Generalized Wintgen-type inequality for submanifolds in S-space-forms

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Abstract. In this paper, we obtain the generalized Wintgen inequality for C-totally real submanifolds in S-space form. The advantage with this result is that we have two inequalities in only one. We introduce bi-slant submanifolds in S-space form. We give a non trivial example. Further, we discuss the Wintgen inequality for bi-slant submanifolds in the same ambient space and derive its applications in various slant cases.

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1 Introduction

The Wintgen inequality (1979) is the sharp geometric inequality for surfaces in the Euclidian space, E^4, involving the Gauss curvature (intrinsic invariant), the normal curvature and squared mean curvature (extrinsic invariant), respectively. De Smet et al [23] conjectured a generalized Wintgen inequality for submanifolds of arbitrary dimension and codimension in Riemannian space form. This conjecture was proved by [16] and Ge and Tang [14], independently. Later, this conjecture was been proved in different space forms, in complex and Sasakian space forms ([18], [19]), Golden Riemannian space form [12], Bochner Kahler space form [1]. Recently, Mohd et al derived a generalized Wintgen-type inequality for submanifolds in generalized space forms, they extended this inequality to the case of bi-slant submanifolds in generalized space forms and derived some applications in various slant cases [3].

On the other hand, Yano, [24], introduced the notion of f-structure on a $(2n+s)$-dimensional manifold as a tensor field of type $(1,1)$ and rank $2n$ satisfying $f^3 + f = 0$. Almost complex, in even dimension ($s = 0$) and almost contact, in odd dimension ($s = 1$) structures are well known examples of f-structure. The existence of such structure is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times O(s)$, [4]. Recently, Najma [21] established new results of squared mean curvature and Ricci curvature for the submanifolds of S-space form that is the

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generalization of complex and contact structure. Kim, [15], obtained a basic inequality for submanifolds of an S-space form tangent to structure vector fields. The notion of bi-slant submanifolds of an almost hermitian manifold or almost contact manifold was introduced as a natural generalisation of CR-submanifold, hemi-slant submanifold, semi-slant submanifold, [6]. In [17], [20], the authors have studied CR-submanifolds of S-manifolds. Motivated by the work above, we establish the generalized Wintgen-type inequality for submanifolds in S-space form that is the genaralization of Sasakian space form and Kahler space form, ([18], [19]). This paper is organized as follows. In section 2, we recall some necessary background on f-structures, S-manifolds and S-space forms. In section 3, we established the generalized Wintgen-type inequality for submanifolds of S-space form. In section 4, we give a non trivial example of bi-slant submanifolds of S-space forms, the generalized Wintgen-type inequality for the same ambient space and derive its applications in various slant cases.

2 Preliminaries

Yano showed that almost complex and almost contact structures can be generalized as f-structures on a smooth manifold of dimension $2n+s$. The idea for the f-structure is to consider a tensor field with condition $f^3+f=0$, of type $(1,1)$ and rank $2n$.

Let $\bar{M}^{2n+s}$ be a smooth manifold along an f-structure of rank $2n$. We take s structural vectors fields $\xi_1, \xi_2, \ldots, \xi_s$ on $\bar{M}$ such as

$$f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0, \quad f^2 = -I + \sum_{\alpha=1}^{s} \xi_\alpha \otimes \eta_\alpha, \tag{2.1}$$

where η_α and ξ_α are dual forms to each other, therefore, complemented frames exist on f-structures. For an f-manifold, we define a Riemannian metric as

$$g(X,Y) = g(fX,fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \quad X,\ Y \in \Gamma(T\bar{M}). \tag{2.2}$$

