Contactomorphisms and Lychagin 1-form

Fidele Balibuno Luganda, Alain Musesa Landa

To the memory of Professor Temo Beko de Loso, who taught us Differential Geometry

Abstract. In [5], Lychagin constructed a smooth chart from the group of contactomorphisms. From this chart, Lychagin showed that the connected component of the identity of the group of contactomorphism is arc wise connected. In [2], it has been shown that the contactomorphism group enjoys the fragmentation property. Herein, we introduce the Lychagin 1-form and the relation between the contactomorphism and the Lychagin 1-form defined on the 1-jet bundle. By the way, we characterize the contact diffeomorphisms by means of the closedness of the Lychagin 1-form. This characterization is known by geometers long time ago through the joint work of Sniatyki and Tulczjew [8]. We give the same characterization in this paper using the geometry of Legendre submanifolds through the Lychagin chart. This last result is the analogue of the Sniatyki-Tulczjew characterization of symplectomorphisms in contact geometry[1].

M.S.C. 2010: 53D10, 70H15, 53D12.
Key words: Lychagin 1-form; Legendre submanifold; contactisation; Lychagin chart; Lychagin neighbourhood.

1 Introduction

Contact geometry is an odd dimension geometry living on an odd dimension manifold. It is the odd counterpart of sympletic geometry. It arose in the works of Christian Huygens, Henri Poincaré, Henri Cartan, in the 1800^{ths}.
In this paper, we focus on the local geometry of the contactomorphisms group.
In other words, we study the local behaviour of the group of contact diffeomorphisms at the identity.
By the way, we study the Lychagin chart by Legendre submanifold geometry seen, in the contact setting, as the analogue of the Weinstein chart for Lagrangian submanifolds.

Balkan Journal of Geometry and Its Applications, Vol.27, No.1, 2022, pp. 11-23.

In fact, A. Weinstein[[9], [10]] generalized the Darboux and the Moser theorems. He made the observation that the local geometry of Lagrangian submanifolds is governed by the zero section of the cotangent bundle.

In counterpart, Lychagin, inspired by the work of A.Weinstein, showed that the local geometry of Legendre submanifolds is governed by the zero section of the 1-jet bundle $J^1(M)$ over a smooth manifold M.

In fact, it is well known by geometers longtime ago that the Weinstein chart is usefull in the fragmentation of the symplectomorphism group. [2]
Indeed, it allows to prove that the group of symplectomorphisms is arc wise connected.

In counterpart, geometers have also shown the usefulness of the Lychagin chart in the fragmentation of the contactomorphisms group.

Indeed, they have shown that the group of contactomorphisms is arc wise connected. Their proof rely on the Lychagin chart.
However, in this paper, our objectives are twofold:
the first is to state and prove the new characterization of the Sniatyki-Tulczjew theorem of the Lychagin chart,
the second is to link the Lychagin chart to the contact flux homomorphism in the flux geometry setting.

This paper is organized as follows:

1. Introduction
2. A brief review of contact geometry and Lychagin chart
3. Statement of the main results
4. Contact flux homomorphism associated with the Lychagin chart
5. Conclusion and Perspective

2 A brief review of contact geometry and the Lychagin chart

Definition 2.1. A contact form on a $(2n+1)$-dimensional manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ (n times) is a volume form.
The pair (M, α) with M a $C^\infty - (2n+1)$ manifold and α a contact 1-form is called a contact manifold.

Definition 2.2. Two contact forms α_1 et α_2 are said to be equivalent if there is a positive function μ such that $\alpha_1 = \mu\alpha_2$. An equivalence class of contact forms is called a contact structure.

Theorem 2.1. *(Reeb) Let α be a contact 1-form on a C^∞-manifold M of odd dimension. There exists a unique vector field ζ on M such that:*

$$i(\zeta)\alpha = 1$$
$$i(\zeta)d\alpha = 0$$

Definition 2.3. The vector field ζ is the characteristic vector field called the *Reeb vector field* associated to the contact 1-form α

Definition 2.4. A contact structure is defined as a hyperplane field $E = ker\alpha \subset TM$.

