Screen bi-slant lightlike submanifolds of indefinite Kaehler manifolds

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Abstract. The purpose of present paper is to introduce the study of screen bi-slant lightlike submanifolds of indefinite Kaehler manifolds. We obtain a characterization result for the existence of screen bi-slant lightlike submanifolds of indefinite Kaehler manifolds. Then, we derive a necessary and sufficient condition for the induced connection on a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold to be a metric connection. Further, we derive the integrability conditions for the various distributions associated with such lightlike submanifolds. Finally, we emphasize on the study of totally umbilical screen bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

Key words: screen bi-slant lightlike submanifold; totally umbilical lightlike submanifold; indefinite complex space form.

1 Introduction

The two well known classes of submanifolds namely, holomorphic and totally real submanifolds of an almost Hermitian manifold, arise due to the action of almost complex structure $J$ when for every vector field $Y(\neq 0)$ tangent to any point $p \in N$, the angle becomes 0 or $\pi/2$, respectively, between the tangent space $T_pN$ and $JY$. Then, Chen [4, 5] as a generalization of holomorphic and totally real submanifolds introduced the notion of slant submanifold in 1990. In this continuation, Lotta [11, 12] investigated the concept of slant submanifolds in contact geometry. After that, Carriazo [2] introduced the geometry of bi-slant submanifolds of almost Hermitian manifolds as well as almost contact metric manifolds as a generalization of slant submanifolds. On the similar note, several new generalized classes of slant submanifolds namely, pseudo-slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds came into existence and the subject matter was dealt in detail by Carriazo [3], Papaghiuc [13] and Sahin [15].

On the other hand, the study of lightlike submanifolds due to its interesting geometric properties attracted many geometers since last two decades. One may note
that, the geometry of lightlike submanifolds has broad application area and has been successfully employed in the theory of black holes, asymptotically flat spacetimes, Killing horizon and electromagnetic as well as radiation fields (see, [6] and [9]). Then, Sahin [14, 17], initiated the study of slant lightlike submanifolds in indefinite almost Hermitian manifolds and indefinite Sasakian manifolds. Afterwards, some more generalizations of slant lightlike submanifolds viz. screen slant lightlike submanifolds, screen pseudo-slant lightlike submanifolds, semi-slant lightlike submanifolds of indefinite Hermitian manifolds were considered and developed by others (for details, see [10, 16, 18, 19]). But the theory of bi-slant submanifolds is yet to be explored in lightlike geometry.

Therefore in this paper, we introduce screen bi-slant lightlike submanifolds in indefinite Kaehler manifolds and justify the existence of this type of submanifolds by giving a characterization result. Then, we derive a necessary and sufficient condition for the induced connection on a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold to be a metric connection. Further, we establish some integrability conditions for the distributions associated with such lightlike submanifolds. Finally, we emphasize on the study of totally umbilical screen bi-slant lightlike submanifolds of indefinite Kaehler manifolds.

2 Preliminaries

In the present section, we will review essential formulas and notations of lightlike submanifolds following [6].

Assume a submanifold \((N, g)\) of semi-Riemannian manifold \((\bar{N}^{m+n}, \bar{g})\) such that \(\bar{g}\) is metric with index \(q\) satisfying \(m, n \geq 1\) and \(1 \leq q \leq m + n - 1\). If the metric \(\bar{g}\) is degenerate on \(T N\), then \(T_pN\) and \(T_pN^\perp\) both becomes degenerate and there exists a radical (null) subspace \(\text{Rad}(T_pN)\) such that \(\text{Rad}(T_pN) = T_pN \cap T_pN^\perp\).

If \(\text{Rad}(T N) : p \in N \rightarrow \text{Rad}(T_pN)\) is a smooth distribution on \(N\) with rank \(r > 0\), \(1 \leq r \leq n\), then \(N\) is called an \(r\)-lightlike submanifold of \(\bar{N}\). While the radical distribution \(\text{Rad}(T N)\) of \(T N\) is defined as:

\[
\text{Rad}(T N) = \cup_{p \in N} \{\xi \in T_pN | g(u, \xi) = 0, \forall u \in T_pN, \xi \neq 0\}.
\]

Further, \(S(T N)\) be the screen distribution in \(T N\) such that \(T N = \text{Rad}(T N) \perp S(T N)\) and similarly \(S(T N^\perp)\) is the screen transversal vector bundle in \(T N^\perp\) such that 

\[
TN^\perp = \text{Rad}(T N) \perp S(T N^\perp).
\]

Moreover, there exists a local null frame \(\{N^i\}\) of null sections with values in the orthogonal complement of \(S(T N^\perp)\) in \(S(T N^\perp)^\perp\) such that

\[
\bar{g}(N^i, \xi_j) = \delta_{ij}, \quad \bar{g}(N^i, N^j) = 0, \quad \text{for } i, j \in \{1, 2, ..., r\},
\]

where \(\{\xi_i\}\) is any local basis of \(\Gamma(\text{Rad}(T N))\). It implies that \(\text{tr}(T N)\) and \(\text{ltr}(T N)\), respectively, be the vector bundles in \(T \bar{N}|_N\) and \(S(T N^\perp)^\perp\) with the property

\[
\text{tr}(T N) = \text{ltr}(T N) \perp S(T N^\perp),
\]

and

\[
T \bar{N}|_N = TN \oplus \text{tr}(T N) = S(T N) \perp (\text{Rad}(T N) \oplus \text{ltr}(T N)) \perp S(T N^\perp).
\]
Further, the Gauss and Weingarten formulas, for \( Y_1, Y_2 \in \Gamma(TN) \) and \( V \in \Gamma(tr(TN)) \), are given by

\[
(2.3) \quad \bar{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h(Y_1, Y_2), \quad \bar{\nabla}_{Y_1}V = -A_VY_1 + \nabla^\bot_{Y_1}V,
\]

where \{\( h(Y_1, Y_2), \nabla^\bot_{Y_1}V \)\} and \{\( \nabla_{Y_1}Y_2, A_VY_1 \)\} belong to \( \Gamma(tr(TN)) \) and \( \Gamma(TN) \), respectively and \( \bar{\nabla} \) represents the Levi-Civita connection on \( \bar{N} \). Here, \( h \) is a symmetric-Civita connection on \( N \). For curvature tensors \( \bar{\nabla} \) which implies that the induced connection \( \nabla \) is a metric connection on \( T(N) \) and \( A_V \) is linear shape operator on \( N \). In view of decomposition given by (2.2), the Gauss and Weingarten formulas become

