Ricci solitons on 3-dimensional $C_{12}$-manifolds

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Abstract. In the present paper we study 3-dimensional $C_{12}$-manifolds admitting Ricci solitons and generalized Ricci solitons and then we introduce a new generalization of $\eta$-Ricci soliton. We give a class of examples.

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1 Introduction

In the classification of D. Chinea and C. Gonzalez [4] of almost contact metric manifolds there is a class $C_{12}$-manifolds which can be integrable but never normal. Recently, in [7], The authors have developed a systematic study of the curvature of the Chinea-Gonzalez class $C_5 \oplus C_{12}$ and obtain some classification theorems for those manifolds that satisfy suitable curvature conditions. This class is defined by using a certain function $\alpha$ and when this function vanishes the class $C_5 \oplus C_{12}$ reduces to class $C_{12}$.

Recently, in [2], the authors have study some properties of three dimensional $C_{12}$-manifolds and construct some relations between class $C_{12}$ and other classes as $C_6$ and $C_2 \oplus C_9$ or $|C|$.

Here, we investigate these manifolds to construct Ricci soliton and generalized Ricci soliton. It is shown that if in a 3-dimensional $C_{12}$-manifolds the metric is Ricci soliton, where potential vector field $V$ is collinear with the characteristic vector field $\xi$, then the manifold is $\eta$-Einstein. We also prove that an $\eta$-Einstein 3-dimensional $C_{12}$-manifold with

$$S = \mu g + \sigma \eta \otimes \eta \quad \mu + \sigma = - \text{div} \psi \quad V = \beta \xi \quad \text{and} \quad \text{grad} \beta = \beta \psi - \sigma \xi$$

admits a Ricci soliton. On the other hand, it is shown that any 3-dimensional $C_{12}$-manifold with $|\psi|^2 - 2 \text{div} \psi - \frac{\xi}{2} = 0$ satisfies the generalized Ricci soliton equation.

This paper is organized in the following way:

Section 2, is devoted to some basic definitions for 3-dimensional $C_{12}$-manifold. In Section 3, we obtain some results for a 3-dimensional $C_{12}$-manifold admitting Ricci soliton. In the last section, we present a study on 3-dimensional $C_{12}$-manifold which
satisfies the generalized Ricci soliton equation and we give concrete examples. Finally, we introduce a generalization of \( \eta \)-Ricci soliton and we prove the existence through several examples.

2 Preliminaries

The notion of Ricci soliton was introduced by Hamilton [10] in 1982. A Ricci soliton is a natural generalization of an Einstein metric. A pseudo-Riemannian manifold \((M, g)\) is called a Ricci soliton if it admits a smooth vector field \(V\) (potential vector field) on \(M\) such that

\[
(2.1) \quad (L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,
\]

where \(L_X g\) is the Lie-derivative of \(g\) along \(X\) given by:

\[
(2.2) \quad (L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y),
\]

\(\lambda\) is a constant and \(X, Y\) are arbitrary vector fields on \(M\).

A Ricci soliton is said to be shrinking, steady or expanding according to \(\lambda\) being negative, zero or positive, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with \(V\) zero or Killing.

The generalized Ricci soliton equation in Riemannian manifold \((M, g)\) is defined by (see [12]):

\[
(2.3) \quad L_X g = -2c_1X^\flat \odot X^\flat + 2c_2 S + 2\lambda g,
\]

where \(X^\flat(Y) = g(X, Y)\) and \(c_1, c_2, \lambda \in \mathbb{R}\).

Equation (2.3), is a generalization of Killing’s equation \((c_1 = c_2 = \lambda = 0)\), Equation for homotheties \((c_1 = c_2 = 0)\), Ricci soliton \((c_1 = 0, c_2 = -1)\), Cases of Einstein-Weyl \((c_1 = 1, c_2 = -1/n, \lambda = 0)\), Vacuum near-horizon geometry equation \((c_1 = 1, c_2 = 1/2)\), and is also a generalization of Einstein manifolds (For more details, see [1], [5], [8], [9], [12]).

An odd-dimensional Riemannian manifold \((M^{2n+1}, g)\) is said to be an almost contact metric manifold if there exist on \(M\) a \((1,1)\)-tensor field \(\varphi\), a vector field \(\xi\) (called the structure vector field) and a 1-form \(\eta\) such that

\[
(2.4) \quad \eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for any vector fields \(X, Y\) on \(M\). In particular, in an almost contact metric manifold we also have \(\varphi^2 \xi = 0\) and \(\eta \circ \varphi = 0\).