A consequence of (2.1) and (2.2) is

$$g(fX,X) = 0, \quad g(fX,Y) = -g(X,fY). \tag{2.3}$$

An f-structure is normal, if there exist complemented frames and $[f,f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$, where $[f,f]$ is the Nijenhuis torsion of f. Consider the fundamental 2-form B defined as $B(X,Y) = g(X,fY)$. A metric f- structure which is normal and $d\eta_1 = d\eta_2 = \cdots = d\eta_s = B$ is know as an S-structure. A smooth manifold along with an S-structure is known as an S-manifold. Blair described such types of manifolds in [4]. In the case $s=1$, an S-manifold is a Sasakian manifold. In the case $s=0$, an $S-$manifold is a Kahler manifold. For $s \geq 2$ examples of $S-$manifold are given in [1] For the Riemannian connection $\bar{\nabla}$ of g of an $S-$manifold $\bar{M}^{2n+s}$, the following were also proved in [4]

$$\bar{\nabla}_X \xi_\alpha = -fX, \quad X \in \Gamma(T\bar{M}), \quad \alpha = 1, \ldots, s. \tag{2.4}$$

$$(\bar{\nabla}_X f)Y = \sum_{\alpha=1}^{s}[g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X], \quad X,\ Y \in \Gamma(T\bar{M}). \tag{2.5}$$

Let L denote the distribution determined by $-f^2$ and M the complementary distribution. M is determined by f^2+I and spanned by $\xi_1, \ldots, \xi_s$. If $X \in L$, then $\eta_\alpha(X) = 0$ for any α and if If $X \in M$, then $fX = 0$.

A plane section π is called an invariant $f-$section if it is determined by a vector $X \in L(p)$, $p \in \bar{M}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X, fX)$ called an invariant f-sectional curvature is a constant c, then its curvature tensor has the form

$$\begin{aligned}\bar{R}(X,Y)Z = &\sum_{\alpha,\beta=1}^{s}\{\eta_\alpha(X)\eta_\beta(Z)f^2Y - \eta_\alpha(Y)\eta_\beta(Z)f^2X \\ &- g(fX,fZ)\eta_\alpha(Y)\xi_\beta + g(fY,fZ)\eta_\alpha(X)\xi_\beta\} \\ &+ \frac{c+3s}{4}\{-g(fY,fZ)f^2X + g(fX,fZ)f^2Y\} \\ &+ \frac{c-s}{4}\{g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ\},\end{aligned} \tag{2.6}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

Then the S-manifold will be denoted by $\bar{M}^{2n+s}(c)$ and it is said to be S-space form. As example of S-space form, we mention the euclidian space and hyperbolic space [4].

Let N be a submanifold with an induced metric g of a real dimension m in an S-space form, $\bar{M}^{2n+s}(c)$. If $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $\bar{M}^{2n+s}(c)$ and N, respectively, then the fundamental formulas of Gauss and Weingarten are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where $X, Y \in \Gamma(TN)$, $\xi \in \Gamma(TN)^\perp$ and $\nabla^\perp$ represents the normal connection. Recall that, in the above basic formulas, h denotes the second fundamental form and A_ξ is the shape operator, they are connected by

$$g(h(X,Y),\xi) = g(A_\xi X, Y).$$

Let R be the Riemannian curvature tensor of N^m. We will use the convention $R(X,Y,Z,W) = g(R(X,Y)W,Z)$, for all $X,Y,Z,W \in \Gamma(TN)$. Then the Gauss equation is given by

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{2.7}$$

for all $X, Y, Z, W \in \Gamma(TN)$, and the Ricci equation by

$$\bar{R}(X,Y,\eta,\xi) = R^\perp(X,Y,\eta,\xi) + g([A_\xi, A_\eta]X,Y), \tag{2.8}$$

for all $X, Y \in \Gamma(TN)$ and $\xi, \eta \in \Gamma(TN)^\perp$.

A submanifold N of an S-space form $\bar{M}(c)$ is called C-totally real submanifold if ξ_α, $\alpha = 1, 2, \ldots, s$ is normal to N, and a consequence of this is that $f(T_pN) \subset T_pN^\perp$, for all $p \in N$ [22].

For a vector field $X \in T_pN$, $p \in N$, it can be written as $fX = PX + QX$, where PX is tangent component of fX and QX is a normal component of fX. If $P = 0$, then the submanifold is said to be an anti-invariant submanifold and if $Q = 0$, the submanifold is said to be an invariant submanifold. Let $\{e_1, e_2, \ldots, e_m\}$ and $\{e_{m+1}, e_{m+2}, \ldots, e_{2n+s}\}$ be a tangent orthonormal frame and normal orthonormal frame respectively on N.

