Ricci solitons on 3-dimensional C_{12}-manifolds

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Abstract. In the present paper we study 3-dimensional C_{12}-manifolds admitting Ricci solitons and generalized Ricci solitons and then we introduce a new generalization of η-Ricci soliton. We give a class of examples.

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1 Introduction

In the classification of D. Chinea and C. Gonzalez [4] of almost contact metric manifolds there is a class C_{12}-manifolds which can be integrable but never normal. Recently, in [7], The authors have developed a systematic study of the curvature of the Chinea-Gonzalez class $C_5 \oplus C_{12}$ and obtain some classification theorems for those manifolds that satisfy suitable curvature conditions. This class is defined by using a certain function α and when this function vanishes the class $C_5 \oplus C_{12}$ reduces to class C_{12}.

Recently, in [2], the authors have study some properties of three dimensional C_{12}-manifolds and construct some relations between class C_{12} and other classes as C_6 and $C_2 \oplus C_9$ or $|C|$.

Here, we investigate these manifolds to construct Ricci soliton and generalized Ricci soliton. It is shown that if in a 3-dimensional C_{12}-manifolds the metric is Ricci soliton, where potential vector field V is collinear with the characteristic vector field ξ, then the manifold is η-Einstein. We also prove that an η-Einstein 3-dimensional C_{12}-manifold with

$$S = \mu g + \sigma \eta \otimes \eta \quad \mu + \sigma = -\mathrm{div}\psi \quad V = \beta\xi \quad \textit{and} \quad \mathrm{grad}\beta = \beta\psi - \sigma\xi$$

admits a Ricci soliton. On the other hand, it is shown that any 3-dimensional C_{12}-manifold with $|\psi|^2 - 2\mathrm{div}\psi - \frac{r}{2} = 0$ satisfies the generalized Ricci soliton equation.

This paper is organized in the following way:
Section 2, is devoted to some basic definitions for 3-dimensional C_{12}-manifold. In Section 3, we obtain some results for a 3-dimensional C_{12}-manifold admitting Ricci soliton. In the last section, we present a study on 3-dimensional C_{12}-manifold which

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satisfies the generalized Ricci soliton equation and we give concrete examples. Finally, we introduce a generalization of η-Ricci soliton and we prove the existence through several examples.

2 Preliminaries

The notion of Ricci soliton was introduced by Hamilton [10] in 1982. A Ricci soliton is a natural generalization of an Einstein metric. A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if it admits a smooth vector field V (potential vector field) on M such that

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{2.1}$$

where $\mathcal{L}_X g$ is the Lie-derivative of g along X given by:

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y), \tag{2.2}$$

λ is a constant and X, Y are arbitrary vector fields on M.
A Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with V zero or Killing.

The generalized Ricci soliton equation in Riemannian manifold (M, g) is defined by (see [12]):

$$\mathcal{L}_X g = -2c_1 X^\flat \odot X^\flat + 2c_2\,\mathrm{S} + 2\lambda g, \tag{2.3}$$

where $X^\flat(Y) = g(X, Y)$ and $c_1, c_2, \lambda \in \mathbb{R}$.
Equation (2.3), is a generalization of Killing's equation ($c_1 = c_2 = \lambda = 0$), Equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton ($c_1 = 0$, $c_2 = -1$), Cases of Einstein-Weyl ($c_1 = 1$, $c_2 = \frac{-1}{n-2}$), Metric projective structures with skew-symmetric Ricci tensor in projective class ($c_1 = 1$, $c_2 = \frac{-1}{n-1}$, $\lambda = 0$), Vacuum near-horzion geometry equation ($c_1 = 1$, $c_2 = \frac{1}{2}$), and is also a generalization of Einstein manifolds (For more details, see [1], [5], [8], [9], [12]).

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi \quad \textit{and} \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

The fundamental 2-form ϕ is defined by $\phi(X, Y) = g(X, \varphi Y)$. It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0, \tag{2.5}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.6}$$

Given an almost contact structure, one can associate in a natural manner an almost CR-structure $(\mathcal{D}, \varphi|_{\mathcal{D}})$, where $\mathcal{D} := Ker(\eta) = Im(\varphi)$ is the distribution of rank $2n$ transversal to the characteristic vector field ξ. If this almost CR-structure is integrable (i.e., $N_\varphi = 0$) the manifold M^{2n+1} is said to be CR-integrable. It is known that normal almost contact manifolds are CR-manifolds.