\[
(2.4) \quad \bar{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2),
\]

\[
(2.5) \quad \bar{\nabla}_{Y_1}N' = -A_NY_1 + \nabla^\bot_{Y_1}N' + D^s(Y_1, N'),
\]

\[
(2.6) \quad \bar{\nabla}_{Y_1}W = -A_WY_1 + D^l(Y_1, W) + \nabla^\bot_{Y_1}W,
\]

where \( Y_1, Y_2 \in \Gamma(TN), N' \in \Gamma(ltr(TN)) \) and \( W \in \Gamma(S(TN^2)) \). Furthermore, employing Eqs. (2.4)-(2.6), we derive

\[
(2.7) \quad g(A_WY_1, Y_2) = \bar{g}(h^s(Y_1, Y_2), W) + \bar{g}(Y_2, D^l(Y_1, W)).
\]

\[
(2.8) \quad \bar{g}(A_WY_1, N') = \bar{g}(D^s(Y_1, N'), W).
\]

Suppose that \( P \) denotes the projection morphism of \( TN \) on its screen distribution \( S(TN) \), then some new geometric objects of \( S(TN) \) on \( N \) are given as

\[
(2.9) \quad \nabla_{Y_1}P Y_2 = \nabla^\bot_{Y_1}P Y_2 + h^s(Y_1, PY_2), \quad \nabla_{Y_1}\xi = -A_N^sY_1 + \nabla^\bot_{Y_1}\xi,
\]

where \{\( h^s(Y_1, PY_2), \nabla^\bot_{Y_1}\xi \)\} \( \in \Gamma(Rad(TN)) \) and \{\( \nabla^\bot_{Y_1}PY_2, A_N^sY_1 \)\} \( \in \Gamma(S(TN^2)) \). Further, employing Eqs. (2.5), (2.6) and (2.9), we attain

\[
(2.10) \quad \bar{g}(h^l(Y_1, PY_2), \xi) = g(A_N^sY_1, PY_2).
\]

Let \( \bar{\nabla} \) is a metric connection on \( \bar{N} \), therefore for \( Y_1, Y_2, Y_3 \in \Gamma(TN) \), one has

\[
(2.11) \quad (\nabla_{Y_1}g)(Y_2, Y_3) = \bar{g}(h^l(Y_1, Y_2), Y_3) + \bar{g}(h^s(Y_1, Y_3), Y_2),
\]

which implies that the induced connection \( \nabla \) on \( N \) is not a metric connection. For curvature tensors \( \bar{R} \) of \( \bar{\nabla} \), the equation of Codazzi is given by

\[
(2.12) \quad (\bar{R}(Y_1, Y_2)Y_3) = (\nabla_{Y_1}h^l)(Y_2, Y_3) - (\nabla_{Y_2}h^l)(Y_1, Y_3) + D^l(Y_1, h^s(Y_2, Y_3)) - D^l(Y_2, h^s(Y_1, Y_3)) + D^s(Y_1, h^l(Y_2, Y_3)) - D^s(Y_2, h^l(Y_1, Y_3)),
\]

where

\[
(2.13) \quad (\nabla_{Y_1}h^s)(Y_2, Y_3) = \nabla^\bot_{Y_1}h^s(Y_2, Y_3) - h^s(\nabla_{Y_1}Y_2, Y_3) - h^s(Y_2, \nabla_{Y_1}Y_3),
\]

\[
(2.14) \quad (\nabla_{Y_1}h^l)(Y_2, Y_3) = \nabla^\bot_{Y_1}h^l(Y_2, Y_3) - h^l(\nabla_{Y_1}Y_2, Y_3) - h^l(Y_2, \nabla_{Y_1}Y_3),
\]

for any \( Y_1, Y_2, Y_3 \in \Gamma(TN) \).
Definition 2.1. [1]. An indefinite almost Hermitian manifold \((\bar{N}, \bar{g}, \bar{J})\) is said to be an indefinite Kaehler manifold if

\[
\bar{J}^2 = -I, \quad \bar{g}(\bar{J}Y_1, \bar{J}Y_2) = \bar{g}(Y_1, Y_2), \quad (\nabla_{Y_1} \bar{J})Y_2 = 0, \quad \forall Y_1, Y_2 \in \Gamma(T\bar{N}),
\]

where \(\nabla\) be the Levi-Civita connection defined on \(\bar{N}\).

An indefinite Kaehler manifold along with constant holomorphic sectional curvature \(c\) is known as indefinite complex space form and it is denoted by \(\bar{N}(c)\). Then, its curvature tensor is given by

\[
\bar{R}(Y_1, Y_2)Y_3 = c\left\{\bar{g}(Y_2, Y_3)Y_1 - \bar{g}(Y_1, Y_3)Y_2 + \bar{g}(JY_2, Y_3)JY_1 - \bar{g}(JY_1, Y_3)JY_2 + 2\bar{g}(Y_1, JY_2)JY_3\right\},
\]

for \(Y_1, Y_2, Y_3 \in \Gamma(T\bar{N})\).

3 Screen bi-slant lightlike submanifolds

Firstly, we state the following lemma given by Sahin [14], which is very useful for the coming part of this article.

Lemma 3.1. Let \(N\) be an \(r\)-lightlike submanifold of an indefinite Kaehler manifold \(\bar{N}\) of index \(2q\) (provided \(2q < \dim(N)\)). Then, screen distribution \(S(TN)\) of lightlike submanifold \(N\) is Riemannian.

Definition 3.1. Assume that \(N\) be a \(q\)-lightlike submanifold of an indefinite Kaehler manifold \(\bar{N}\) with index \(2q\). Then, \(N\) is said to be a screen bi-slant lightlike submanifold of \(\bar{N}\), if the following conditions are satisfied:

(i) \(\text{Rad}(TN)\) is invariant with respect to \(\bar{J}\), that is, \(\bar{J}\text{Rad}(TN) = \text{Rad}(TN)\).