The mean curvature vector field is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i). \tag{2.9}$$

The norm of the squared mean curvature of the submanifolds is defined by

$$\| H \|^2 = \frac{1}{m^2}\sum_{r=m+1}^{2n+s}\left(\sum_{i=1}^{m} h_{ii}^r\right)^2. \tag{2.10}$$

Further,

$$\| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)), \tag{2.11}$$

and

$$\| P \|^2 = \sum_{i,j=1}^{m} g^2(Pe_i, e_j). \tag{2.12}$$

3 Generalized Wintgen inequality for C-totally real submanifolds in S-space form

We denote by K and $R^\perp$ the sectional curvature function and the normal curvature tensor on N. Then the normalized scalar ρ is given by

$$\rho = \frac{2\tau}{m(m-1)} = \frac{2}{m(m-1)}\sum_{1 \le i < j \le m} K(e_i, e_j),$$

where τ is a scalar curvature, and the normalized normal scalar curvature is given by [16]

$$\rho^\perp = \frac{2\tau^\perp}{m(m-1)} = \frac{2}{m(m-1)}\sqrt{\sum_{1 \le i < j \le m}\ \sum_{1 \le r < s \le 2n+s-m} (R^\perp(e_i, e_j, \xi_r, \xi_s))^2}.$$

Following [16], we put

$$K_N = \frac{1}{4}\sum_{r,s=1}^{2n+s-m} trace[A_r, A_s]^2,$$

and call it the scalar normal curvature of N. The normalized scalar normal curvature is given by

$$\rho_N = \frac{2}{m(m-1)}\sqrt{K_N}.$$

Obviously

$$\begin{aligned} K_N &= \frac{1}{4}\sum_{r,s=1}^{2n+s-m} trace[A_r, A_s]^2 \\ &= \sum_{1\le r<s\le 2n+s-m}\ \sum_{1\le i<j\le m} g([A_r, A_s]e_i, e_j)^2. \end{aligned} \tag{3.1}$$

In terms of the component of the second fundamental form, we can express K_N by the formula

$$\sum_{1\le r<s\le 2n+s-m}\ \sum_{1\le i<j\le m} (\sum_{k=1}^{m} h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})^2. \tag{3.2}$$

Lemma 3.1. *Let N be a C-totally real submanifold in $S-$space form $\bar{M}(c)$. Then*

$$\rho_N \le \parallel H \parallel^2 - \rho + \frac{c+3s}{4}. \tag{3.3}$$

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+s}\}$ the shape operators of N in $\bar{M}$ take the form

$$A_{e_{m+1}} = \begin{pmatrix} f_1 & b & 0 & \ldots & 0 \\ b & f_1 & 0 & \ldots & 0 \\ 0 & 0 & f_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & f_1 \end{pmatrix},$$

$$A_{e_{m+2}} = \begin{pmatrix} f_2+b & 0 & 0 & \ldots & 0 \\ 0 & f_2-b & 0 & \ldots & 0 \\ 0 & 0 & f_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & f_2 \end{pmatrix},$$

$$A_{e_{m+3}} = \begin{pmatrix} f_3 & 0 & 0 & \ldots & 0 \\ 0 & f_3 & 0 & \ldots & 0 \\ 0 & 0 & f_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & f_3 \end{pmatrix},$$

where f_1, f_2, f_3 and b are real functions, and $A_{e_{m+4}} = \cdots = A_{e_{2n+s}} = 0$.

Proof. Let N be a C-totally real submanifold in S-space form $\bar{M}(c)$. We choose $\{e_1, e_2, \ldots, e_m\}$ and $\{e_{m+1}, e_{m+2}, \ldots, e_{2n+s}\}$ as orthonormal frame and orthonormal

normal frame on N. respectively. From (2.6), we put $X = e_i$, $Y = e_j$, $Z = e_j$ and $W = e_i$, $i < j$, we have

$$\bar{R}(e_i, e_j, e_j, e_i) = \frac{1}{8}(c + 3s).m(m-1)$$

Using Gauss equation, we infer

$$\frac{1}{8}(c + 3s)m(m-1) = \tau - m^2 \parallel H \parallel^2 + \parallel h \parallel^2 .$$