The following proposition gives an example of a contact manifold.

Proposition 2.2. *Let (M_1, α_1) and (M_2, α_2) be contact manifolds. Then, the product*

$\hat{M} = M_1 \times M_2 \times \mathbb{R}^+$ is a contact manifold with contact 1-form $\hat{\alpha} = \lambda\pi_1^*\alpha_1 + \mu\pi_2^*\alpha_2$.

The π_i $(i = 1, 2)$ are the projections on the i^{th} factor.
In particular, the product $\hat{M} = M \times M \times \mathbb{R}^+$ is a contact manifold with contact 1-form $\hat{\alpha} = \lambda\pi_1^*\alpha - \pi_2^*\alpha$.
It is clear, by a straightforward calculations, that the Reeb vector field $\hat{\zeta}$ of $\hat{\alpha} = \lambda\pi_1^*\alpha + \mu\pi_2^*\alpha$ describes the dynamic of the contact manifold $\hat{M} = M_1 \times M_2 \times \mathbb{R}^+$, i.e, the Lie derivative of $\hat{\alpha}$ in the direction of $\hat{\zeta}$ gives $L_{\hat{\zeta}}\hat{\alpha} = 0$.

Among the other examples of contact manifolds is the 1-jet bundle over a C^∞-manifold denoted by $\pi : J^1(M) \longrightarrow M$. It plays a crucial role in the construction of the Lychagin chart.

Example 2.5. Let λ_M be the Liouville 1-form on the cotangent bundle T^*M. The pair $(J^1(M), \alpha^* = \pi_1^*\lambda_M + dz)$ is a contact manifold whose contact 1-form is $\alpha^* = \pi_1^*\lambda_M + dz$.

The above procedure stands for the contactisation of the cotangent bundle T^*M.

Definition 2.6. Let (M, α) be a contact manifold. A diffeomorphism $\varphi : M \longrightarrow M$ is said to be a contactomorphism if $\varphi^*\alpha = \lambda\alpha$ with λ a strictly positive function. The set of contactomorphisms is a group of infinite dimension denoted by $Diff_\alpha^\infty(M)$.
V. Lychagin [5] and O. Spăcil [7] have shown that this group is arcwise connected using Legendre submanifolds geometry we explore in the sequel.
Denote by $\Gamma_\varphi = \{(x, y, z) : y = \varphi(x)\}$ the graph of the diffeomorphism φ. About Legendre submanifolds, we have:

Definition 2.7. Let N be a submanifold of the contact manifold (M, α). An immersion $j : N \hookrightarrow M$ is said to be a Legendre immersion if $j^*\alpha = 0$ and $dimN = n$.
The submanifold $j(N)$ is called a Legendre submanifold and is everywhere tangent to the contact structure.
In this paper, we state and prove the analogue, in contact setting, of the characterization of contactomorphisms by Legendre submanifolds. Notice that the theorem of characterization of symplectomorphisms have been stated and proved by Sniatyki and W. Tulczjew in their joint work [8]. Their result have been extended to Weinstein 1-form in [1].

Theorem 2.3. *(Sniatyki-Tulczjew)*
A diffeomorphism φ is a contactomorphism iff Γ_ϕ is a Legendre submanifold.

Proof. 1. The condition is necessary:

Assume that, $j : M \hookrightarrow \Gamma_\varphi \subset M \times M \times \mathbb{R}_+^*$ is an immersion of the graph Γ_φ into the contact manifold $M \times M \times \mathbb{R}^+$. Then, setting $\hat{\alpha} = \lambda\pi_1^*\alpha - \pi_2^*\alpha$, we have:

$$\begin{aligned} j^*\hat{\alpha} &= j^*(\lambda\pi_1^*\alpha - \pi_2^*\alpha) \\ &= \lambda j^*\pi_1^*\alpha - j^*\pi_2^*\alpha \\ &= \lambda(\pi_1 \circ j)^*\alpha - (\pi_2 \circ j)^*\alpha \\ &= \lambda\alpha - \phi^*\alpha \\ &= \lambda\alpha - \lambda\alpha \\ &= 0 \end{aligned}$$

i.e., the graph Γ_φ is a Legendre submanifold.

