mu_{HT} = d\mu_\alpha$. Thus, by (2.1), $\Pi^F_{\nabla r} = \Pi^\alpha_{\nabla r}$. Since (M, α) is a complete, simply connected noncompact harmonic Riemannian manifold with horospheres of constant mean curvature, that is, Π^α_∞ exists; thereby, Π^F_∞ exists. Hence, $(M, F = \alpha\, \phi(s),\, d\mu_{HT})$ is an AHF-manifold.
(3) It follows directly from Theorem 3.4. □

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