(ii) There exists non-degenerate orthogonal distributions \(D_1\) and \(D_2\) on \(N\) such that \(S(TN) = D_1 \perp D_2\).

(iii) For each non-zero vector field tangent to \(D_i\) for \(i = 1, 2\), at \(y \in U \subset N\), the angle \(\theta_i(Y)\) between \(JY\) and the vector space \(D_{iy}\) is constant, that is, it is independent of the choice of \(y \in U \subset N\) and \(Y \in D_{iy}\).

This constant angle \(\theta_i(Y)\) for \(i = 1, 2\), is known as slant angle of the distribution \(D_i\), respectively. Moreover, a screen bi-slant lightlike submanifold is said to be proper if \(D_i \neq \{0\}\) and \(\theta_i \neq 0, \pi/2\), for \(i = 1, 2\).

In view of Definition 3.1, the tangent bundle \(TN\) of \(N\) has the following decomposition

\[
TN = \text{Rad}(TN) \perp D_1 \perp D_2.
\]

For any \(Y \in \Gamma(TN)\), we write

\[
\bar{J}Y = fY + \omega Y.
\]
where $fY$ and $\omega Y$ are the tangential and transversal components of $\bar{J}Y$, respectively. Similarly, for any $V \in \Gamma(tr(TN))$,

\begin{equation}
\bar{J}V = tV + nV, \tag{3.3}
\end{equation}

where $tV$ and $nV$ are the tangential and transversal components $\bar{J}V$, respectively. Consider $\phi_1, \phi_2$ and $\phi_3$ be the projections of $TM$ on $D_1, D_2$ and $Rad(TN)$, respectively. Similarly, Consider $\eta_1, \eta_2$ and $\eta_3$ be the projections of $tr(TN)$ on $\bar{J}D_1, \bar{J}D_2$ and $ltr(TN)$, respectively, where $\bar{J}D_1$ and $\bar{J}D_2$ are non-degenerate subbundles in $S(TN^\perp)$. Then, for $Y \in \Gamma(TN)$, we have

\begin{equation}
Y = \phi_1 Y + \phi_2 Y + \phi_3 Y. \tag{3.4}
\end{equation}

On applying $\bar{J}$ to (3.4), we obtain

\[ \bar{J}Y = \bar{J}\phi_1 Y + \bar{J}\phi_2 Y + \bar{J}\phi_3 Y, \]

which yields

\begin{equation}
\bar{J}Y = f\phi_1 Y + \omega\phi_1 Y + f\phi_2 Y + \omega\phi_2 Y + \bar{J}\phi_3 Y. \tag{3.5}
\end{equation}

Further, (3.5) can be written as

\begin{equation}
\bar{J}Y = fY + \omega\phi_1 Y + \omega\phi_2 Y, \tag{3.6}
\end{equation}

where $fY = f\phi_1 Y + f\phi_2 Y + \bar{J}\phi_3 Y$. Also, for $W \in \Gamma(tr(TN))$, we have

\begin{equation}
W = \eta_1 W + \eta_2 W + \eta_3 W. \tag{3.7}
\end{equation}

then applying $\bar{J}$ to (3.7), we obtain

\[ \bar{J}W = \bar{J}\eta_1 W + \bar{J}\eta_2 W + \bar{J}\eta_3 W, \]

which yields

\begin{equation}
\bar{J}W = t\eta_1 W + n\eta_1 W + t\eta_2 W + n\eta_2 W + \bar{J}\eta_3 W. \tag{3.8}
\end{equation}

Differentiating (3.5) and then by equating the tangent components and transversal components, we obtain

\begin{equation}
(\nabla_X f)Y = A_{\omega Y} X + th^s(X,Y), \tag{3.9}
\end{equation}

\begin{equation}
\bar{J}h^l(X,Y) = h^l(X, fY) + D^l(X, \omega Y) \tag{3.10}
\end{equation}

and

\begin{equation}
(\nabla_X \omega)Y = -h^s(X, fY) + nh^s(X,Y), \tag{3.11}
\end{equation}

where $(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y$ and $(\nabla_X \omega)Y = \nabla^s\omega Y - \omega\nabla_X Y$, for $X,Y \in \Gamma(TN)$. 


Lemma 3.2. For a screen bi-slant lightlike submanifold \( \tilde{N} \), \( \omega D_1 \) and \( \omega D_2 \) are orthogonal.

Proof. Let \( X \in \Gamma(D_1) \) and \( Y \in \Gamma(D_2) \); then
\[
\tilde{g}(\omega X, \omega Y) = \tilde{g}(JX - fX, \omega Y) = \tilde{g}(JX, \omega Y) = -\tilde{g}(X, J\omega Y) = -\tilde{g}(X, \omega fY + \omega Y) = 0,
\]
which completes the proof. \( \square \)

Thus, it is clear that there exist \( \mu \subset S(TN^\perp) \) such that
\[
(3.12) \quad T\tilde{N} = S(TN) \perp \{\text{Rad}(TN) \oplus \text{ltr}(TN)\} \perp \{\omega(D_1) \perp \omega(D_2) \perp \mu\}.
\]

Theorem 3.3. (Existence Theorem) Consider \( N \) be a q-lightlike submanifold of an indefinite Kaehler manifold \( \tilde{N} \). Then \( N \) is a screen bi-slant lightlike submanifold, if and only if,

(i) \( \text{ltr}(TN) \) is invariant with respect to \( J \).

(ii) the screen distribution \( S(TN) \) can be split as \( S(TN) = D_1 \perp D_2 \).

(iii) \( f^2 \phi_i Y = -\cos^2 \theta_i(\phi_i Y), \quad \text{for } i=1,2 \) and \( Y \in \Gamma(S(TN)) \).

(iv) \( t\omega \phi_i Y = -\sin^2 \theta_i(\phi_i Y), \quad \text{for } i=1,2 \) and \( Y \in \Gamma(S(TN)) \).