2. The condition is sufficient:

Assume, conversely, that Γ_φ is a Legendre submanifold. Let show that the diffeomorphism φ is a contactomorphism. A straightforward calculation shows that:

$$\begin{aligned} 0 = j^*\hat{\alpha} &= j^*(\lambda\pi_1^*\alpha - \pi_2^*\alpha) \\ &= \lambda j^*\pi_1^*\alpha - j^*\pi_2^*\alpha \\ &= \lambda(\pi_1 \circ j)^*\alpha - (\pi_2 \circ j)^*\alpha \\ &= \lambda\alpha - \varphi^*\alpha \end{aligned}$$

i.e., $\varphi^*\alpha = \lambda\alpha$. In other words, φ is a contactomorphism.

□

V. Lychagin [5] and T. Duchamp [4] investigated further the Legendre submanifolds geometry. In particular, when the sections of the 1-jet bundle $J^1(M)$ are holonomic, i.e, the 1-jet of a function. V. Lychagin stated and proved the following theorem:

Definition 2.8. Let $f : P \longrightarrow \mathbb{R}$ be a smooth function. A section $s_f : P \longrightarrow J^1P$ is said to be holonomic if $s_f(x) = j^1(f)(x)$

Before proving it, let state and prove the following lemma:

Lemma 2.4. *The assignement σ defined by $\sigma(\varphi)(s)(p) = J_p^1(f_{\varphi(p)})$ is a covariant functor.*

Proof. 1. $\sigma(id) = id_{\sigma(p)}$
Since we have:

$$\begin{aligned} \sigma(id_p)(s)(p) &= J_p^1(f_p) \\ &= s(p) \\ &= id_{\sigma(p)}(s)(p) \end{aligned}$$

Therefore:

$$\sigma(id_p) = id_{\sigma(p)}$$

2. Let show that:
   $\sigma(\varphi \circ \psi) = \sigma(\psi) \circ \sigma(\varphi)$
   we have:

$$\begin{aligned}\sigma(\varphi \circ \psi)(s)(p) &= J^1_p(f_{(\varphi\circ\psi)(p)} \\ &= J^1_p(f_{\varphi(\psi(p))}) \\ &= \sigma(\psi)[\sigma(\varphi)(s)(p)] \\ &= [\sigma(\psi) \circ \sigma(\varphi)](s)(p)\end{aligned}$$

   i.e., $\sigma(\varphi \circ \psi) = \sigma(\psi) \circ \sigma(\varphi)$. Therefore, σ is a covariant functor.

□

Theorem 2.5. *(Lychagin) [2] The image $s(P) \subset J^1P$ of a section $s : P \longrightarrow J^1P$ is a Legendre submanifold if and only if it is holonomic.*

Proof. The condition is necessary:

Assume that the image $s(P) \subset J^1P$ of a section $s : P \longrightarrow J^1P$ is a Legendre submanifold. Let show that the image $s(P) \subset J^1P$ is holonomic.
So, let $\sigma(P)$ denote the space of all sections of the 1-jet bundle $\pi : J^1P \longrightarrow P$ over a smooth manifold P.

For $s \in \sigma(P)$ and $p \in P$, there exists a germ f_p of smooth function defined near p such that $s(p)$ is the 1-jet of f_p at p.
We write $s(p) = j^1_p(f_p)$.

Therefore, by definition, the Legendre submanifold $s(P)$ is holonomic.