The fundamental 2-form \(\phi\) is defined by \(\phi(X, Y) = g(X, \varphi Y)\). It is known that the almost contact structure \((\varphi, \xi, \eta)\) is said to be normal if and only if

\[
(2.5) \quad N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0,
\]

for any \(X, Y\) on \(M\), where \(N_\varphi\) denotes the Nijenhuis torsion of \(\varphi\), given by

\[
(2.6) \quad N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].
\]
Given an almost contact structure, one can associate in a natural manner an almost CR-structure \((\mathcal{D}, \varphi|_{\mathcal{D}})\), where \(\mathcal{D} := \text{Ker}(\eta) = \text{Im}(\varphi)\) is the distribution of rank 2n transversal to the characteristic vector field \(\xi\). If this almost CR-structure is integrable (i.e., \(N_\varphi = 0\)) the manifold \(M^{2n+1}\) is said to be CR-integrable. It is known that normal almost contact manifolds are CR-manifolds.

In the classification of D. Chinea and C. Gonzalez [4], the almost contact metric structures have been completely classified. The \(C_5 \oplus C_{12}\) class was recently discussed by S. de Candia and M. Falcitelli [7]. We just recall the defining relations of \(C_5 \oplus C_{12}\) class, which will be used in this study.

The \(C_5 \oplus C_{12}\)-manifolds can be characterized by:

\[
(\nabla_X \varphi)Y = \alpha (g(X, Y)\xi - \eta(Y)\varphi X) - \eta(X)((\nabla_\xi \eta)(\varphi Y)\xi + \eta(Y)\varphi \nabla_\xi \xi).
\]

(2.7)

It is known that any almost contact metric manifold \((\varphi, \xi, \eta, g)\) from \(C_5 \oplus C_{12}\) class satisfies (see [7])

\[
\begin{align*}
\nabla_X \xi &= -\alpha \varphi^2 X + \eta(X)\nabla_\xi \xi, \\
d\eta &= \eta \wedge \nabla_\xi \eta, \\
d(\nabla_\xi \eta) &= -\alpha \nabla_\xi \eta + \nabla_\xi (\nabla_\xi \eta) \wedge \eta,
\end{align*}
\]

(2.8)

where \(\text{dim} \, M = 2n+1\) and \(\alpha = -\frac{1}{2n} \delta \eta\). Furthermore, if \(\text{dim} \, M \geq 5\), the Lee form of \(M\) is \(\omega = -\alpha \eta\) and it is closed. Applying (2.8), one has

\[
d\alpha = \xi(\alpha) \eta + \alpha \nabla_\xi \eta.
\]

In this paper, we will focus on the class \(C_{12}\). So, putting \(\alpha = 0\), \(\omega = -\nabla_\xi \eta\) and if \(\psi\) is the vector field given by \(\omega(X) = g(X, \psi)\) for all \(X\) vector field on \(M\), from formula (2.7) \(M\) is of class \(C_{12}\) if and only if

\[
(\nabla_X \varphi)Y = \eta(X)(\omega(\varphi Y)\xi + \eta(Y)\varphi \psi).
\]

Moreover, from (2.8) it follow,

\[
\begin{align*}
\nabla_X \xi &= -\eta(X)\psi, \\
d\eta &= \omega \wedge \eta, \\
d\omega &= 0.
\end{align*}
\]

Notice that \(\nabla_\xi \xi = -\psi\).

In [2], we have given a characterization of class \(C_{12}\) as follows:

**Theorem 2.1.** An almost contact metric manifold is of class \(C_{12}\) if and only if there exists a 1-form \(\omega\) such that

\[
d\eta = \omega \wedge \eta \quad \text{and} \quad N_\varphi = 0.
\]

Now, we denote by \(R, S, r\) the curvature tensor, the Ricci curvature and the scalar curvature respectively, which are defined for all \(X, Y, Z \in \mathfrak{X}(M)\) by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,
\]

(2.13)
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(2.14) \[ S(X, Y) = \sum_{i=1}^{2n+1} g(R(e_i, X)Y, e_i), \]

(2.15) \[ r = \sum_{i=1}^{2n+1} S(e_i, e_i), \]

with \( \{e_1, \ldots, e_{2n+1}\} \) is a local orthonormal basis. The divergence of a vector field \( X \) on \( M \) is defined by:

(2.16) \[ \text{div} \psi = \sum_{i=1}^{2n+1} g(\nabla_{e_i} \psi, e_i). \]

(For more details of previous definitions, see for example [11]).