In the classification of D. Chinea and C. Gonzalez [4], the almost contact metric structures have been completely classified. The $C_5 \oplus C_{12}$ class was recently discussed by S. de Candia and M. Falcitelli [7]. We just recall the defining relations of $C_5 \oplus C_{12}$ class, which will be used in this study.

The $C_5 \oplus C_{12}$-manifolds can be characterized by:

$$\begin{aligned} (\nabla_X \varphi)Y &= \alpha\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big) \\ &\quad -\eta(X)\big((\nabla_\xi \eta)(\varphi Y)\xi + \eta(Y)\varphi \nabla_\xi \xi\big). \end{aligned} \tag{2.7}$$

It is known that any almost contact metric manifold (φ, ξ, η, g) from $C_5 \oplus C_{12}$ class satisfies (see [7])

$$\begin{cases} \nabla_X \xi = -\alpha\varphi^2 X + \eta(X)\nabla_\xi \xi, \\ \mathrm{d}\eta = \eta \wedge \nabla_\xi \eta, \\ \mathrm{d}(\nabla_\xi \eta) = -\big(\alpha \nabla_\xi \eta + \nabla_\xi(\nabla_\xi)\eta\big) \wedge \eta, \end{cases} \tag{2.8}$$

where $\dim M = 2n+1$ and $\alpha = -\frac{1}{2n}\delta\eta$. Furthermore, if $\dim M \geq 5$, the Lee form of M is $\omega = -\alpha\eta$ and it is closed. Applying (2.8), one has

$$\mathrm{d}\alpha = \xi(\alpha)\eta + \alpha\nabla_\xi \eta. \tag{2.9}$$

In this paper, we will focus on the class C_{12}. So, putting $\alpha = 0$, $\omega = -\big(\nabla_\xi \xi\big)^\flat = -\nabla_\xi \eta$ and if ψ is the vector field given by $\omega(X) = g(X, \psi)$ for all X vector field on M, from formula (2.7) M is of class C_{12} if and only if

$$(\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big). \tag{2.10}$$

Moreover, from (2.8) it follow,

$$\begin{cases} \nabla_X \xi = -\eta(X)\psi, \\ \mathrm{d}\eta = \omega \wedge \eta, \\ \mathrm{d}\omega = 0. \end{cases} \tag{2.11}$$

Notice that $\nabla_\xi \xi = -\psi$.

In [2], we have given a characterization of class C_{12} as follows:

Theorem 2.1. *An almost contact metric manifold is of class C_{12} if and only if there exists a 1-form ω such that*

$$\mathrm{d}\eta = \omega \wedge \eta \qquad \mathrm{d}\phi = 0 \qquad \textit{and} \qquad N_\varphi = 0. \tag{2.12}$$

Now, we denote by R, S, r the curvature tensor, the Ricci curvature and the scalar curvature respectively, which are defined for all $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.13}$$

$$S(X,Y)=\sum_{i=1}^{2n+1} g\big(R(e_i,X)Y,e_i\big), \tag{2.14}$$

$$r=\sum_{i=1}^{2n+1} S(e_i,e_i), \tag{2.15}$$

with $\{e_1,...,e_{2n+1}\}$ is a local orthonormal basis . The divergence of a vector field X on M is defined by:

$$\mathrm{div}\psi=\sum_{i=1}^{2n+1} g(\nabla_{e_i}\psi,e_i). \tag{2.16}$$

(For more details of previous definitions, see for example [11]).
Then, from Corollary 3.1 of [7] we have,

$$R(X,Y)\xi=-2\mathrm{d}\eta(X,Y)\psi-\eta(Y)\nabla_X\psi+\eta(X)\nabla_Y\psi, \tag{2.17}$$

$$S(X,\xi)=-\eta(X)\mathrm{div}\psi. \tag{2.18}$$

Proposition 2.2. *In a 3-dimensional C_{12}-manifold, Ricci tensor and curvature tensor are given respectively by*

$$\begin{aligned} S(X,Y) &= \big(\frac{r}{2}+div\psi\big)g(X,Y)+\big(|\psi|^2-2div\psi-\frac{r}{2}\big)\eta(X)\eta(Y)\\ &- \omega(X)\omega(Y)-g(\nabla_X\psi,Y), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} R(X,Y)Z &= \big(|\psi|^2-2div\psi-\frac{r}{2}\big)\eta(Z)\big(\eta(Y)X-\eta(X)Y\big)\\ &- g(Y,Z)\Big(\omega(X)\psi+\nabla_X\psi-\big(2div\psi+\frac{r}{2}\big)X\Big)\\ &+ g(X,Z)\Big(\omega(Y)\psi+\nabla_Y\psi-\big(2div\psi+\frac{r}{2}\big)Y\Big)\\ &+ \big(|\psi|^2+2div\psi-\frac{r}{2}\big)\Big(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\Big)\xi\\ &- \omega(Z)\big(\omega(Y)X-\omega(X)Y\big)+g(\nabla_X\psi,Z)Y-g(\nabla_Y\psi,Z)X. \end{aligned} \tag{2.20}$$