Proof. Assume that \( N \) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \tilde{N} \). Suppose that \( JN' \in \Gamma(\text{Rad}(TN)) \); then we get \( JJN' = -N' \in \Gamma(\text{ltr}(TN)) \), since \( \text{Rad}(TN) \) is invariant with respect to \( J \), therefore we get a contradiction, which implies that \( JN' \notin \Gamma(\text{Rad}(TN)) \). Now, employing Eqs. (2.15) and (3.2), for \( Y \in \Gamma(S(TN)) \) and \( N' \in \Gamma(\text{ltr}(TN)) \), we acquire
\[
\tilde{g}(JN', Y) = -\tilde{g}(N', JY) = 0.
\]

Therefore, \( JN' \) does not belong to \( S(TN) \). In addition, for \( W \in \Gamma(S(TN^\perp)) \) and \( N' \in \Gamma(\text{ltr}(TN)) \), employing Eqs. (2.15) and (3.2), we obtain,
\[
\tilde{g}(JN', W) = -\tilde{g}(N', JW) = 0,
\]
which implies that \( JN' \) does not belong to \( S(TN^\perp) \). Moreover, the angle between \( \tilde{J}\phi_i Y \) and \( D_i \) is constant for \( i = 1,2 \). Therefore, for \( Y \in \Gamma(S(TN)) \), we have
\[
(3.13) \quad \cos \theta_i(\phi_i Y) = \frac{\tilde{g}(\tilde{J}\phi_i Y, f\phi_i Y)}{|\tilde{J}\phi_i Y||f\phi_i Y|} = \frac{-\tilde{g}(\phi_i Y, \tilde{J}f\phi_i Y)}{|\phi_i Y||f\phi_i Y|} = \frac{-\tilde{g}(\phi_i Y, f^2 \phi_i Y)}{|\phi_i Y||f\phi_i Y|}.
\]

On the other hand, we also have
\[
(3.14) \quad \cos \theta_i(\phi_i Y) = \frac{|f\phi_i Y|}{|\tilde{J}\phi_i Y|}, \quad \text{for } i = 1,2.
\]

Thus, from Eqs. (3.13) and (3.14), we get
\[
(3.15) \quad \cos^2 \theta_i(\phi_i Y) = \frac{-\tilde{g}(\phi_i Y, f^2 \phi_i Y)}{|\phi_i Y|^2}.
\]

As we know that, \( \theta_i(\phi_i Y) \) is constant on \( D_i \), for \( i = 1,2 \). Thus, we conclude that
\[
(3.16) \quad f^2 \phi_i Y = -\cos^2 \theta_i(\phi_i Y),
\]
which proves (iii). Next, for $Y \in \Gamma(S(TN))$, we have

$$\bar{J}\phi_i Y = f\phi_i Y + \omega\phi_i Y \quad \text{for} \quad i = 1, 2.$$  

Further, applying $\bar{J}$, the above equation yields

$$-\phi_i Y = f^2\phi_i Y + \omega f\phi_i Y + t\omega\phi_i Y + n\omega\phi_i Y.$$  

Then, equating the tangential components on both sides, we acquire

$$-\phi_i Y = f^2\phi_i Y + t\omega\phi_i Y.$$  

As $N$ is a screen bi-slant lightlike submanifold, thus using (3.16), we have $f^2\phi_1 Y = -\cos^2 \theta_1(\phi_1 Y)$ and $f^2\phi_2 Y = -\cos^2 \theta_2(\phi_2 Y)$, thus we attain

$$t\omega\phi_i Y = -\sin^2 \theta_i(\phi_i Y), \quad \text{for} \quad i = 1, 2,$$  

which proves the assertion (iv). Conversely, we can prove that $\text{Rad}(TN)$ is invariant in a similar way that $\text{ltr}(TN)$ is invariant. Further, from Lemma 3.1, it is clear that $S(TN)$ is Riemannian; then, for $i = 1, 2$, we have

$$g(f\phi_i Y, f\phi_i Y) = -g(f^2\phi_i Y, \phi_i Y) = \cos^2 \theta_i(\phi_i Y)g(\phi_i Y, \phi_i Y),$$  

for $Y \in \Gamma(S(TN))$, which further gives

$$\cos^2 \theta_i(\phi_i Y) = \frac{g(f\phi_i Y, f\phi_i Y)}{g(\phi_i Y, \phi_i Y)}.$$  

On the other hand, from condition (iv), one has $t\omega\phi_i Y = -\sin^2 \theta_i(\phi_i Y)$, for $i = 1, 2$. Further, employing (3.19), we infer

$$f^2\phi_i Y = -(1 - \sin^2 \theta_i(\phi_i Y)) = -\cos^2 \theta_i(\phi_i Y).$$  

Then, following the similar steps of assertion (iii), the proof follows. □

**Corollary 3.4.** Let $N$ be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{N}$; then one has

$$g(f\phi_i X, f\phi_i Y) = \cos^2 \theta_i g(\phi_i X, \phi_i Y)$$  

and

$$\bar{g}(\omega\phi_i X, \omega\phi_i Y) = \sin^2 \theta_i g(\phi_i X, \phi_i Y),$$  

for $i = 1, 2$ and $X, Y \in \Gamma(TN)$.

From (2.11), it can be noticed that the induced connection $\nabla$ on a lightlike submanifold is not necessarily a metric connection. Therefore, in the next characterization result, we present a necessary and sufficient condition under which the induced connection $\nabla$ on a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{N}$ becomes a metric connection.
Theorem 3.5. A necessary and sufficient condition for the induced connection $\nabla$ on a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{N}$ to be a metric connection is that
\[
\nabla_X \bar{J} Y \in \Gamma(\text{Rad}(TN)) \quad \text{and} \quad th^*(X, \bar{J} Y) = 0,
\]
for $X \in \Gamma(TN)$ and $Y \in \Gamma(\text{Rad}(TN))$.

Proof. Assume that $N$ be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{N}$. Since $\bar{J}$ is an almost complex structure on $\bar{N}$, therefore for $X \in \Gamma(TN)$ and $Y \in \Gamma(\text{Rad}(TN))$, we have $\nabla_X Y = -\bar{J} \nabla_X \bar{J} Y$. Further, considering (2.15), we get $\nabla_X Y = -\bar{J} \nabla_X \bar{J} Y$. Now, employing (2.3) and then equating the tangential components on both sides, we derive
\[
(3.26) \quad \nabla_X Y = -\bar{J}(\nabla_X \bar{J} Y) - th^*(X, \bar{J} Y).
\]
Hence, from (3.26), $\nabla_X Y \in \Gamma(\text{Rad}(TN))$ if and only if $\nabla_X \bar{J} Y \in \Gamma(\text{Rad}(TN))$ and $th^*(X, \bar{J} Y) = 0$, which proves the result. \[\square\]

Theorem 3.6. Let $N$ be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold $\bar{N}$; then

(i) the slant distributions $D_i$ is integrable, if and only if,
\[
\nabla_X f Y - \nabla_Y f X - A_{\omega \phi_i} Y X + A_{\omega \phi_i} X Y \in \Gamma(D_i),
\]
for $X, Y \in \Gamma(D_i)$ and $i = 1, 2$.