On the other hand, we have

$$\begin{aligned} m^2 \parallel H \parallel^2 &= \sum_{r=m+1}^{2n+s} (\sum_{i=1}^{m} h_{ii}^r)^2 \\ &= \frac{1}{m-1} \sum_{r=m+1}^{2n+s} \sum_{1=i<j}^{m} (h_{ii}^r - h_{jj}^r)^2 + \frac{2m}{m-1} \sum_{r=m+1}^{2n+s} \sum_{1=i<j}^{m} h_{ii}^r h_{jj}^r. \end{aligned} \tag{3.4}$$

Furthermore, from [13], we have

$$\sum_{r=m+1}^{2n+s} \sum_{1=i<j}^{m} (h_{ii}^r - h_{jj}^r)^2 + 2m \sum_{r=m+1}^{2n+s} (\sum_{1=i<j}^{m} h_{ij}^r)^2 \geq [\sum_{r=m+1}^{2n+s} \sum_{1=i<j}^{m} (h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s)]^{1/2}. \tag{3.5}$$

Combining (3.2), (3.4) (3.5), we have

$$m^2 \parallel H \parallel^2 - m^2 \rho_N \geq \frac{2m}{m-1} \sum_{r=m+1}^{2n+s} \sum_{1=i<j}^{m} h_{ii}^r h_{jj}^r - (h_{ij}^r)^2. \tag{3.6}$$

From the relation (2.7) and (3.6), we get

$$m^2 \parallel H \parallel^2 - m^2 \rho_N \geq \frac{2m}{m-1} [\tau - \frac{1}{8}(c + 3s)m(m-1)]. \tag{3.7}$$

Then

$$\parallel H \parallel^2 - \rho_N \geq \rho - \frac{1}{4}(c + 3s). \tag{3.8}$$

Finally, analysing the case of equality in (3.5), we deduce that the equality holds in the inequality (3.3), at some point $p \in N$ if and only if there exist an orthonormal basis of T_pN and an orthonormal basis of $T_pN^{\perp}$ such that the shape operators take the form desired. □

As an immediate consequence of Lemma 1, we deduce the following results of [18], [19]

Corollary 3.2. *Let N be a totally real submanifold of Kahler space form $\bar{M}^{2n}(c)$. Then*

$$\rho_N \leq \parallel H \parallel^2 - \rho + \frac{c}{4}. \tag{3.9}$$

Corollary 3.3. *Let N be a totally real submanifold of Sasakian space form $\bar{M}^{2n+1}(c)$. Then*

$$\rho_N \leq \| H \|^2 - \rho + \frac{c+3}{4}. \tag{3.10}$$

The main result of this section is the following

Theorem 3.4. *Let N be a totally real submanifold of S-space $\bar{M}$. Then*

$$(\rho^{\perp})^2 \leq (\| H \|^2 - \rho + \frac{c+3s}{4})^2 + \frac{4}{m(m-1)}(\rho - \frac{c+3s}{4}).\frac{c-s}{4} + \frac{(c-s)^2}{8m(m-1)}.$$

Proof. Let N be a totally real submanifold of S-space form $\bar{M}$. We choose $\{e_1, \dots, e_m\}$ an orthonormal frame on N. From (2.6), we put $X = e_j$, $Y = e_i$, $Z = \xi$ and $W = \eta$, we have

$$\bar{R}(e_i, e_j, \xi, \eta) = \frac{c-s}{4}\{g(e_i, f\xi)g(fe_j, \eta) - g(e_j, f\xi)g(fe_i, \eta)\},$$

without loss of generality, we can suppose that $\eta = fe_k$ and $\xi = fe_l$.

Then

$$\bar{R}(e_i, e_j, \xi, \eta) = \frac{c-s}{4}\{\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\}, \tag{3.11}$$

where γ_{il} is the Kronecker symbol.