Proof. Suppose that $(M,\varphi,\xi,\psi,\eta,\omega,g)$ is a 3-dimensional C_{12}-manifold.
Setting $Y=Z=\xi$ in the well known formula (which holds for any 3-dimensional Riemannian manifold [3]):

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\\ &- \frac{r}{2}\big(g(Y,Z)X-g(X,Z)Y\big), \end{aligned} \tag{2.21}$$

where Q is the Ricci operator defined by

$$S(X,Y)=g(QX,Y). \tag{2.22}$$

We get

$$R(X,\xi)\xi = QX - (\mathrm{div}\psi)X + 2(\mathrm{div}\psi)\eta(X)\xi + \frac{r}{2}\varphi^2 X. \tag{2.23}$$

Again, Setting $Y = \xi$ in formula (2.17), we obtain

$$R(X,\xi)\xi = -g(\nabla_\xi \xi, X)\psi - \nabla_X \psi + \eta(X)\nabla_\xi \psi. \tag{2.24}$$

On the other hand, we have

$$\begin{aligned} 2\mathrm{d}\omega(\xi, X) = 0 \Leftrightarrow g(\nabla_\xi \psi, X) &= g(\nabla_X \psi, \xi) \\ &= -g(\psi, \nabla_X \xi) \\ &= \omega(\psi)\eta(X), \end{aligned}$$

which gives

$$\nabla_\xi \psi = \omega(\psi)\xi. \tag{2.25}$$

So, using (2.11) and (2.25) in formula (2.24) we get

$$R(X,\xi)\xi = -\omega(X)\psi - \nabla_X \psi + |\psi|^2 \eta(X)\xi. \tag{2.26}$$

In view of (2.23) and (2.26), we obtain

$$QX = -\omega(X)\psi - \nabla_X \psi + (\mathrm{div}\psi + \frac{r}{2})X + (|\psi|^2 - 2\mathrm{div}\psi - \frac{r}{2})\eta(X)\xi. \tag{2.27}$$

Finally, equation (2.19) follows from (2.27) and (2.22). Using (2.22) and (2.27) in (2.21), the curvature tensor in a 3-dimensional C_{12}-manifold is given by (2.20). □

Example 2.1. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by

$$g = \mathrm{e}^{2f} \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where $f = f(y)$, $\tau = \tau(x)$ and $\rho = \rho(x, y)$ are functions on $\mathbb{R}^3$ with $f' = \frac{\partial f}{\partial y}$. Further, we define an almost contact metric (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad \xi = \mathrm{e}^{-f} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = \mathrm{e}^f(-\tau, 0, 1).$$

The fundamental 1-form η and the 2-form ϕ have the forms,

$$\eta = \mathrm{e}^f(dz - \tau dx) \qquad and \qquad \phi = -2\rho^2 \mathrm{e}^{2f} dx \wedge dy,$$

and hence

$$\mathrm{d}\eta = f'\mathrm{e}^f\Big(\tau dx \wedge dy + dy \wedge dz\Big),$$

$$\mathrm{d}\phi = 0.$$

By a direct computation the non trivial components of $N^{(1)\ i}_{kj}$ are given by

$$N^{(1)\ 3}_{12} = \tau f', \quad N^{(1)\ 3}_{23} = f'.$$

But, $\forall i, j, k \in \{1, 2, 3\}$

$$(N_\varphi)^i_{kj} = 0,$$

implying that the structure (φ, ξ, η, g) is CR-integrable.
Therefore, to continue studying this example, it suffices to take $f' \neq 0$ to ensure that the structure is CR-integrable not normal.