(ii) the distribution $\text{Rad}(TN)$ is integrable, if and only if,
\[
h^*(X, \bar{J} Y) = h^*(\bar{J} X, Y),
\]
for $X, Y \in \Gamma(\text{Rad}(TN))$.

Proof. For $X, Y \in \Gamma(D_i)$ for $i = 1, 2$, using (3.9), we derive
\[
(3.27) \quad \nabla_X f Y - A_{\omega \phi_i} Y X - th^*(X, Y) = f \nabla_X Y,
\]
by interchanging the role of $X$ and $Y$ in above equation, we get
\[
(3.28) \quad \nabla_Y f X - A_{\omega \phi_i} X Y - th^*(Y, X) = f \nabla_Y X.
\]
Therefore, from Eqs. (3.27) and (3.28), we obtain
\[
\nabla_X f Y - \nabla_Y f X - A_{\omega \phi_i} Y X + A_{\omega \phi_i} X Y = f[X, Y],
\]
which proves the assertion (i).
For $X, Y \in \Gamma(\text{Rad}(TN))$, using (3.11), we have
\[
(3.29) \quad h^*(X, \bar{J} Y) - nh^*(X, Y) = \omega \nabla_X Y.
\]
On interchanging the role of $X$ and $Y$ in (3.29), we get
\[
(3.30) \quad h^*(Y, \bar{J} X) - nh^*(Y, X) = \omega \nabla_Y X.
\]
Further, from Eqs. (3.11) and (3.29), we obtain
\[
h^*(X, \bar{J} Y) - h^*(\bar{J} X, Y) = \omega[X, Y],
\]
from which assertion (ii) follows. \[\square\]
Further, employing (2.4), we get
\[ g(h^s(X, Y), \omega Z) = g(h^s(X, Z), \omega Y), \]
for \( X \in \Gamma(TN), Y, Z \in \Gamma(S(TN)) \) and \( N' \in \Gamma(ltr(TN)) \).

**Proof.** Employing (3.9) for \( Y \in \Gamma(Rad(TN)), X \in \Gamma(TN) \) and \( N' \in \Gamma(ltr(TN)) \), we obtain \( \bar{g}((\nabla_X f)Y, N') = 0 \). Further, for \( Y \in \Gamma(S(TN)) \), we have \( \bar{g}((\nabla_X f)Y, N') = \bar{g}(A_{\omega Y}X, N') \). Moreover, using (2.8) for \( X \in \Gamma(S(TN)) \), we acquire
\[ \bar{g}((\nabla_X f)Y, N') = \bar{g}(D^s(X, N'), \omega Y). \]

On the other hand, for \( X, Y \in \Gamma(TN) \) and \( Z \in \Gamma(S(TN)) \), we derive
\[ \bar{g}((\nabla_X f)Y, Z) = g(A_{\omega Y}X, Z) - g(h^s(X, Y), \omega Z), \]
then employing (2.7), we get
\[ \bar{g}((\nabla_X f)Y, Z) = g(h^s(X, Z), \omega Y) - g(h^s(X, Y), \omega Z). \]

Thus, from Eqs. (3.31) and (3.32), the assertions follow. \( \square \)

**Theorem 3.8.** Let \( N \) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \tilde{N} \). If \( (\nabla_X f)Y = 0 \), for \( X \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \), then the induced connection \( \nabla \) is a metric connection.

**Proof.** Using the hypothesis and (3.9), we have \( th^s(X, Y) = 0 \) for \( X \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \). Therefore, \( g(th^s(X, Y), Z) = 0 \), for \( X, Z \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \). Thus, we attain
\[ \bar{g}(Jh^s(X, Y), Z) = 0, \]
which further gives
\[ \bar{g}(h^s(X, Y), \omega \phi_i Z) = 0 \quad \text{for} \quad i = 1, 2. \]
Moreover, by using (2.4) for \( X \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \), we have
\[ \bar{g}(\omega \phi_i \nabla_X Y, Jh^s(X, Y)) = g(\omega \phi_i \nabla_X Y, \bar{J}h^s(X, Y)) \]
Since \( ltr(TN) \) is invariant, then employing Eqs. (2.15) and (3.5), we acquire
\[ \bar{g}(\omega \phi_i \nabla_X Y, Jh^s(X, Y)) = g(\omega \phi_i \nabla_X Y, \bar{J}h^s(X, Y) - g(\omega \phi_i \nabla_X Y, \omega \phi_i \nabla_X Y). \]
Further, employing (2.4), we get
\[ \bar{g}(\omega \phi_i \nabla_X Y, Jh^s(X, Y)) = g(\omega \phi_i \nabla_X Y, h^s(X, JY) - g(\omega \phi_i \nabla_X Y, \omega \phi_i \nabla_X Y). \]
Then, using (3.34), we obtain \( \bar{g}(\omega \phi_i \nabla_X Y, Jh^s(X, Y)) = -\bar{g}(\omega \phi_i \nabla_X Y, \omega \phi_i \nabla_X Y) \) and further employing (3.25) for \( X \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \), we get
\[ \bar{g}(\omega \phi_i \nabla_X Y, Jh^s(X, Y)) = -\sin^2 \theta_i g(\phi_i \nabla_X Y, \phi_i \nabla_X Y). \]
Now, using Eqs. (2.15) and (3.5), we have
\[ \bar{g}(\omega \phi_i \nabla_X Y, \bar{J}h^s(X, Y)) = -\bar{g}(f \phi_i \nabla_X Y, \bar{J}h^s(X, Y)), \]
for \( X \in \Gamma(TN) \) and \( Y \in \Gamma(Rad(TN)) \) and then employing (3.33), we obtain
\begin{equation}
(3.36) \hspace{1cm} \bar{g}(\omega \phi_i \nabla_X Y, \bar{J}h^s(X, Y)) = 0. 
\end{equation}

Further, from Eqs. (3.35) and (3.36), we attain \( \sin^2 \theta_i g(\phi_i \nabla_X Y, \phi_i \nabla_X Y) = 0 \) for \( i = 1, 2 \). Since \( \theta_i \neq 0 \) and \( S(TN) \) is Riemannian, therefore we have \( \phi_i \nabla_X Y = 0 \).