The condition is sufficient:

Suppose the image $s(P) \subset J^1P$ of a section $s : P \longrightarrow J^1P$ is holonomic and let α_p be the contact 1-form on J^1P. Let show that the image $s(P)$ is a Legendre submanifold i.e., $s^*\alpha_p = 0$.

We use herein the functorial arguments and lemma [2.4]. So, let the covariant functor σ be defined by the relation:

$\sigma(\varphi)(s)(p) = j^1_{\varphi(p)}(f_{\varphi(p)})$ for $\varphi \in \sigma(Q)$, $s \in \sigma(P)$ and $p \in P$.

In particular let $s : P \longrightarrow J^1P$ be a section of J^1P inducing a covariant functor $\sigma(s) : \sigma(J^1P) \longrightarrow \sigma(P)$ which is one to one i.e., for every $s \in \sigma(P)$, there exists one and only one $U \in \sigma(J^1P)$, $\sigma(s)(U) = s$.

Notice that we have the decompositon
$\sigma(P) = \Omega^1(P) \oplus C^\infty(P)$ since the locally trivial bundles category admits the

direct sums.
The universal section U assumes the following expression:

$$U(x) = (\sum_i p_i dq_i, u)$$

So, let D be a natural transformation. Using the Darboux coordinates, we have:

$$\begin{aligned} D(U) &= \sum_i p_i dq_i - du \\ &= \alpha_p \text{ and since} \end{aligned}$$

$s = \sigma(s)(U)$, it is a well known fact that $s = j^1 f$ if and only if $s \in KerD$.

Moreover we have the following commutative diagram:

$$\begin{array}{ccc} \sigma(J^1 P) & \xrightarrow{D} & \wedge^1(J^1(P)) \\ {\scriptstyle \sigma(s)}\downarrow & & \downarrow{\scriptstyle s^*} \\ \sigma(P) & \xrightarrow[D]{} & \wedge^1(P) \end{array}$$

Figure 1:

$$\text{i.e., } D \circ \sigma(s) = s^* D$$

Therefore,

$$\begin{aligned} 0 = D(s) &= D \circ \sigma(s)(U) \\ &= s^* D(u) = s^* \alpha_p \end{aligned}$$

i.e., $s^* \alpha_p = 0$.

□

A local model of Legendre submanifold of the contact manifold $M \times M \times \mathbb{R}^+$ is provided by the diagonal:

$\Delta \times \{1\} = \{(x, x, 1) \in M \times M \times \mathbb{R}^+, \varphi = id\}$ where Δ stands for the diagonal in $M \times M$.

It procures another example of a Legendre submanifold of $M \times M \times \mathbb{R}^+$.
Herein, we call the Sniatyki-Tulczjew theorem the first characterization of contactomorphism by Legendre submanifold and we look for another characterization of contactomorphisms by means of the closedness of the Lychagin 1-form, we explain below.
However, whenever ϕ is a contactomorphism isotopic to the identity, the Legendre immersion will be denoted by the pair (id, φ).

3 Statement of the main results

3.1 One form as sections of the cotangent bundle [1]

Lagrangian submanifolds of the cotangent bundle are obtained this way:

Theorem 3.1. *Let $\alpha : M \longrightarrow T^*M$ be a 1-form.*
The following statements are equivalent:

1) The image $\alpha(M) \subset T^*M$ of a 1-form α is a lagrangian submanifold of T^*M.

2) α is a closed 1-form.

We have already demonstrated this theorem in [1]. Herein, we prove it by functorial arguments to shorten the proof. We need the following lemma:

Lemma 3.2. *Let α be a 1-form on T^*M, σ a functor and d the De Rham exterior differential. We denote by λ_M the Liouville 1-form on T^*M.*

Then the formula

$$\alpha^* \circ d_{T^*M} = d_M \circ \sigma(\alpha)$$

holds.

Proof. We follow the proof of [[2],p.144].