In order to define the closed 1-form ω, putting $\omega = adx + bdy + cdz$ where a, b and c are functions on $\mathbb{R}^3$, and using formulas $\mathrm{d}\eta = \omega \wedge \eta$ and $\omega(\xi) = 0$, we can check that is very simply as follows:

$$\omega = f'\ dy, \tag{2.28}$$

notice that $\mathrm{d}\omega = 0$.
Knowing that ω is the g-dual of ψ i.e. $\omega(X) = g(X, \psi)$, we have immediately that

$$\psi = \frac{f'}{\rho^2}\mathrm{e}^{-2f}\frac{\partial}{\partial y}. \tag{2.29}$$

Thus, $(\varphi, \xi, \psi, \eta, \omega, g)$ becomes a C_{12} structure on $\mathbb{R}^3$.
Now we have

$$\left\{e_1 = \frac{\mathrm{e}^{-f}}{\rho}\Big(\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial z}\Big), \quad e_2 = \frac{\mathrm{e}^{-f}}{\rho}\frac{\partial}{\partial y}, \quad e_3 = \xi = \mathrm{e}^{-f}\frac{\partial}{\partial z}\right\}$$

form an orthonormal basis. To verify result in formula (2.10), the non zero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1}e_1 = -\frac{(f'\rho + \rho_2)}{\rho^2\mathrm{e}^f}e_2, \quad \nabla_{e_1}e_2 = \frac{(f'\rho + \rho_2)}{\rho^2\mathrm{e}^f}e_1,$$

$$\nabla_{e_2}e_1 = \frac{\rho_1}{\rho^2\mathrm{e}^f}e_2, \quad \nabla_{e_2}e_2 = -\rho_1\frac{}{\rho^2\mathrm{e}^f}e_1,$$

$$\nabla_{e_3}e_2 = \frac{f'}{\rho\ \mathrm{e}^f}e_3, \quad \nabla_{e_2}e_2 = -\frac{f'}{\rho\ \mathrm{e}^f}e_2.$$

Then, one can easily check that for all $i, j \in \{1, 2, 3\}$

$$\begin{aligned}(\nabla_{e_i}\varphi)e_j &= \nabla_{e_i}\varphi e_j - \varphi\nabla_{e_i}e_j \\ &= \eta(e_i)\big(\omega(\varphi e_j)\xi + \eta(e_j)\varphi\psi\big).\end{aligned}$$

3 Ricci soliton

In this section, we consider a 3-dimensional C_{12}-manifold M admitting a Ricci soliton defined by (2.1). Let V be a pointwise collinear vector field with the structure vector field ξ, that is $V = \beta\xi$, where β is a function on M. From (2.1) we write

$$g(\nabla_X \beta\xi, Y) + g(\nabla_Y \beta\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{3.1}$$

for all X and Y vector fields on M. Then, we have

$$\begin{aligned} &X(\beta)\eta(Y) + \beta g(\nabla_X \xi, Y) + Y(\beta)\eta(X) \\ &+\beta g(\nabla_Y \xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \end{aligned}$$

which implies

$$\begin{aligned} &X(\beta)\eta(Y) - \beta\eta(X)\omega(Y) + Y(\beta)\eta(X) \\ &-\beta\eta(Y)\omega(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \end{aligned} \tag{3.2}$$

by virtue of (2.11). By putting $Y = \xi$ in (3.2) and using (2.18) we obtain

$$X(\beta) - \beta\omega(X) + \big(\xi(\beta) - 2\mathrm{div}\psi + 2\lambda\big)\eta(X) = 0. \tag{3.3}$$

Taking $X = \xi$ in the previous equation gives

$$\xi(\beta) = \mathrm{div}\psi - \lambda. \tag{3.4}$$

If we replace (3.4) in (3.3), we get

$$X(\beta) = \beta\omega(X) + \big(\mathrm{div}\psi - \lambda\big)\eta(X), \tag{3.5}$$

again, if we replace (3.5) in (3.2), we obtain

$$S(X,Y) = -\lambda g(X,Y) + \big(\lambda - \mathrm{div}\psi\big)\eta(X)\eta(Y), \tag{3.6}$$

for all X and Y vector fields on M. Hence we have

Theorem 3.1. *Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a 3-dimensional C_{12}-manifold. If M admits a Ricci soliton and V is pointwise collinear with the structure vector field ξ, then M is an η-Einstein manifold.*
In addition, if $\lambda = \mathrm{div}\psi =$ constant then M is an Einstein manifold.