Then, from Corollary 3.1 of [7] we have,

(2.17) \[ R(X, Y)\xi = -2d\eta(X,Y)\psi - \eta(Y)\nabla_X \psi + \eta(X)\nabla_Y \psi, \]

(2.18) \[ S(X, \xi) = -\eta(X)\text{div} \psi. \]

**Proposition 2.2.** In a 3-dimensional $C_{12}$-manifold, Ricci tensor and curvature tensor are given respectively by

\[
S(X, Y) = \left(\frac{r}{2} + \text{div} \psi\right) g(X, Y) + (|\psi|^2 - 2\text{div} \psi - \frac{r}{2}) \eta(X)\eta(Y) - \omega(X)\omega(Y) - g(\nabla_X \psi, Y),
\]

and

\[
R(X, Y)Z = \left(|\psi|^2 - 2\text{div} \psi - \frac{r}{2}\right) \eta(Z)\eta(Y)X - \eta(X)Y - g(Y, Z)\left(\omega(X)\psi + \nabla_X \psi - \text{div} \psi + \frac{r}{2}X\right) + g(X, Z)\left(\omega(Y)\psi + \nabla_Y \psi - \text{div} \psi + \frac{r}{2}Y\right) - \omega(Z)\omega(X)X + g(\nabla_X \psi, Z)Y - g(\nabla_Y \psi, Z)X.
\]

**Proof.** Suppose that \( (M, \varphi, \xi, \psi, \eta, \omega, g) \) is a 3-dimensional $C_{12}$-manifold. Setting \( Y = Z = \xi \) in the well known formula (which holds for any 3-dimensional Riemannian manifold [3]):

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2} g(Y, Z)X - g(X, Z)Y,
\]

where \( Q \) is the Ricci operator defined by

(2.22) \[ S(X, Y) = g(QX, Y). \]
We get
\[ R(X, \xi)\xi = QX - (\text{div}\psi)X + 2(\text{div}\psi)\eta(X)\xi + \frac{r}{2}\varphi^2X. \]

Again, setting \( Y = \xi \) in formula (2.17), we obtain
\[ R(X, \xi)\xi = -g(\nabla_\xi X, \psi) - \nabla_X \psi + \eta(X)\nabla_\xi \psi. \]

On the other hand, we have
\[ 2d\omega(\xi, X) = 0 \iff g(\nabla_\xi \psi, X) = g(\nabla_X \psi, \xi) = -g(\psi, \nabla_X \xi) = \omega(\psi)\eta(X), \]
which gives
\[ \nabla_\xi \psi = \omega(\psi)\xi. \]

So, using (2.11) and (2.25) in formula (2.24) we get
\[ R(X, \xi)\xi = -\omega(X)\psi - \nabla_X \psi + |\psi|^2\eta(X)\xi. \]

In view of (2.23) and (2.26), we obtain
\[ QX = -\omega(X)\psi - \nabla_X \psi + (\text{div}\psi + \frac{r}{2})X + (|\psi|^2 - 2\text{div}\psi - \frac{r}{2})\eta(X)\xi. \]

Finally, equation (2.19) follows from (2.27) and (2.22). Using (2.22) and (2.27) in (2.21), the curvature tensor in a 3-dimensional \( C_{12} \)-manifold is given by (2.20).

**Example 2.1.** We denote the Cartesian coordinates in a 3-dimensional Euclidean space \( \mathbb{R}^3 \) by \( (x, y, z) \) and define a symmetric tensor field \( g \) by
\[ g = e^{2f} \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix}, \]
where \( f = f(y), \tau = \tau(x) \) and \( \rho = \rho(x, y) \) are functions on \( \mathbb{R}^3 \) with \( f' = \frac{\partial f}{\partial y} \). Further, we define an almost contact metric \((\varphi, \xi, \eta)\) on \( \mathbb{R}^3 \) by
\[ \varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \tau & 0 \end{pmatrix}, \quad \xi = e^{-f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = e^f(-\tau, 0, 1). \]

The fundamental 1-form \( \eta \) and the 2-form \( \phi \) have the forms,
\[ \eta = e^f(dz - \tau dx) \quad \text{and} \quad \phi = -2\rho^2 e^{2f} dx \wedge dy, \]
and hence
\[ d\eta = f' e^f \left( \tau dx \wedge dy + dy \wedge dz \right), \]
By a direct computation the non trivial components of $N^{(1)}_{kj}$ are given by

$$N^{(1)}_{12} = \tau f', \quad N^{(1)}_{23} = f'.$$

But, $\forall i, j, k \in \{1, 2, 3\}$

$$(N_{\phi'})^j_{kj} = 0,$$

implying that the structure $(\phi, \xi, \eta, g)$ is CR-integrable.