Then, \( \nabla_X Y \in \Gamma(Rad(TN)) \), which means \( Rad(TN) \) is parallel. Hence, the theorem is proved following [6].

\[ \square \]

## 4 Foliations determined by distributions

**Theorem 4.1.** Let \( N \) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{N} \). Then, the screen distribution defines a totally geodesic foliation, if and only if, \( JA_{\omega \phi_i}X - A_{\omega f \phi_i}X \) for \( i = 1, 2 \) has no components in \( Rad(TN) \) for \( X, Y \in \Gamma(D_i) \).

**Proof.** Using Eqs. (2.4) and (2.15), we have \( \bar{g}(\nabla_X Y, N') = \bar{g}(\nabla_X JY, JN') \), for \( X, Y \in \Gamma(D_i) \) for \( i = 1, 2 \) and \( N' \in \Gamma(\text{tr}(TN)) \). Thus, from Eqs. (2.6) and (3.5), we obtain
\[ \bar{g}(\nabla_X Y, N') = \bar{g}(\nabla_X f \phi_1 Y, JN') - \bar{g}(A_{\omega \phi_i}X, JN'). \]

Again using Eqs. (2.4), (2.6), (2.15) and (3.5), we derive
\[ \bar{g}(\nabla_X Y, N') = -\bar{g}(\nabla_X f^2 \phi_i Y, N') - \bar{g}(A_{\omega f \phi_i}X, N') - \bar{g}(A_{\omega \phi_i}X, JN'), \]

thus in view of Theorem 3.3, we get
\[ \bar{g}(\nabla_X Y, N') = \cos^2 \theta \bar{g}(\nabla_X Y, N') - \bar{g}(A_{\omega f \phi_i}X, N') - \bar{g}(A_{\omega \phi_i}X, JN') \]
\[ \sin^2 \theta \bar{g}(\nabla_X Y, N') = -\bar{g}(A_{\omega f \phi_i}X, N') - \bar{g}(A_{\omega \phi_i}X, JN'), \]

for \( i = 1, 2 \). Hence, the proof is completed. \[ \square \]

**Theorem 4.2.** Let \( N \) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{N} \). Then, \( Rad(TN) \) defines a totally geodesic foliation, if and only if,
\[ \bar{g}(h^l(X, f \phi_1 Z) + h^l(X, f \phi_2 Z), JY) = -\bar{g}(D^l(X, \omega \phi_1 Z) + D^l(X, \omega \phi_2 Z), JY), \]

for \( X, Y \in \Gamma(Rad(TN)) \) and \( Z \in \Gamma(S(TN)) \).

**Proof.** Let \( N \) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{N} \). To prove that \( Rad(TN) \) defines a totally geodesic foliation, it is sufficient to show that \( \nabla_X Y \in \Gamma(Rad(TN)) \), for every \( X, Y \in \Gamma(Rad(TN)) \). Since \( \nabla \) is a metric connection, using Eqs. (2.4) and (2.15), for every \( X, Y \in \Gamma(Rad(TN)) \) and for every \( Z \in \Gamma(S(TN)) \), we obtain
\begin{equation}
(4.1) \hspace{1cm} g(\nabla_X Y, Z) = \bar{g}(\nabla_X Y, Z) = -\bar{g}(Y, \nabla_X Z) \\
= -\bar{g}(\nabla_X JZ, JY).
\end{equation}
In view of (3.5), we get
\[
\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X (f\phi_1 Z + \omega \phi_1 Z + f\phi_2 Z), \bar{J}Y)
\]
\[
= \bar{g}(h^i(X, f\phi_1 Z), \bar{J}Y) - \bar{g}(D^i(X, \omega \phi_1 Z), \bar{J}Y)
\]
\[
(4.2)
- \bar{g}(h^i(X, f\phi_2 Z), \bar{J}Y) - \bar{g}(D^i(X, \omega \phi_2 Z), \bar{J}Y),
\]
which completes the proof. \(\square\)

**Theorem 4.3.** Let \(N\) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \(\bar{N}\). Then, \(D_i\) for \(i=1,2\) defines a totally geodesic foliation, if and only if,
\[
\bar{g}(f Y, A_{JN} X) = -\bar{g}(A_{\omega Y} X, \bar{J}N')
\]
for \(X, Y \in \Gamma(D_i)\) where \(i = 1, 2\) and \(N' \in \Gamma(ltr(TN))\).

**Proof.** Let \(N\) be a screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \(\bar{N}\). To prove that \(D_i\) for \(i = 1, 2\) defines a totally geodesic foliation, it is sufficient to show that \(\nabla_X Y \in D_i\) for \(i = 1, 2\), for every \(X, Y \in \Gamma(D_i)\) for \(i = 1, 2\). Since \(\nabla\) is a metric connection, using Eqs. (2.4) and (2.15), for every \(X, Y \in \Gamma(D_i)\) for \(i = 1, 2\) and \(N' \in \Gamma(ltr(TN))\), we obtain
\[
\bar{g}(\nabla_X Y, N') = \bar{g}(\nabla_X Y, N') = \bar{g}(\nabla_X \bar{J}Y, \bar{J}N')
\]
\[
= -\bar{g}(JY, \nabla_X \bar{J}N')
\]
\[
(4.3)
= g(f Y, A_{JN} X) - \bar{g}(\omega Y, D^s(X, \bar{J}N')).
\]
Further, employing (2.8), we obtain
\[
\bar{g}(\nabla_X Y, N') = g(f Y, A_{JN} X) - \bar{g}(A_{\omega Y} X, \bar{J}N'),
\]
which completes the proof. \(\square\)