Let assume the converse, that is, let M be a 3-dimensional η-Einstein C_{12}-manifold with $V = \beta\xi$. Then we can write

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.7}$$

where a and b are scalars and X, Y are vector fields on M. From (2.2) we have

$$\begin{aligned} (\mathcal{L}_V g)(Y,Y) &= g(\nabla_X V, Y) + g(\nabla_Y V, X) \\ &= X(\beta)\eta(Y) + Y(\beta)\eta(X) - \beta\eta(X)\omega(Y) - \beta\eta(Y)\omega(X), \end{aligned}$$

which implies that

$$\begin{aligned}(\mathcal{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y) &= 2(a+\lambda)g(X,Y)\\ &+ \eta(X)\big(b\eta(Y)-\beta\omega(Y)+Y(\beta)\big)\\ &+ \eta(Y)\big(b\eta(X)-\beta\omega(X)+X(\beta)\big).\end{aligned}$$

From the previous equation it is obvious that M admits a Ricci soliton (g,V,λ) if

$$a+\lambda=0 \qquad \textit{and} \qquad b\eta(Y)-\beta\omega(Y)+Y(\beta)=0.$$

Equating the right hand sides of (3.7) and (2.18) and taking $X=Y=\xi$ gives

$$a+b=-\mathrm{div}\psi,$$

Thus, we get

Theorem 3.2. *Let $(M,\varphi,\xi,\psi,\eta,\omega,g)$ be a 3-dimensional C_{12}-manifold with* $\mathrm{div}\psi$ *is constant. If M is an η-Einstein manifold with $S=ag+b\eta\otimes\eta$ and $a+b=-\mathrm{div}\psi$, then the manifold admits a Ricci soliton $(g,\beta\xi,a)$ with* $\mathrm{grad}\beta=\beta\psi-b\xi$.

4 Generalized Ricci soliton

In this section we will study the generalized Ricci soliton equation (2.3) on a C_{12}-manifold of dimension three. let's start with our main result

Theorem 4.1. *Any three-dimensional C_{12}-manifold satisfies the generalized Ricci soliton equation (2.3) with $X=\psi$, $c_1=1$, $c_2=-1$ and $\lambda=|\psi|^2-\mathrm{div}\psi$ if and only if*

$$|\psi|^2-2\mathrm{div}\psi-\frac{r}{2}=0. \tag{4.1}$$

Proof. Suppose that $(M,\varphi,\xi,\psi,\eta,\omega,g)$ is a C_{12}-manifold of dimension three which satisfies the generalized Ricci soliton equation (2.3) with $X=\psi$, that is, for all $Y,Z\in\Gamma(TM)$

$$\big(\mathcal{L}_\psi g\big)(Y,Z)=-2c_1\omega(Y)\omega(Z)+2c_2S(Y,Z)+2\lambda g(Y,Z). \tag{4.2}$$

Since ω is closed then $g(\nabla_Y\psi,Z)=g(\nabla_Z\psi,Y)$. Therefore, we can express the generalized soliton equation as

$$\nabla_Y\psi=-c_1\omega(Y)\psi+c_2QY+\lambda Y. \tag{4.3}$$

Now, from (2.27) we get

$$\nabla_Y\psi=-\omega(Y)\psi-QY+(\mathrm{div}\psi+\frac{r}{2})Y+(|\psi|^2-2\mathrm{div}\psi-\frac{r}{2})\eta(Y)\xi. \tag{4.4}$$

In view of (4.4) and (4.3) the proof is complete. □

Proposition 4.2. *Let $(M,\varphi,\xi,\psi,\eta,\omega,g)$ be a C_{12}-manifold of dimension three which satisfies the generalized Ricci soliton equation (2.3) with $X=\psi$. If $|\psi|=1$ then $r=constant$.*

Proof. The proof is direct, it suffices to use Theorem 4.1. □

Example 4.1. Let's go back to the class of the previous examples . With simple but long calculations, we can get the following:

$$|\psi|^2 = \frac{f'^2}{\rho^2}\mathrm{e}^{-2f}, \qquad \mathrm{div}\psi = \frac{\mathrm{e}^{-2f}}{\rho^2}(f'^2 + f''), \qquad \lambda = \frac{f''\mathrm{e}^{-2f}}{\rho^2},$$

$$r = \frac{2\mathrm{e}^{-2f}}{\rho^4}(\rho_1^2 - \rho\rho_{11} + \rho_2^2 - \rho\rho_{22} - 2f''\rho^2 - f'^2\rho^2),$$

where $\rho_i = \frac{\partial\rho}{\partial x_i}$. Then, the condition (4.1) gives the following differential equation

