A note on the compatibility of $G_2$-structures with symplectic structures

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Abstract. In this paper we study the relationship between $G_2$-structures and 8-dimensional symplectic structures. We introduce the notion of compatibility of these structures. It is shown that a 7-manifold with $G_2$ structure can be embedded into an 8-dimensional symplectic manifold and with additional conditions, this symplectic structure can be chosen compatible with $G_2$-structure.


Key words: $G_2$-structure; symplectic structure.

1 Introduction

In the classification of Riemannian holonomy groups due to Berger, there are two exceptional cases: $G_2$ and $Spin(7)$. In this paper we concern with manifolds of exceptional holonomy group $G_2$. The compact, simple and simply connected Lie group $G_2$ can be defined as the group of linear transformations of $\mathbb{R}^7$ that preserves the Euclidean metric and a vector cross product. A $G_2$-structure (or an almost $G_2$-structure) on a 7-dimensional manifold $Q$ is a nondegenerate three form $\Omega$ on it. A $G_2$-structure induces a unique Riemannian metric $g$ on $Q$. If furthermore $\text{Hol}(g) \subseteq G_2$, then $Q$ is called a $G_2$-manifold.

The geometry of $G_2$-manifolds has been studied extensively in several papers ([8],[4],[5],[11]). Akbulut and Salur in [1] studied the relationship between Calabi-Yau geometry and $G_2$ geometry. By definition a Calabi-Yau manifold is a Kähler manifold $X$ with $c_1(X) = 0$(of course there are some inequivalent definitions). Thus a Calabi-Yau manifold is a special symplectic manifold. On the other hand the relation between symplectic geometry and contact geometry is obvious. So it is natural to expect a connection between $G_2$ geometry from one hand and symplectic geometry and contact geometry from another hand. In [2] the relationship between $G_2$ geometry and contact geometry has been studied. The relationship between $G_2$ geometry and symplectic geometry emerged in [9] for the first time. In [9], by using methods of spin geometry, Fernandez and Gray showed that $T^*Q \times \mathbb{R}$ admits a closed $G_2$-structure,
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when $Q$ is an oriented 3-dimensional manifold and in [7], Cho and Salur computed this $G_2$-structure as $\Omega = Re\Theta + \omega \wedge dt$, where $\Theta$ is a certain complex valued 3-form and $\omega$ is the standard symplectic form on $T^*Q$.

In this paper we investigate the connection between symplectic structures and $G_2$-structures. The paper is organized as follows:

In section 2 we present some preliminaries. In section 3, the compatibility of symplectic structures with $G_2$-structures and its relation with compatibility of contact structures and $G_2$-structures, will be study. In particular the following theorems will be proved.

**Theorem.** Let $(Q, \alpha)$ be a 7-dimensional contact manifold and $\Omega$ be a $G_2$-structure on $Q$ compatible with $\alpha$. Then $\Omega$ is compatible with symplectic form $\omega = d(e^\theta \alpha)$ on $M = Q \times \mathbb{R}$, where $\theta$ denotes the coordinate on $\mathbb{R}$.

**Theorem.** Let $(Q, \Omega)$ be a hypersurface of symplectic manifold $(M, \omega)$ and $\omega$ is compatible with $\Omega$. If furthermore $Q$ is of contact type then $\Omega$ is compatible with contact structure of $Q$.

In section 4 the existence of symplectic structures on $Q \times \mathbb{R}$ and $Q \times S^1$ is discussed, when $Q$ is a 7-manifold with $G_2$-structure. The main results of this section are as follows:

**Theorem.** Let $Q$ be a 7-dimensional manifold with a $G_2$-structure $\Omega$. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with $\Omega$. The same statement is true for $M = Q \times S^1$.

**Theorem.** Let $Q$ be a connected 7-dimensional manifold with a $G_2$-structure. Then $M = Q \times \mathbb{R}$ is a symplectic manifold. The same statement is true for $M = Q \times S^1$, when $Q$ is furthermore noncompact.

**Theorem.** In previous Theorem, if $R$ is a vector field on $Q$ such that $\iota_R \varphi$ is exact, then $Q \times \mathbb{R}$ and $Q \times S^1$ admits a symplectic structure compatible with $\varphi$.

2 Preliminaries

2.1 $G_2$-structures

In this section $V$ is a finite dimensional real vector space and $(\cdot, \cdot)$ is an inner product on $V$.