From (3.11) and (2.8),

$$g(R^{\perp}(e_i, e_j)\eta, \xi) = \frac{c-s}{4}\{\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\} - g([A_r, A_s]e_i, e_j). \tag{3.12}$$

From this, we get

(3.13)

$$\begin{aligned}(\tau^{\perp})^2 = \sum_{i,j=1}^{m} g(R^{\perp}(e_i, e_j)\eta, \xi)^2 &= (\frac{c-s}{4}\{\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\})^2 \\ &\quad - 2\frac{c-s}{4}\{\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\}g([A_r, A_s]e_i, e_j) + (g([A_r, A_s]e_i, e_j))^2 \\ &= \frac{m^2(m-1)^2}{4}\rho_N + \frac{m(m-1)}{2}(\frac{c-s}{4})^2 + (\frac{c-s}{4})(- \| h \|^2 + m^2 \| H \|^2).\end{aligned}$$

On other hand (2.7) give us

$$m^2 \| H \|^2 - \| h \|^2 = 2\tau - \frac{(c+3s)m(m-1)}{4}, \tag{3.14}$$

or equivalently,

$$m^2 \| H \|^2 - \| h \|^2 = m(m-1)(\rho - \frac{c+3s}{4}). \tag{3.15}$$

By substituting (3.15) in (3.13) we obtain

$$(\rho^{\perp})^2 \leq (\rho^N)^2 + \frac{4}{m(m-1)}(\rho - \frac{c+3s}{4}).\frac{c-s}{4} + \frac{(c-s)^2}{8m(m-1)}.$$

Taking account of lemma 3.1, it follows that

$$(\rho^{\perp})^2 \leq (\| H \|^2 - \rho + \frac{c+3s}{4})^2 + \frac{4}{m(m-1)}(\rho - \frac{c+3s}{4}).\frac{c-s}{4} + \frac{(c-s)^2}{8m(m-1)}. \quad \square$$

Remark 3.1. For integral submanifolds with N normal to the structure vector fields, we have the same inequality.

4 Generalized Wintgen inequality for bi-slant submanifolds in S-space form

In this section, we suppose that the structure vector fields ξ_α, $\alpha = 1, \ldots, s$, are tangent to N.

A submanifold N in an almost contact metric manifold $\bar{M}$ is said to be Slant if for any differentiable function f on N, and any non zero vector field X on N, linearly independent on ξ angle between fX and T_pM is a constant $\theta \in [0, \frac{\pi}{2}]$, called the slant angle of N in $\bar{M}$. Recall that both invariant and anti-invariant submanifolds are particular examples of slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively, moreover, if $0 < \theta < \frac{\pi}{2}$, then N is said to be a θ-slant submanifold or proper slant submanifold. A submanifold in an almost hermitian manifold $\bar{M}$ is said to be slant if for any differentiable function f on N and any non zero vector field X on N, linearly independent on ξ angle between fX and T_pM is a constant $\theta \in [0, \frac{\pi}{2}]$.

Combining these two concepts lead us to the introduction of bi-slant submanifolds for S-space forms

Definition 4.1. A submanifold N tangent to structure vector field of an S-space $\bar{M}$ is said to be a bi-slant submanifold, if there exist three orthogonal distribution D_1, D_2 and $D_3 = span\{\xi_1, \xi_2, \ldots, \xi_s\}$ such that

1) $TN = D_1 \oplus D_2 \oplus D_3$,

2) D_i, is the slant distribution with slant angle θ_i, for any $i = 1, 2$.

3) $fD_1 \perp D_2$ and $fD_2 \perp D_1$.

4.1 Examples of bi-slant submanifolds of S-space form

As example of bi-slant submanifold in an S-space form, for $s = 0$, we have the class of slant submanifold but also the class of semi-slant submanifold, hemi-slant submanifold and CR-submanifold [2].
Now we give a nontrivial example of proper bi-slant submanifold.
For any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$,

$$x(u, v, w, t, z_1, z_2) = (u, 0, w, v\cos\theta_1, v\sin\theta_1, t\cos\theta_2, t\sin\theta_2, z_1, z_2),$$

defines a 6−dimensional bi-slant submanifold N, with slant angle θ_1, θ_2 in $R^{10}(-3s)$ with its S-structure given by