Let $\sigma(M)$ denote the space of all sections of the cotangent bundle $\pi : T^*M \longrightarrow M$ over a smooth manifold M.
Any section $\alpha : M \longrightarrow T^*M$ of π induces a mapping $\sigma(\alpha) : \sigma(T^*M) \longrightarrow \sigma(M)$ which is one-to-one [[2],p.144].

Hence, there is a unique 1-form λ_M called the Liouville 1-form such that $\sigma(\alpha)(\lambda_M) = \alpha$ and such that the following diagram is commutative:

$$\begin{array}{ccc} \sigma(T^*M) & \xrightarrow{d_{T^*M}} & \Omega^1(T^*M) \\ {\scriptstyle \sigma(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ \sigma(M) & \xrightarrow[d_M]{} & \Omega^1(M) \end{array}$$

Figure 2:

i.e.,

$$\alpha^* \circ d_{T^*M} = d_M \circ \sigma(\alpha) \tag{3.1}$$

Therefore:

$$\begin{aligned} \alpha^* d_{T^*M} \lambda_M &= d_M \sigma(\alpha)(\lambda_M) \\ &= d_M \alpha \end{aligned}$$

In other words, $\alpha^*\Omega_M = d_M\alpha$.
Herein, the differential d stands for the natural transformation and $\Omega_M = d\lambda_M$ is the standard symplectic structure on the cotangent bundle T^*M. □

Proof of the theorem 3.1

1. The condition is necessary:

   Assume that $\alpha : M \longrightarrow T^*M$ is a lagrangian immersion. In other words, we suppose that the image $\alpha(M)$ is a lagrangian submanifold. By the formula (3.1), we deduce that: $0 = \alpha^*\Omega_M = d\alpha$ i.e., α is a closed 1-form.

2. The condition is sufficient:

   Assume, conversely, that α is a closed 1-form. Then, $\alpha^*\Omega_M = d\alpha = 0$ i.e., $\alpha^*\Omega_M = 0$ i.e., α is a lagrangian immersion.

   In other words, the image $\alpha(M)$ is a lagrangian submanifold.

3.2 The Lychagin chart

A. Weinstein exhibited the Darboux-Moser theorem. In fact, this theorem is the well known Lagrangian tubular neighborhood theorem called in [2] the Kostant-Weinstein-Sternberg theorem. Through it, he showed that the local model for Lagrangian submanifold of symplectic manifolds is the zero section of the cotangent bundle.

Following these ideas, V. Lychagin adapted these techniques to show that the local model for Legendre submanifolds is the zero section of the manifold $J^1(M)$ of 1-jet bundle.
In particular, he proved the following theorem which is the Legendre tubular neighborhood theorem and a generalization of the Gray-Martinet stability theorem in contact geometry setting.

Theorem 3.3. *(Lychagin) [2] Let L be a Legendre submanifold of a contact manifold (M, α). Then, there is a diffeomorphism*
$\psi : U \longrightarrow W$ where U is a tubular neighborhood of the zero section $s_0 : L \longrightarrow J^1(L)$ in $J^1(L)$, W is the tubular neighborhood of the zero section in $M \times M \times \mathbb{R}^+$, $\psi/L_0 = id_M$ and $\psi^*\theta = \hat{\alpha}$ where θ is the standard 1-form on $J^1(M)$ and $\hat{\alpha}$ is the contact 1-form on $M \times M \times \mathbb{R}^+$.
The map ψ is called the Kostant map.
The material of the following theorem can be found in [2].