**Definition 2.1.** A skew symmetric bilinear map

\[
(2.1) \quad V \times V \to V : (u, v) \mapsto u \times v
\]

is called a cross product if it satisfies

\[
(u \times v, u) = (u \times v, v) = 0,
\]

\[
|u \times v|^2 = |u|^2|v|^2 - (u, v)^2
\]

for all $u, v \in V$.

It is well known that if $V$ admits a non vanishing cross product, then dimension of $V$ is 3 or 7.
Lemma 2.1. If \( \times \) be a cross product on \( V \), then the map \( \Omega : V \times V \times V \to \mathbb{R} \), defined by
\[
\Omega(u, v, w) = \langle u \times v, w \rangle,
\]
is an alternating 3-form the so called the associative calibration of \( V \).

Definition 2.2. Let \( V \) be a finite dimensional real vector space. A 3-form \( \Omega \in \Lambda^3 V^* \) is called nondegenerate if, \( \iota_v\Omega = 0 \) implies that \( v = 0 \). An inner product on \( V \) is called compatible with \( \Omega \) if the map (2.1) defined by (2.2) is a cross product.

Theorem 2.2. Let \( V \) be a 7-dimensional real vector space and \( \Omega \in \Lambda^3 V^* \). Then:
(i) \( \Omega \) is nondegenerate if and only if it admits a compatible inner product.
(ii) The inner product in (i), if it exists, is uniquely determined by \( \Omega \).
(iii) If \( \Omega_1, \Omega_2 \in \Lambda^3 V^* \) are nondegenerate, then there is an automorphism \( g : V \to V \) such that \( g^*\Omega_2 = \Omega_1 \).
(iv) If \( \Omega \) is compatible with the inner product \( \langle \cdot, \cdot \rangle \), then there is an orientation on \( V \) such that the associated volume form \( d\text{vol} \in \Lambda^7 V^* \) satisfies
\[
\iota_u \Omega \wedge \iota_v \Omega \wedge \Omega = 6 \langle u, v \rangle d\text{vol}
\]
for all \( u, v \in V \).

Example 2.3. Identify \( \mathbb{R}^7 \) with \( \text{Im}O \) of imaginary part of octonions. then for \( u, v \in \mathbb{R}^7 \)
\[
\Omega(u, v) = \text{im} u v
\]
defines a cross product with respect to the standard inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^7 \). The associated calibration \( \Omega_0 \) reads
\[
\Omega_0 = e^{123} + e^{145} + e^{167} + e^{167} + e^{246} - e^{275} - e^{347} - e^{356}
\]
where \( e^{ijk} = dx_i \wedge dx_j \wedge dx_k \).

Let \( \langle \cdot, \cdot \rangle \) be an inner product space endowed with a cross product \( \times \) and \( \Omega \) be it’s associated calibration. The sub group of \( \text{Gl}(V) \) that preserve \( \Omega \) is denoted by
\[
G(V, \Omega) = \{ g \in \text{Gl}(V) : g^*\Omega = \Omega \}.
\]
The group \( G(\mathbb{R}^7, \Omega_0) \) will be denoted simply by \( G_2 \). According Theorem 2.4(iii), for an arbitrary nondegenerate 3-form \( \Omega \) on a 7-dimensional vector space \( V \), The group \( G(V, \Omega) \) is isomorphic to \( G_2 \).

Definition 2.4. A nondegenerate 3-form \( \Omega \) on a smooth 7-dimensional manifold \( Q \) is called a \( G_2 \)-structure(or an almost \( G_2 \)-structure).

Remark 2.5. By Theorem 2.4(i, iv) a \( G_2 \) structure \( \Omega \) on \( Q \) induces a unique Riemannian metric and a unique orientation on \( Q \). Thus each tangent space \( T_pQ \) of \( Q \) admits a cross product defined by (2.2).

For more information about \( G_2 \)-structures we refer to [13] and [10].
2.2 Almost symplectic structures and Gromov’s Theorem

Let $M$ be a $2n$-dimensional smooth manifold. A nondegenerate two form $\omega$ on $M$ is called an almost symplectic structure. If furthermore $\omega$ is closed, then $\omega$ is called a symplectic structure on $M$. It is well known that an almost symplectic manifold $(M, \omega)$ admits almost complex structures $J$ tamed by $\omega$, i.e., $\omega(v, Jv) > 0$ for all nonzero $v$ in $TM$. The space of such almost complex structures is contractible. The following theorem, due to Gromov, states that an almost symplectic structure is homotopic to a symplectic structure. For a proof of this theorem we refer to Theorem 7.34 of [12].