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha}, \alpha = 1, 2$$
$$\eta_\alpha = \frac{1}{2}(dz_\alpha - \sum_{i=1}^{4} y_i dx_i), \alpha = 1, 2$$
$$fX = \sum_{i=1}^{4} Y^i \frac{\partial}{\partial x_i} - \sum_{i=1}^{4} X^i \frac{\partial}{\partial y_i} + (\sum_{i=1}^{4} Y^i Y_i)(\sum_{\alpha=1}^{2} \frac{\partial}{\partial z_\alpha})$$
$$g = \sum_{\alpha=1}^{2} \eta_\alpha \otimes \eta_\alpha + \sum_{i=1}^{4} (dx_i \otimes dx_i + dy_i \otimes dy_i),$$

where

$$X = \sum_{i=1}^{4} (X^i \frac{\partial}{\partial x_i} + Y^i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{2} Z^\alpha \frac{\partial}{\partial z_\alpha}.$$

Furthemore, it is easy to see that

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \cos\theta_1 \frac{\partial}{\partial y_1} + \sin\theta_1 \frac{\partial}{\partial y_2}, e_3 = \frac{\partial}{\partial x_3},$$
$$e_4 = \cos\theta_2 \frac{\partial}{\partial y_3} + \sin\theta_2 \frac{\partial}{\partial y_4}, e_5 = \frac{\partial}{\partial z_1}, \quad e_6 = \frac{\partial}{\partial z_2}.$$

From a local orthonormal frame of T_pN, if we define $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$ then $g(fe_1, e_2) = \cos\theta_1$ $g(fe_3, e_4) = \cos\theta_2$ proving that the distribution D_1 is θ_1-slant and the distribution D_2 is θ_2- slant.

4.2 Wintgen inequality

Theorem 4.1. *Let N be a bi-slant submanifold in S-space form $\bar{M}$, with slant angle θ_i and $dimD_i = d_i$, $i = 1, 2$. Then*

$$\parallel H \parallel^2 - \rho_N \geq \rho - \frac{c + 3s}{4} + \frac{s(c + 3s - 4)}{2m} - \frac{3(c - s)}{4m(m-1)}(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2). \tag{4.1}$$

Proof. Let N be a bi-slant submanifold in S-space form. We choose $\{e_1, e_2, \ldots, e_m\}$, where $m = d_1 + d_2 + s$, and $\{e_{m+1}, e_{m+2}, \ldots, e_{2n}\}$ as orthonormal frame and orthonormal normal frame on N respectively. From (2.6), we take $X = e_i$, $Y = e_j$, $Z = e_j$ and $W = e_i$, $i < j$.

$$\begin{aligned}\bar{R}(e_i, e_j, e_j, e_i) = &\sum_{\alpha,\beta=1}^{s} \{g(fe_i, fe_i)\eta_\alpha(e_j)\eta_\beta(e_j)) - g(fe_i, fe_j)\eta_\alpha(e_j)\eta_\beta(e_i)) \\ &+ g(fe_j, fe_j)\eta_\alpha(e_i)\eta_\beta(e_i)) - g(fe_j, fe_i)\eta_\alpha(e_i)\eta_\beta(e_j))\} \\ &+ \frac{c + 3s}{4}\{g(fe_i, fe_i)g(fe_j, fe_j) - g(fe_i, fe_j)g(fe_j, fe_i)\} \\ &+ \frac{c - s}{4}\{g(e_i, fe_i)g(e_j, fe_j) - g(e_i, fe_j)g(e_j, fe_i) - 2g(e_i, fe_j)g(e_j, fe_i)\}.\end{aligned}$$

By using (2.2) and (2.3) in the above equation, we get

$$\begin{aligned}\bar{R}(e_i,e_j,e_j,e_i) &= g(e_i,e_i)\eta_\alpha^2(e_j) - \eta_\alpha^2(e_j)\eta_\gamma^2(e_i) + \eta_\alpha(e_j)\eta_\beta(e_i)\eta_\gamma(e_i)\eta_\gamma(e_j)\\ &+ g(e_j,e_j)\eta_\alpha^2(e_i) - \eta_\alpha^2(e_i)\eta_\gamma^2(e_j) + \eta_\alpha(e_i)\eta_\beta(e_j)\eta_\gamma(e_i)\eta_\gamma(e_j)\\ &+ \frac{c+3s}{4}(g(e_i,e_i)g(e_j,e_j) + \eta_\gamma^2(e_j)\eta_\gamma^2(e_i)\\ &- g(e_i,e_i)\eta_\gamma^2(e_j) - g(e_{ij},e_j)\eta_\gamma^2(e_i) - (\eta_\gamma(e_j)\eta_\gamma(e_i))^2) + \frac{c-s}{4}(3g^2(Pe_i,e_j)),\end{aligned}$$