Therefore, to continue studying this example, it suffices to take $f' \neq 0$ to ensure that the structure is CR-integrable not normal.

In order to define the closed 1-form $\omega$, putting $\omega = adx + bdy + cdz$ where $a, b$ and $c$ are functions on $\mathbb{R}^3$, and using formulas $d\eta = \omega \wedge \eta$ and $\omega(\xi) = 0$, we can check that is very simply as follows:

$$\omega = f' \ dy,$$

notice that $d\omega = 0$.

Knowing that $\omega$ is the $g$-dual of $\psi$ i.e. $\omega(X) = g(X, \psi)$, we have immediately that

$$\psi = \frac{f'}{\rho^2} e^{-2f} \frac{\partial}{\partial y}.$$

Thus, $(\phi, \xi, \psi, \eta, \omega, g)$ becomes a $C_{12}$ structure on $\mathbb{R}^3$.

Now we have

$$\left\{ e_1 = \frac{e^{-f}}{\rho} \left( \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial z} \right), \quad e_2 = \frac{e^{-f}}{\rho} \frac{\partial}{\partial y}, \quad e_3 = \xi = e^{-f} \frac{\partial}{\partial z} \right\}$$

form an orthonormal basis. To verify result in formula (2.10), the non zero components of the Levi-Civita connection corresponding to $g$ are given by:

$$\nabla_{e_1} e_1 = -\frac{(f' \rho + \rho_2)}{\rho^2 e_f} e_2, \quad \nabla_{e_1} e_2 = \frac{(f' \rho + \rho_2)}{\rho^2 e_f} e_1,$$

$$\nabla_{e_2} e_1 = \frac{\rho_1}{\rho^2 e_f} e_2, \quad \nabla_{e_2} e_2 = -\frac{\rho_1}{\rho^2 e_f} e_1,$$

$$\nabla_{e_3} e_2 = \frac{f'}{\rho e_f} e_3, \quad \nabla_{e_2} e_3 = -\frac{f'}{\rho e_f} e_2.$$

Then, one can easily check that for all $i, j \in \{1, 2, 3\}$

$$(\nabla_{e_i} \phi) e_j = \nabla_{e_i} \phi e_j - \phi \nabla_{e_i} e_j = \eta(e_i)(\omega(\phi e_j)\xi + \eta(e_j)\phi).$$
3 Ricci soliton

In this section, we consider a 3-dimensional $C_{12}$-manifold $M$ admitting a Ricci soliton defined by (2.1). Let $V$ be a pointwise collinear vector field with the structure vector field $\xi$, that is $V = \beta \xi$, where $\beta$ is a function on $M$. From (2.1) we write

$$g(\nabla_X \beta \xi, Y) + g(\nabla_Y \beta \xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

for all $X$ and $Y$ vector fields on $M$. Then, we have

$$X(\beta)\eta(Y) + \beta g(\nabla_X \xi, Y) + Y(\beta)\eta(X) + \beta g(\nabla_Y \xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

which implies

$$X(\beta)\eta(Y) - \beta \eta(X)\omega(Y) + Y(\beta)\eta(X) - \beta \eta(Y)\omega(X) + 2\lambda g(X, Y) = 0,$$

by virtue of (2.1). By putting $Y = \xi$ in (3.2) and using (2.18) we obtain

$$X(\beta) - \beta \omega(X) + (\xi(\beta) - 2\text{div}\psi + 2\lambda)\eta(X) = 0.$$

Taking $X = \xi$ in the previous equation gives

$$\xi(\beta) = \text{div}\psi - \lambda.$$

If we replace (3.4) in (3.3), we get

$$X(\beta) = \beta \omega(X) + (\text{div}\psi - \lambda)\eta(X),$$

again, if we replace (3.5) in (3.2), we obtain

$$S(X, Y) = -\lambda g(X, Y) + (\lambda - \text{div}\psi)\eta(X)\eta(Y),$$

for all $X$ and $Y$ vector fields on $M$. Hence we have

**Theorem 3.1.** Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a 3-dimensional $C_{12}$-manifold. If $M$ admits a Ricci soliton and $V$ is pointwise collinear with the structure vector field $\xi$, then $M$ is an $\eta$-Einstein manifold.