**Theorem 2.3.** (Gromov’s Theorem) Let $M$ be an open $2n$ dimensional manifold. Let $\tau$ be an almost symplectic structure on $M$ and $a \in H^2(M, \mathbb{R})$. There exists a family of almost symplectic forms $\tau_t$ on $M$ such that $\tau_0 = \tau$ and $\tau_1$ is a symplectic form that represents the class $a$.

2.3 Almost contact structures

Let $M$ be an $(2n + 1)$ dimensional smooth manifold. An almost contact structure on $M$ is a triple $(J, R, \alpha)$ consists of a field $J$ of endomorphisms of the tangent bundle, a vector field $R$ and a 1-form $\alpha$ satisfying

1) $\alpha(R) = 1$,
2) $J^2(X) = -X + \alpha(X)R$,

for all $X$ in $TM$.

Let $(J, R, \alpha)$ be an almost contact structure on $M$. A Riemannian metric $g$ on $M$ is called a compatible metric if

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v),$$

for all $u, v$ in $TM$. An **almost contact metric structure** on $M$ is a quadruple $(J, R, \alpha, g)$, where $(J, R, \alpha)$ is an almost contact symplectic structure and $g$ is a compatible metric.

It is well known that every manifold with an almost contact structure admits a compatible metric. For more details we refer to [3].

3 Compatibility of $G_2$-structures and symplectic structures

In [2], two kind of compatibility of contact structures and $G_2$ structures on a manifold, when both of them exist, has been defined. Here we need one of them, the so called $A$-compatibility, which we simply call it compatible.

**Definition 3.1.** Let $\Omega$ be a $G_2$-structure on 7-dimensional manifold $Q$. A contact structure $\xi$ on $Q$ is said to be compatible with $\Omega$ if there exist a vector field $R$ on $Q$, a contact form $\alpha$ for $\xi$ and a nonzero function $f : Q \to \mathbb{R}$ such that $d\alpha = \iota_R \Omega$ and $fR$ is the Reeb vector field of a contact form for $\xi$.

In this section we consider a hypersurface of a symplectic 8-dimensional manifold, which admits a $G_2$-structure. We want to know how these two structures are related.
Definition 3.2. Let \((M, \omega)\) be an eight dimensional (almost) symplectic manifold and \(Q\) be a hypersurface of \(M\) with \(G_2\)-structure \(\Omega\). The (almost) symplectic form \(\omega\) is called compatible with \(\Omega\) if there is a vector field \(R : Q \rightarrow TQ\) satisfying

\[
j^*(\omega) = \iota_R \Omega,
\]

where \(j : Q \hookrightarrow M\) is the inclusion map.

The following example explains the motivation of this definition.

Example 3.3. Let \((x_1, ..., x_8)\) denotes the coordinates on \(\mathbb{R}^8\) and consider the symplectic form \(\omega\) on \(\mathbb{R}^8\) as follows:

\[
\omega = dx_1 \wedge dx_8 + dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7.
\]

Consider \(\mathbb{R}^7\) as a hypersurface in \(\mathbb{R}^8\) with coordinates \((x_1, ..., x_7)\). Let \(\Omega_0\) be the standard \(G_2\)-structure on \(\mathbb{R}^7\). If \(R = \frac{\partial}{\partial x_1}\), we have

\[
\iota_R \Omega_0 = dx_2 \wedge dx_3 + dx_4 \wedge dx_5 + dx_6 \wedge dx_7 = j^*(\omega),
\]

where \(j : \mathbb{R}^7 \rightarrow \mathbb{R}^8\) is defined by \(j(x_1, ..., x_7) = (x_1, ..., x_7, 0)\).

Theorem 3.1. Let \((Q, \alpha)\) be a 7-dimensional contact manifold and \(\Omega\) be a \(G_2\)-structure on \(Q\) compatible with \(\alpha\). Then \(\Omega\) is compatible with symplectic form \(\omega = d(\theta^\alpha)\) on \(M = Q \times \mathbb{R}\), where \(\theta\) denotes the coordinate on \(\mathbb{R}\).