## 5 Totally umbilical screen bi-slant lightlike submanifolds

**Definition 5.1.** [7] A lightlike submanifold \((N, \bar{g})\) of a semi-Riemannian manifold \((\bar{N}, \bar{g})\) is called totally umbilical, if there exists transversal curvature vector field \(H \in \Gamma(tr(TN))\) on \(N\) such that
\[
h(Y_1, Y_2) = H\bar{g}(Y_1, Y_2),
\]
for \(Y_1, Y_2 \in \Gamma(TN)\). Using Eqs. (2.4) and (2.6), clearly \(N\) is totally umbilical, if and only if, there exist smooth vector fields \(H^i \in \Gamma(ltr(TN))\) and \(H^s \in \Gamma(S(TN^\perp))\) such that
\[
h^i(Y_1, Y_2) = H^i\bar{g}(Y_1, Y_2), \quad h^s(Y_1, Y_2) = H^s\bar{g}(Y_1, Y_2), \quad \text{and} \quad D^i(Y_1, W) = 0,
\]
for \(Y_1, Y_2 \in \Gamma(TN)\) and \(W \in \Gamma(S(TN^\perp))\).

On the other hand, a lightlike submanifold is totally geodesic, if and only if, \(h(Y_1, Y_2) = 0\), for \(Y_1, Y_2 \in \Gamma(TN)\). Thus, a lightlike submanifold is totally geodesic, if and only if, \(H^i = 0\) and \(H^s = 0\).
**Theorem 5.1.** Consider \( N \) be a totally umbilical screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{N} \). Then, at least one of the following statements is true:

(a) \( N \) is an anti-invariant submanifold.

(b) \( D_i = \{0\} \) for \( i = 1, 2 \).

(c) If \( N \) is a proper screen bi-slant lightlike submanifold, then \( H^s \in \Gamma(\mu) \).

**Proof.** For a totally umbilical screen bi-slant lightlike submanifold \( N \) of \( \bar{N} \), employing (5.1), for \( Z \in \Gamma(D_i) \) for \( i = 1, 2 \), \( \Phi Z \in D_1 \) or \( D_2 \), we have

\[ h(\phi Z, \phi Z) = g(\phi Z, \phi Z)H. \]  

Then, using Eqs. (2.3) and (3.24), we obtain

\[ \bar{\nabla}_{\phi Z}Z - \nabla_{\phi Z}Z = \cos^2 \theta g(Z, Z)H, \]  

which yields

\[ J\bar{\nabla}_{\phi Z}Z - \bar{\nabla}_{\phi Z}\omega Z - \nabla_{\phi Z}\phi Z = \cos^2 \theta g(Z, Z)H. \]  

Further, using Eqs. (2.4) and (2.6), we derive

\[ \cos^2 \theta g(Z, Z) = \bar{\nabla}_{\phi Z}Z + A_{\omega Z}Z - D^l(\phi Z, \omega Z) - \nabla_{\phi Z}\phi Z. \]  

\( \Phi Z \in \Gamma(D_1) \) or \( D_2 \) and employing (3.2), (3.3) and (5.4), we obtain

\[ \cos^2 \theta g(Z, Z) = \phi_{\omega Z}Z + a_{\omega Z}Z + g(\phi Z, Z) + g(\phi Z, Z)\bar{J}H^1 + g(\phi Z, Z)H^s + g(\phi Z, Z)nH^s \]  

\[ + A_{\omega Z}Z - \nabla_{\phi Z}\omega Z - D^l(\phi Z, \omega Z) - \nabla_{\phi Z}\phi Z. \]  

Taking the inner product with respect to \( \omega Z \) in above equation, we get

\[ \cos^2 \theta g(Z, Z)\bar{g}(H^s, \omega Z) = \bar{g}(\nabla_{\phi Z}Z, \omega Z) - \bar{g}(\nabla_{\phi Z}\omega Z, \omega Z). \]  

Since (3.24) holds for \( i = 1, 2 \), \( \phi_i X = \phi_i Y \in \Gamma(D_i) \) and taking the covariant derivative with respect to \( \phi Z \), we get

\[ \bar{g}(\nabla_{\phi Z}\omega Z, \omega Z) = \sin^2 \theta g(\nabla_{\phi Z}Z, Z). \]  

Now, using Eqs. (3.25) and (5.8) in (5.7), we obtain

\[ \cos^2 \theta g(Z, Z)\bar{g}(H^s, \omega Z) = 0. \]  

Thus, (5.9) implies that either \( Z = 0 \) or \( \theta = \pi/2 \) or \( H^s \in \Gamma(\mu) \). Hence, the result follows.

**Theorem 5.2.** Every totally umbilical proper screen bi-slant lightlike submanifold of an indefinite Kaehler manifold \( \bar{N} \) is totally geodesic.
Proof. As $\bar{N}$ is an indefinite Kaehler manifold therefore for $Z \in \Gamma(D_i)$ where $i = 1, 2$, employing (2.15), we acquire $\nabla_Z JZ = J\nabla_Z Z$, which implies that

$$\nabla_Z \phi Z + h^i(Z, \phi Z) + h^s(Z, \phi Z) - A\omega Z + \nabla_Z \omega Z + D^i(Z, \omega Z)$$

(5.10)

$$= \phi \nabla_Z Z + \omega \nabla_Z Z + \bar{J} h^i(Z, Z) + th^s(Z, Z) + nh^s(Z, Z).$$

Using (5.2) and equating the tangential components on both sides, the above equation becomes

$$\nabla_Z \phi Z = \phi \nabla_Z Z + \bar{J} h^i(Z, Z) + th^s(Z, Z).$$

(5.11)

Taking the inner product with respect to $\bar{J} \xi \in \Gamma(\text{Rad}(TN))$ on both sides of (5.10), we obtain

$$g(A\omega Z, \bar{J} \xi) + \bar{g}(h^i(Z, Z), \xi) = 0.$$

(5.12)

Then, employing (2.7), we have

$$\bar{g}(h^s(Z, \bar{J} \xi), \omega Z) + \bar{g}(\bar{J} \xi, D^i(Z, \omega Z)) + \bar{g}(h^i(Z, Z), \xi) = 0.$$

(5.13)

In view of (5.2), the above equation reduces to

$$\bar{g}(H^s, \omega Z)g(Z, \bar{J} \xi) + \bar{g}(H^i, \xi)g(Z, Z) = 0.$$

(5.14)