In addition, if $\lambda = \text{div}\psi = \text{constant}$ then $M$ is an Einstein manifold.

Let assume the converse, that is, let $M$ be a 3-dimensional $\eta$-Einstein $C_{12}$-manifold with $V = \beta \xi$. Then we can write

$$S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y),$$

where $a$ and $b$ are scalars and $X, Y$ are vector fields on $M$. From (2.2) we have

$$(\mathcal{L}_V g)(Y, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) = X(\beta)\eta(Y) + Y(\beta)\eta(X) - \beta \eta(X)\omega(Y) - \beta \eta(Y)\omega(X),$$
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which implies that

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 2(a + \lambda)g(X,Y)$$

$$+ \eta(X)(b\eta(Y) - \beta\omega(Y) + Y(\beta))$$

$$+ \eta(Y)(b\eta(X) - \beta\omega(X) + X(\beta)).$$

From the previous equation it is obvious that $M$ admits a Ricci soliton $(g, V, \lambda)$ if

$$a + \lambda = 0 \quad \text{and} \quad b\eta(Y) - \beta\omega(Y) + Y(\beta) = 0.$$ 

Equating the right hand sides of (3.7) and (2.18) and taking $X = Y = \xi$ gives

$$a + b = -\text{div}\psi,$$

Thus, we get

**Theorem 3.2.** Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a 3-dimensional $C_{12}$-manifold with div$\psi$ is constant. If $M$ is an $\eta$-Einstein manifold with $S = ag + b\eta \otimes \eta$ and $a + b = -\text{div}\psi$, then the manifold admits a Ricci soliton $(g, \beta\xi, a)$ with grad$\beta = \beta\psi - b\xi$.

### 4 Generalized Ricci soliton

In this section we will study the generalized Ricci soliton equation (2.3) on a $C_{12}$-manifold of dimension three. Let’s start with our main result

**Theorem 4.1.** Any three-dimensional $C_{12}$-manifold satisfies the generalized Ricci soliton equation (2.3) with $X = \psi$, $c_1 = 1$, $c_2 = -1$ and $\lambda = |\psi|^2 - \text{div}\psi$ if and only if

$$(4.1) \quad |\psi|^2 - 2\text{div}\psi - \frac{r}{2} = 0.$$ 

**Proof.** Suppose that $(M, \varphi, \xi, \psi, \eta, \omega, g)$ is a $C_{12}$-manifold of dimension three which satisfies the generalized Ricci soliton equation (2.3) with $X = \psi$, that is, for all $Y, Z \in \Gamma(TM)$

$$\mathcal{L}_\psi g(Y, Z) = -2c_1\omega(Y)\omega(Z) + 2c_2S(Y, Z) + 2\lambda g(Y, Z).$$

Since $\omega$ is closed then $g(\nabla Y \psi, Z) = g(\nabla Z \psi, Y)$. Therefore, we can express the generalized soliton equation as

$$(4.3) \quad \nabla Y \psi = -c_1\omega(Y)\psi + c_2QY + \lambda Y.$$ 

Now, from (2.27) we get

$$\nabla Y \psi = -\omega(Y)\psi - QY + (\text{div}\psi + \frac{r}{2})Y + (|\psi|^2 - 2\text{div}\psi - \frac{r}{2})\eta(Y)\xi.$$ 

In view of (4.4) and (4.3) the proof is complete. 

**Proposition 4.2.** Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a $C_{12}$-manifold of dimension three which satisfies the generalized Ricci soliton equation (2.3) with $X = \psi$. If $|\psi| = 1$ then $r = \text{constant}$. 
\textbf{Proof.} The proof is direct, it suffices to use Theorem 4.1. \hfill \Box

\textbf{Example 4.1.} Let’s go back to the class of the previous examples. With simple but long calculations, we can get the following:

\[
|\psi|^2 = \frac{f'^2}{\rho^2} e^{-2f}, \quad \text{div}\psi = \frac{e^{-2f}}{\rho^2} \left( f'^2 + f'' \right), \quad \lambda = \frac{f'' e^{-2f}}{\rho^2},
\]

\[r = \frac{2e^{-2f}}{\rho^4} \left( \rho^2 - \rho \rho_{11} + \rho^2 - \rho \rho_{22} - 2f'' \rho^2 - f'^2 \rho^2 \right),\]

where $\rho_i = \frac{\partial \rho}{\partial x_i}$. Then, the condition (4.1) gives the following differential equation

\[\rho^2 - \rho \rho_{11} + \rho^2 - \rho \rho_{22} = 0.\]

Henceforth, we can construct a non-trivial generalized Ricci soliton.