Proof. By assumption, there is a vector field \(R\) on \(Q\) such that \(\iota_R \Omega = d\alpha\). but \(d\alpha = j^*(\omega)\). \(\square\)

Example 3.4. Let \(Q\) be a 3-dimensional oriented Riemannian manifold. Consider the coordinates \((x_1, x_2, x_3, y_1, y_2, y_3)\) on the cotangent bundle \(T^*Q\). Assume \(\omega = -d\lambda_{\text{can}}\) be the standard symplectic form on \(T^*Q\), where \(\lambda_{\text{can}} = \sum y_i dx_i\) is the canonical 1-form on \(T^*Q\). If \(t\) denotes the coordinate on \(\mathbb{R}\), define the 3-form \(\Omega\) on \(T^*Q\) by

\[
\Omega = Re(\Theta) + dt \wedge \omega,
\]

where \(\Theta = (dx_1 + idy_1) \wedge (dx_2 + idy_2 \wedge (dx_3 + idy_3))\) is a complex valued \((3, 0)\) form on \(T^*Q\). In [7] it is shown that \(\Omega\) is a \(G_2\)-structure on \(T^*Q \times \mathbb{R}\). On the other hand it is easy to see that \(\alpha = dt + \lambda_{\text{can}}\) defines a contact structure on \(T^*Q \times \mathbb{R}\) with the Reeb field \(\frac{\partial}{\partial t}\). This contact structure is compatible with \(\Omega\). Thus \(\Omega\) is compatible with symplectic structure \(\omega = d(\theta^\alpha)\) on \(M = T^*Q \times \mathbb{R}^2\).

Definition 3.5. A compact and orientable hypersurface \(Q\) of a symplectic manifold \((M, \omega)\) is called of contact type if there exists a 1-form \(\alpha\) on \(Q\) satisfying

1) \(d\alpha = j^*(\omega)\),
2) \(\alpha(\xi) \neq 0\) for \(0 \neq \xi \in \mathcal{L}_Q\),

where \(j : Q \hookrightarrow M\) is the inclusion map and \(\mathcal{L}_Q\) is the canonical line bundle of \(Q\).

Theorem 3.2. Let \((Q, \Omega)\) be a hypersurface of symplectic manifold \((M, \omega)\) and \(\omega\) is compatible with \(\Omega\). If furthermore \(Q\) is of contact type then \(\Omega\) is compatible with contact structure of \(Q\).
Proof. Since $Q$ is of contact type then there exists a 1-form $\alpha$ on $Q$ such that $d\alpha = j^* (\omega)$ and since $\omega$ is compatible with $\Omega$, there is a vector field $R$ on $Q$ such that

$$\iota_R \Omega = j^*(\omega) = d\alpha.$$ 

Moreover $\iota_R d\alpha = 0$ and since the restriction of $d\alpha$ to $\ker \alpha$ is symplectic, then $\alpha(R) \neq 0$ and so $f_R$ is the Reeb field of $\alpha$, where $f = \frac{1}{\alpha(R)}$. □

Theorem 3.3. Let $(M, \omega)$ be an 8-dimensional symplectic manifold and $Q \subset M$ be a closed (i.e. compact and without boundary) hypersurface of $M$ with a closed $G_2$-structure $\Omega$. If $H^1(Q) = 0$, then $\omega$ is not compatible with $\Omega$. 

Proof. Since $j^*(\omega)$ is closed and $H^1(Q) = 0$, then $j^*(\omega) = d\alpha$ for some 1-form $\alpha$ on $Q$. If $\omega$ is compatible with $\Omega$, then there is a vector field $R$ on $Q$ such that $\iota_R \omega = d\alpha$.

Thus $g(R, R)vol_Q = (\iota_R \omega) \wedge (\iota_R \omega) \wedge \omega$ is exact and hence $\int_Q g(R, R) vol_Q = 0$, which is a contradiction. □

4 $G_2$-structures and existence of symplectic structures

In this section we show that if $Q$ admits a $G_2$-structure, then $Q \times \mathbb{R}$ and $Q \times S^1$ admit a symplectic structure, and hence $Q$ can be embedded in a symplectic manifold.

Lemma 4.1. Let $(2n+1)$-dimensional manifold $Q$ admits an almost contact structure. Then $Q \times \mathbb{R}$ and $Q \times S^1$ admit an almost complex structure.