Theorem 3.4 ([2], [3])**.** *Let $\mathcal{L}_\alpha(M)$ be the Lie algebra of contact fields and φ be a contactomorphism isotopic to the identity in the C^1-topology. Let f_ϕ belongs to $C^\infty(M)$ and assume that $\zeta : \mathcal{L}_\alpha(M) \longrightarrow C^\infty(M)$ is a Lie algebra isomorphism. Then there exists a smooth chart defined by:*

$$\Omega : \mathcal{W} \subset Diff_\alpha(M) \longrightarrow L_\alpha(M)$$
$$\varphi \rightsquigarrow \Omega(\varphi) = \zeta^{-1}(f_\varphi)$$

Proof. Let φ be a contactomorphism isotopic to the identity in the C^1topology. Then the Legendre submanifold Γ_φ is close enough to the diagonal $\Delta \times \{1\}$. Since ψ is a contactomorphism, it preserves the Legendre submanifolds i.e., $\psi(\Gamma_\varphi)$ is a Legendre submanifold in $J^1(M)$ close enough to the zero section of the 1-jet bundle $J^1(M)$. A. Banyaga, in [2] defined the Lychagin chart by means of the map $\Omega(\varphi) = \zeta^{-1}(f_\varphi)$ i.e., by contact vector field.

In this work, however, we define the Lychagin chart by a closed 1-form we further explain in the sequel. □

The local geometry of the contactomorphisms group is described this way:

Theorem 3.5. *Let φ be a contactomorphism C^1-close to the identity and let the pair (Ω, ν) be the Lychagin chart with ν the Lychagin neighborhood in $Diff^\infty_\alpha(M)_0$. Then, at the identity, $\Omega(id_M) = 0_M$ with 0_M the zero section of $J^1(M)$.*

Proof. The existence of the Lychagin chart follows from the construction below. In fact, a straightforward calculation gives the following commutative diagram:

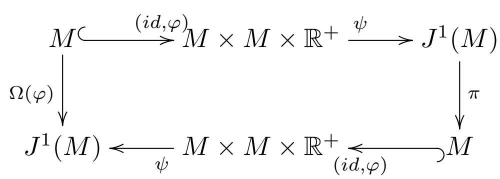

Figure 3: Existence of the Lychagin chart.

i.e, $\Omega(\varphi) = (\psi \circ (id, \varphi)) \circ (\pi \circ \psi \circ (id, \varphi))$. Hence, at the identity, we have, that the local geometry of the Lychagin chart, is given by:

$$\begin{aligned}\Omega(id_M) &= (\psi \circ (id, id)) \circ (\pi \circ \psi \circ (id, id)) \\ &= \psi \circ 0_M \circ \pi \circ \psi \circ 0_M \\ &= 0_M\end{aligned}$$

□

We apply [3.1] to state and prove the following:

Theorem 3.6. *Let α be a contactomorphism isotopic to the identity in the C^1-topology. Then, the Lychagin 1-form $\Omega(\varphi)$ is given by the formula:*

$$\Omega(\varphi) = \lambda\alpha - \varphi^*\alpha$$

Proof. Assume that the pair (id, φ) is a Legendre immersion and ψ is the Kostant map in contact geometry setting. Set $\theta_1 = \psi^*\hat{\alpha}$ with $\hat{\alpha}$ the contact 1-form on $M \times M \times \mathbb{R}^+$. One observes that the triangle below is commutative:

i.e., the Legendre immersion is given by the formula:

$$(id, \varphi) = \psi \circ \Omega(\varphi).$$

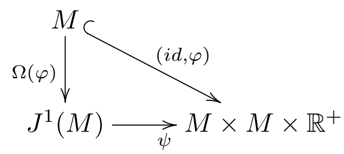

Figure 4:

Then, let $\hat{\alpha} = \lambda\pi_1^*\alpha - \pi_2^*\alpha$ be the contact 1-form on $M \times M \times \mathbb{R}^+$ with π_i $(i = 1, 2)$, the i^{th} projection.