For example:

(1): $\rho = 1$, $\lambda = f'' e^{-2f}$,

\[\nabla_{e_1} \psi = e^{-2f} f'^2 e_1, \quad Q e_1 = -e^{-2f} \left( f'' + f'^2 \right) e_1,\]

\[\nabla_{e_2} \psi = e^{-2f} \left( -f'' + f' \right) e_2, \quad Q e_2 = -2e^{-2f} f'' e_2,\]

\[\nabla_{e_3} \psi = e^{-2f} f'^2 e_3, \quad Q e_3 = -e^{-2f} \left( f' + f'' \right) e_3.\]

(2): $\rho = e^y$, $\lambda = f'' e^{-2(y + f)}$,

\[\nabla_{e_1} \psi = e^{-2(y + f)} \left( f' (f' + 1) \right) e_1, \quad Q e_1 = -e^{-2(y + f)} \left( f'' + f'^2 + f' \right) e_1,\]

\[\nabla_{e_2} \psi = e^{-2(y + f)} \left( -f'' + f' f'' + f' \right) e_2, \quad Q e_2 = -2e^{-2(y + f)} \left( f' + f'^2 \right) e_2,\]

\[\nabla_{e_3} \psi = e^{-2(y + f)} f'^2 e_3, \quad Q e_3 = -e^{-2(y + f)} \left( f'' + f'' \right) e_3.\]

(3): $\rho = e^{-f}$, $\lambda = 0$,

\[\nabla_{e_1} \psi = 0, \quad Q e_1 = 0,\]

\[\nabla_{e_2} \psi = f'' e_2, \quad Q e_2 = -f' e_2,\]

\[\nabla_{e_3} \psi = f'^2 e_3, \quad Q e_3 = -f'^2 e_3.\]

Of course, we must choose $f$ so that $\lambda$ is constant. We can construct further examples of generalized Ricci soliton on a 3-dimensional $C_{12}$-manifold by the similar way.

At the end of this section, we present the concept of the generalized $\eta$-Ricci soliton as a generalization of the $\eta$-Ricci soliton given by Cho-Kimura in [6] by the following equation:

\[\mathcal{L}_V g + 2S + 2\lambda g + \mu \eta \otimes \eta = 0,\]

where the tensor product notation $(\eta \otimes \eta)(X, Y) = \eta(X)\eta(Y)$ is used and $\lambda, \mu$ are real constants.

The generalized $\eta$-Ricci soliton equation in Riemannian manifold $(M, g)$ is defined by:

\[\mathcal{L}_X g = -2c_1 X^a \otimes X^b + 2c_2 S + 2\lambda g + \mu \eta \otimes \eta,\]

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$. 
With the same reasoning above, we can express formula (4.7) as follows

\[ \nabla_X \psi = -c_1 \omega(X) \psi + c_2 QX + \lambda X + \mu \eta \otimes \xi. \]

Now, based on equation (4.4), we declare the following result

**Theorem 4.3.** Any 3-dimensional \(C_{12}\)-manifold satisfies the generalized \(\eta\)-Ricci soliton equation with

\[ c_1 = 1 \quad c_2 = -1 \quad \lambda = |\psi|^2 - \text{div} \psi \quad \text{and} \quad \mu = |\psi|^2 - 2 \text{div} \psi - \frac{r}{2}. \]

**Example 4.2.** From Example 4.1, we can construct several non-trivial cases, namely:

1) If \(f = y\) and \(\rho = \frac{4}{e^{2y}-1}\) with \(c \in \mathbb{R}\), then we get

\[ c_1 = 1, \quad c_2 = -1, \quad \lambda = 0, \quad \text{and} \quad \mu = c. \]

2) If \(f = \ln \left(\frac{1}{\sin^2 y}\right)\) and \(\rho = c \sin y\), then we get

\[ c_1 = 1, \quad c_2 = -1, \quad \lambda = -\frac{2}{c^2}, \quad \mu = -\frac{1}{c^2}. \]

Of course, while taking into account the necessary conditions on \(f\) and \(\rho\).

**References**


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