Proof. Let $(J, R, \alpha)$ be an almost contact structure on $Q$ and $g$ be a Riemannian compatible metric. Let $D$ be the sub bundle of $TQ$ generated by $R$ and $H$ be the orthogonal complement of $D$ with respect to $g$. Thus $TQ = H \oplus D \oplus TR$. So, for $X \in T(Q \times \mathbb{R})$, $X$ splits as $X = X_H + bR + a \frac{\partial}{\partial \theta}$, where $X_H \in H$ and $\theta$ denotes the coordinate on $\mathbb{R}$. Define the automorphism $J' : T(Q \times \mathbb{R}) \to T(Q \times \mathbb{R})$ by

$$J'(X_H + bR + a \frac{\partial}{\partial \theta}) = J(X_H) + aR - b \frac{\partial}{\partial \theta}.$$ 

It is easy to see that $J'$ is an almost complex structure on $Q \times \mathbb{R}$. □

Theorem 4.2. Let $Q$ be a 7-dimensional manifold with a $G_2$-structure $\Omega$. Then $M = Q \times \mathbb{R}$ admits an almost symplectic structure compatible with $\Omega$. The same statement is true for $M = Q \times S^1$.

Proof. Let $g_\Omega$ and $\times_\Omega$ denotes, respectively, the Riemannian metric and cross product associative to $\Omega$ on $Q$. Choose a nonzero vector field $R$ on $Q$ with $g_\Omega(R, R) = 1$ and define the 1-form $\alpha$ and endomorphism $J_R : TQ \to TQ$ by

$$\alpha_R(u) = g_\Omega(R, u),$$ 

$$J_R(u) = R \times_\Omega u.$$
The quadruple \((J_R, R, \alpha_R, g_\Omega)\) defines an almost contact metric structure on \(Q\). Let \(J\) be the almost complex structure induced by \(J_R\) on \(Q \times \mathbb{R}\). Let \(\theta\) denotes the coordinate on \(\mathbb{R}\) and define the Riemannian metric \(g\) and the two form \(\omega\) on \(M = Q \times \mathbb{R}\) by
\[
g = g_\Omega + d\theta^2,
\]
\[
\omega(u, v) = g(Ju, v).
\]
\(\omega\) is an almost symplectic structure on \(M\) and for \(u, v\) in \(TQ\) we have
\[
\omega(u, v) = g(Ju, v) = g(R \times u, v) = \Omega(R, u, v).
\]
Thus \(\omega\) and \(\Omega\) are compatible. \(\square\)

**Corollary 4.3.** Every connected 7-dimensional manifold with \(G_2\)-structure can be embedded in an 8-dimensional symplectic manifold.

**Proof.** Let \(Q\) be a 7-dimensional manifold with \(G_2\)-structure. By Theorem 4.2, \(Q \times \mathbb{R}\) and \(Q \times S^1\) admit an almost symplectic structure. Now Gromov’s Theorem follows the assertion. \(\square\)

As in Corollary 4.3 mentioned if \(Q\) admits a \(G_2\)-structure, then \(Q \times \mathbb{R}\) and \(Q \times S^1\) (if \(Q\) is not compact) admit a symplectic structure. It seems to be an open question wether or not every \(G_2\)-structure is compatible with a symplectic structure. We could not find counterexample but also did not see how to prove it.

**Definition 4.1.** (see [6]) Let \(\varphi\) be a closed \(G_2\)-structure on \(Q\). The vector field \(R\) on \(Q\) is called a \(G_2\)-vector field if the flow of \(R\) preserves the \(G_2\)-structure. Also \(R\) is called Rochesterian if \(\iota_R \varphi\) is an exact form.

**Corollary 4.4.** Let \((Q, \varphi)\) is a hypersurface of \((M, \omega)\) and \(\omega\) is compatible with \(\varphi\). If \(\varphi\) is closed and \(\iota_R \varphi = j^*(\omega)\), then \(R\) is a \(G_2\)-vector field.

**Corollary 4.5.** In Theorem 4.2, if \(R\) is a vector field on \(Q\) such that \(\iota_R \varphi\) is exact, then \(Q \times \mathbb{R}\) and \(Q \times S^1\) admits a symplectic structure compatible with \(\varphi\).

In [6] it is shown that there is no Rochesterian vector field on a closed 7-dimensional manifold with a closed \(G_2\)-structure. So in the Corollary 4.4, if \(\omega\) is exact, then \(Q\) is assumed to be noncompact or compact without boundary.

**Corollary 4.6.** In Theorem 4.2, if \(\varphi\) is closed and \(R\) is a \(G_2\)-vector field, then there exists a symplectic form \(\omega\) on \(Q \times \mathbb{R}\) such that \([\omega] = [\pi^*(\iota_R \varphi)]\), where \(\pi : Q \times \mathbb{R} \to Q\) is the projection map. The same result is true for \(Q \times S^1\).

**References**


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