A straightforward calculation gives:

$$\begin{aligned}
\Omega(\varphi)^*\theta_1 &= \Omega(\varphi)^*(\psi)^*\hat{\alpha} \\
&= (\psi \circ \Omega(\varphi))^*(\lambda\pi_1^*\alpha - \pi_2^*\alpha) \\
&= (id, \varphi)^*(\lambda\pi_1^*\alpha - \pi_2^*\alpha) \\
&= (id, \varphi)^*(\lambda\pi_1^*\alpha) - (id, \varphi)^*\pi_2^*\alpha \\
&= \lambda(\pi_1 \circ (id, \varphi))^*\alpha - (\pi_2 \circ (id, \varphi))^*\alpha \\
&= \lambda\alpha - \varphi^*\alpha \\
&= \Omega(\varphi)
\end{aligned}$$

□

Therefore, the relation between the contactomorphism and the Lychagin 1-form is given by the formula:

$$\Omega(\varphi) = \lambda\alpha - \varphi^*\alpha$$

This formula can also be obtained by a family of contact vector fields whose flows are contactomorphisms isotopic to the identity and through the formula of advanced calculus [[2], [6]].

Hence, we see the link between the Lychagin 1-form and contactomorphism. Indeed, we have restated and proved the homotopy formula and obtained the new characterization of contactomorphisms by means of the Lychagin 1-form. We have obtained the following result, thanks to A. Weinstein and V. Lychagin. In the sequel, the 1-form $\Omega(\varphi)$ is called the Lychagin 1-form.

So, among our main results, we have this new characterization of contactomorphisms.

Theorem 3.7. *Let φ be a diffeomorphism C^1-close to the identity so that it's graph Γ_φ is close enough with the diagonal $\Delta \times \{1\}$.*
Then, φ is a contactomorphism if and only if the Lychagin 1-form $\Omega(\varphi)$ is a closed 1-form.

Proof. 1. The condition is necessary:

Assume tha $\Omega(\varphi)$ is closed. Then, we have:

$$\begin{aligned}
0 = d\Omega(\varphi) &= d(\lambda\alpha - \varphi^*\alpha) \\
&= \lambda d\alpha - d\varphi^*\alpha \\
&= \lambda d\alpha - \varphi^* d\alpha
\end{aligned}$$

i.e., $\varphi^* d\alpha = \lambda d\alpha$. In other words, $\varphi^*\alpha = \lambda\alpha$. Therefore, φ is a contactomorphism.

2. The condition is sufficient:
Assume that φ is a contactomorphism i.e.,

$$\begin{aligned} \varphi^*\alpha &= \lambda\alpha \\ \Rightarrow d\varphi^*\alpha &= \lambda d\alpha \\ \text{Hence, we have: } 0 &= \lambda d\alpha - d\varphi^*\alpha \\ &= \lambda d\alpha - \varphi^* d\alpha = d(\lambda\alpha - \varphi^*\alpha) \\ &= d\Omega(\phi) \end{aligned}$$

i.e., $d\Omega(\varphi) = 0$ which means that the Lychagin 1-form is closed.
Thus, the De Rham cohomology class $[\Omega(\varphi)]$ of the Lychagin 1-form $\Omega(\varphi)$ is non trivial.
Therefore, the non trivial cohomology class of the Lychagin 1-form is an obstruction to the diffeomorphism φ to be a contactomorphism. We asked whether the above formula agrees with the local geometry of the Lychagin chart. We have the following corollary:

□

Corollary 3.8. *At the identity:*

$$\Omega(id_M) = 0_M$$

where 0_M *is the zero section of the 1-jet bundle.*

Proof. At the identity and since $\Omega(\varphi) = \lambda\alpha - \varphi^*\alpha$; with $\varphi = id$, we have:

$$\begin{aligned} \Omega(id_M) &= \lambda\alpha - \lambda\alpha \\ &= \lambda\alpha - \lambda id_M^*\alpha \\ &= 0_M \end{aligned}$$

i.e., $\Omega(id_M) = 0_M$.
Thus, again, the formula $\Omega(\varphi) = \lambda\alpha - \varphi^*\alpha$ expresses the local geometry of the Lychagin chart at the identity. □

4 Contact flux homomorphism associated with the Lychagin chart

We introduce the relation between the Lychagin 1-form and the contact flux homomorphism. A Banyaga [2] studied this subject in the flux transverse setting. Herein, we link the flux homomorphism and the Lychagin chart. Our diagrams agree with [[2], page 57] since the Lychagin 1-form is closed.