whence

$$\begin{aligned}2\bar{\tau} &= (2ms-2s) + \frac{c+3s}{4}(m(m-1) - 2ms + 2s) + \frac{3(c-s)}{4} \parallel P \parallel^2\\ &= \frac{1}{4}(c+3s)m(m-1) + \frac{c+3s-4}{4}(2s-2ms) + \frac{3(c-s)}{4} \parallel P \parallel^2 .\end{aligned} \tag{4.2}$$

Since N is bi-slant submanifold on S-space form $\bar{M}$, where $\dim N = m = n_1 + n_2 + s$, we may consider an adapted bi-slant orthonormal frames as follows:

$$e_1, e_2 = \frac{1}{\cos\theta_1}Pe_1, \ldots, e_{n_1-1}, e_{n_1} = \frac{1}{\cos\theta_1}Pe_{n_1-1}$$

$$e_{n_1+1}, e_{n_1+2} = \frac{1}{\cos\theta_2}Pe_{n_1+1}, \ldots, e_{n_1+n_2-1}, e_{n_1+n_2} = \frac{1}{\cos\theta_2}Pe_{n_1+n_2-1},$$

and $e_{n_1+n_2+\alpha} = \xi_\alpha$. Then we have

$$g(e_1, fe_2) = -g(fe_1, e_2) = -g(fe_1, \frac{1}{\cos\theta_1}Pe_1),$$

or,

$$g(e_1, fe_2) = -\frac{1}{\cos\theta_1}g(Pe_1, Pe_1).$$

Now, from [6], we get $g(e_1, fe_2) = -\cos\theta_1$. Similarly,

$$g^2(e_i, fe_{i+1}) = \begin{cases} \cos^2\theta_1 & 1 \leqslant i < n_1\\ \cos^2\theta_2 & n_1+1 \leqslant i < n_1+n_2+2. \end{cases}$$

Hence,

$$\parallel P \parallel^2 = \sum_{i,j=1}^{m} g^2(e_i, fe_j) = (n_1\cos^2\theta_1 + n_2\cos^2\theta_2). \tag{4.3}$$

Then (4.3) in (4.2) give us

$$2\bar{\tau} = \frac{1}{4}(c+3s)m(m-1) + \frac{c+3s-4}{4}(2s-2ms) + \frac{3(c-s)}{4}(n_1\cos^2\theta_1 + n_2\cos^2\theta_2). \tag{4.4}$$

Using (2.7), (3.4), (3.5), and (3.6), we get (4.1). □

The following results is an immediate consequence of Theorem 4.1

Corollary 4.2. *Let N be a semi-slant submanifold of Sasakian space form $(s = 1)$ $\bar{M}$. Then*

$$\rho_N \leq \| H \|^2 - (\rho - \frac{c+3}{4}) - \frac{(c-1)}{2m} + \frac{3(c-1)}{4m(m-1)}(d_1 + d_2 \cos^2\theta_2). \tag{4.5}$$

Corollary 4.3. *Let N be a hemi-slant submanifold of Sasakian space form $(s = 1)$ $\bar{M}$. Then*

$$\rho_N \leq \| H \|^2 - (\rho - \frac{c+3}{4}) - \frac{(c-1)}{2m} + \frac{3(c-1)}{4m(m-1)}(d_1 \cos^2\theta_1). \tag{4.6}$$

Corollary 4.4. *Let N be an anti-invariant submanifold of Sasakian space form $(s = 1)$ $\bar{M}$. Then*

$$\rho_N \leq \| H \|^2 - (\rho - \frac{c+3}{4}) - \frac{(c-1)}{2m}. \tag{4.7}$$

Corollary 4.5. *Let N be an invariant submanifold of Sasakian space form $(s = 1)$ $\bar{M}$. Then*

$$\rho_N \leq \| H \|^2 - (\rho - \frac{c+3}{4}) - \frac{(c-1)}{2m} + \frac{3(c-1)}{4m}. \tag{4.8}$$

We may have the similar results for Kahler space form $(s = 0)$.

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