$$\rho_1^2 - \rho\rho_{11} + \rho_2^2 - \rho\rho_{22} = 0. \tag{4.5}$$

Henceforth, we can construct a non-trivial generalized Ricci soliton.
For example:
(1): $\rho = 1, \quad \lambda = f''\mathrm{e}^{-2f}$,
$\nabla_{e_1}\psi = \mathrm{e}^{-2f}f'^2 e_1, \qquad Qe_1 = -\mathrm{e}^{-2f}(f'' + f'^2)e_1,$
$\nabla_{e_2}\psi = \mathrm{e}^{-2f}(-f'^2 + f'')e_2, \qquad Qe_2 = -2\mathrm{e}^{-2f}f''e_2,$
$\nabla_{e_3}\psi = \mathrm{e}^{-2f}f'^2 e_3, \qquad Qe_3 = -\mathrm{e}^{-2f}(f'^2 + f'')e_3.$

(2): $\rho = \mathrm{e}^y, \quad \lambda = f''\mathrm{e}^{-2(y+f)}$,
$\nabla_{e_1}\psi = \mathrm{e}^{-2(y+f)}(f'(f'+1)e_1, \qquad Qe_1 = -\mathrm{e}^{-2(y+f)}(f'' + f'^2 + f')e_1,$
$\nabla_{e_2}\psi = \mathrm{e}^{-2(y+f)}(-f'^2 + f'' - f')e_2, \qquad Qe_2 = -2\mathrm{e}^{-2(y+f)}(f'' + f'^2)e_2,$
$\nabla_{e_3}\psi = \mathrm{e}^{-2(y+f)}f'^2 e_3, \qquad Qe_3 = -\mathrm{e}^{-2(y+f)}(f'^2 + f'')e_3.$

(3): $\rho = \mathrm{e}^{-f}, \quad \lambda = 0$,
$\nabla_{e_1}\psi = 0, \qquad Qe_1 = 0,$
$\nabla_{e_2}\psi = f''e_2, \qquad Qe_2 = -\big(f'^2 + f''\big)e_2,$
$\nabla_{e_3}\psi = f'^2 e_3, \qquad Qe_3 = -(f'^2 + f'')e_3.$

Of course, we must choose f so that λ is constant. We can construct further examples of generalized Ricci soliton on a 3-dimensional C_{12}-manifold by the similar way.

At the end of this section, we present the concept of the generalized η-Ricci soliton as a generalization of the η-Ricci soliton given by Cho-Kimura in [6] by the following equation:

$$\mathcal{L}_V g + 2S + 2\lambda g + \mu\eta\otimes\eta = 0, \tag{4.6}$$

where the tensor product notation $(\eta\otimes\eta)(X,Y) = \eta(X)\eta(Y)$ is used and λ, μ are real constants.

The generalized η-Ricci soliton equation in Riemannian manifold (M, g) is defined by:

$$\mathcal{L}_X g = -2c_1 X^\flat \odot X^\flat + 2c_2\,\mathrm{S} + 2\lambda g + \mu\eta\otimes\eta, \tag{4.7}$$

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$.

With the same reasoning above, we can express formula (4.7) as follows

$$\nabla_X \psi = -c_1\omega(X)\psi + c_2 QX + \lambda X + \mu\eta \otimes \xi. \tag{4.8}$$

Now, based on equation (4.4), we declare the following result

Theorem 4.3. *Any 3-dimensional C_{12}-manifold satisfies the generalized η-Ricci soliton equation with*

$$c_1 = 1 \quad c_2 = -1 \quad \lambda = |\psi|^2 - \mathrm{div}\psi \quad \textit{and} \quad \mu = |\psi|^2 - 2\mathrm{div}\psi - \frac{r}{2}.$$

Example 4.2. From Example 4.1, we can construct several non-trivial cases, namely:
1) If $f = y$ and $\rho = \frac{4}{\mathrm{e}^{2y} - c}$ with $c \in \mathbb{R}$, then we get

$$c_1 = 1, \quad c_2 = -1, \quad \lambda = 0, \quad \textit{and} \quad \mu = c.$$

2) If $f = \ln\left(\frac{1}{\sin^2 y}\right)$ and $\rho = c\sin y$, then we get

$$c_1 = 1, \quad c_2 = -1, \quad \lambda = -\frac{2}{c^2}, \quad \mu = -\frac{1}{c^2}.$$

Of course, while taking into account the necessary conditions on f and ρ.

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