From Theorem 5.1, we have $H^s \in \Gamma(\mu)$; therefore from (5.14), we obtain

$$\bar{g}(H^i, \xi)g(Z, Z) = 0.$$

(5.15)

Since $D_i$ for $i = 1, 2$ is non-degenerate, therefore $\bar{g}(H^i, \xi) = 0$, which implies

$$H^i = 0.$$

(5.16)

Moreover, $H^s \in \Gamma(\mu)$ for a proper totally umbilical screen bi-slant lightlike submanifold of $\bar{N}$. Therefore, equating the transversal components on both sides of (5.10), we have

$$\omega \nabla_Z Z + nh^s(Z, Z) = h^i(Z, \phi Z) + h^s(Z, \phi Z) + \nabla_Z \omega Z + D^i(X, \omega Z).$$

(5.17)

Then, employing (5.2), we derive

$$\omega \nabla_Z Z + g(Z, Z)nH^s = g(Z, \phi Z)H^i + g(Z, \phi Z)H^s + \nabla_Z \omega Z.$$

(5.18)

By taking the inner product of (5.10) with respect to $\bar{J} H^s$, we obtain

$$g(Z, Z)\bar{g}(H^s, H^s) = \bar{g}(\nabla_Z \omega Z, \bar{J} H^s).$$

(5.19)

Furthermore, one has $\bar{\nabla}_Z \bar{J} H^s = \bar{J} \nabla_Z H^s$ and it implies

$$-A_{\bar{J} H^s} Z + \nabla_Z \bar{J} H^s + D^i(Z, \bar{J} H^s) = -\phi A_{\bar{H}^s} Z - \omega A_{\bar{H}^s} Z + t\nabla_Z H^s + n\nabla_Z H^s + \bar{J} D^i(Z, H^s).$$

(5.20)
Since $\mu$ is invariant and by taking the inner product on both sides with respect to $\omega Z$, we infer
\[
\tag{5.19} \bar{g}(\nabla_Z^*J^*H^*, \omega Z) = -\bar{g}(\omega A_H^*, Z, \omega Z) = -\sin^2 \theta g(A_H^*, Z, Z).
\]
As $\nabla$ is a metric connection, thus we have $(\bar{\nabla}_Z^*\bar{g})(\omega Z, J^*H^*) = 0$, which implies that $\bar{g}(\nabla_Z^*\omega Z, J^*H^*) = -\bar{g}(\nabla_Z^*J^*H^*, \omega Z)$; therefore (5.19) becomes
\[
\tag{5.20} \bar{g}(\nabla_Z^*\omega Z, J^*H^*) = \sin^2 \theta g(A_H^*, Z, Z).
\]
Then, using (5.20) in (5.17), we derive
\[
\tag{5.21} g(Z, Z)\bar{g}(H^*, H^*) = \sin^2 \theta g(A_H^*, Z, Z).
\]
Now, employing (2.7), the above equation yields
\[
g(Z, Z)\bar{g}(H^*, H^*) = \sin^2 \theta \bar{g}(h^*(Z, Z), H^*) = \sin^2 \theta g(Z, Z)\bar{g}(H^*, H^*),
\]
it implies that
\[
(1 - \sin^2 \theta)g(Z, Z)\bar{g}(H^*, H^*) = 0.
\]
Since $N$ is a proper screen bi-slant lightlike submanifold, therefore $\sin^2 \theta \neq 1$ and from the non-degeneracy of $D_i$ for $i = 1, 2$, we derive
\[
\tag{5.22} H^* = 0.
\]
Hence, the result follows from Eqs. (5.16) and (5.22).

**Theorem 5.3.** For a proper totally umbilical screen bi-slant lightlike submanifold $N$ of an indefinite Kaehler manifold $\bar{N}$, $\nabla$ is always a metric connection.

*Proof.* The proof follows directly from Eqs. (5.16) and (2.11).

**Theorem 5.4.** There does not exist any proper totally umbilical screen bi-slant lightlike submanifold in $\bar{N}(c)$, provided, $c \neq 0$.

*Proof.* For $N' \in \Gamma(ltr(TN))$, $Y \in \Gamma(S(TN))$ and $\xi \in \Gamma(Rad(TN))$, using (2.16), we infer
\[
\tag{5.23} \bar{g}(\mathcal{R}(Y, J^*Y)N', \xi) = -\frac{c}{2} \bar{g}(\mathcal{J}N', \xi)g(Y, Y).
\]
Moreover, using (2.12), we attain
\[
\tag{5.24} -\bar{g}(\mathcal{R}(Y, J^*Y)N', \xi) = \bar{g}((\nabla_{J^*Y}h')(Y, N'), \xi) - \bar{g}((\nabla_Y h')(J^*Y, N'), \xi).
\]
Then, from (5.23) and (5.24), we get
\[
\tag{5.25} -\frac{c}{2} \bar{g}(\mathcal{J}N', \xi)g(Y, Y) = -\bar{g}((\nabla_{J^*Y}h')(Y, N'), \xi) + \bar{g}((\nabla_Y h')(J^*Y, N'), \xi).
\]
Since $N$ is totally umbilical; therefore using (5.2), we have
\[
\tag{5.26} (\nabla_Y h')(J^*Y, N') = -\{g(\nabla_Y J^*Y, N') + g(J^*Y, \nabla_Y N')\}H^1.
\]
For \( N' \in \Gamma(\text{ltr}(TN)) \) and \( Y \in \Gamma(S(TN)) \), one has \( \bar{g}(\bar{J}Y, N') = 0 \); then taking covariant derivative with respect to \( Y \), we get \( g(\bar{J}Y, \nabla_Y N') = -g(\nabla_Y \bar{J}Y, N') \). Thus (5.26) reduces to

\[
(5.27) \quad (\nabla_Y h^l)(\bar{J}Y, N') = 0.
\]

Similarly, it follows that

\[
(5.28) \quad (\nabla_{\bar{J}Y} h^l)(Y, N') = 0.
\]

Then, using (5.27) and (5.28), the relation (5.25) yields

\[
-\frac{c}{2} g(Y, Y) g(\bar{J}N', \xi) = 0.
\]

Hence, the non-degeneracy of \( S(TN) \) implies that \( c \neq 0 \), which proves the result. \( \square \)

References


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