Proposition 4.1. *Let* α *be a contact form on an odd dimension manifold* M *i.e., the pair* (M, α) *is a contact manifold. Denote by* $\widetilde{Diff}_\alpha^\infty(M)_0$ *the universal cover of* $Diff_\alpha(M)_0$ *and*

$$\pi : \widetilde{Diff}_\alpha^\infty(M)_0 \longrightarrow Diff_\alpha^\infty(M)_0$$

the projection of $\widetilde{Diff}^{\infty}_{\alpha}(M)_0$ *onto* $Diff^{\infty}_{\alpha}(M)_0$.
Let $Z^1_c(M)$ *be the space of closed 1-forms and* $p : Z^1_c(M) \longrightarrow H^1_c(M)$ *the projection of* $Z^1_c(M)$ *into the De Rham cohomology* $H^1_c(M)$ *with compact support. We denote by* Ω *the Lychagin parametrization. Then, the following formula holds:*

$$\tilde{S}_{\alpha} = p \circ \Omega \circ \pi$$

i.e., contactomorphism isotopic to the identity is seen as a Legendre submanifold of contact manifold $\hat{M} = M \times M \times \mathbb{R}^+$.

Proof. We have to prove that the diagram below is commutative:

$$\begin{array}{ccc} \widetilde{Diff}^{\infty}_{\alpha}(M)_0 & \xrightarrow{\pi} & Diff^{\infty}_{\alpha}(M)_0 \\ \downarrow{\scriptstyle \tilde{S}_{\alpha}} & & \downarrow{\scriptstyle \Omega} \\ H^1_c(M) & \xleftarrow[p]{} & Z^1(M) \end{array}$$

Figure 5:

□

In other words,

$$\tilde{S}_{\alpha} = p \circ \Omega \circ \pi$$

So, let $\{\varphi_t\}$ be the homotopy class of the contact isotopy (φ_t). We have, by direct computation;

$$\begin{aligned} (p \circ \Omega \circ \pi)(\{\phi_t\}) &= (p \circ \Omega)(\pi(\varphi_t)) \\ &= p \circ \Omega(\varphi_t) = [\Omega(\varphi_t)] = \tilde{S}_{\alpha}(\{\varphi_t\}). \end{aligned}$$

Therefore, $\tilde{S}_{\alpha} = p \circ \Omega \circ \pi$. Moreover, the contact flux subgroup is $\Gamma_{\alpha} = \tilde{S}_{\alpha}(\pi_1(Diff^{\infty}_{\alpha}(M))_0)$ where $\pi_1(Diff^{\infty}_{\alpha}(M)_0$ is the first homotopy group of $Diff^{\infty}_{\alpha}(M)_0$.
Since $\pi' \circ \tilde{S}_{\alpha} = S_{\alpha} \circ \pi$ [2], in the commutative diagram below, we have:

$$\begin{array}{ccc} \widetilde{Diff}^{\infty}_{\alpha}(M)_0 & \xrightarrow{\tilde{S}_{\alpha}} & H^1_c(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Diff^{\infty}_{\alpha}(M)_0 & \xrightarrow[S_{\alpha}]{} & H^1_c(M)/\Gamma_{\alpha} \end{array}$$

Figure 6:

$$\begin{aligned} \pi' \circ \tilde{S}_{\alpha} &= \pi' \circ (p \circ \Omega \circ \pi) \\ &= (\pi' \circ p \circ \Omega) \circ \pi \\ &= S_{\alpha} \circ \pi \end{aligned}$$

i.e., $S_{\alpha} = \pi' \circ p \circ \Omega$. □

5 Conclusion and perspective

In this work, we have stated and proved the characterization of contactomorphisms using the Lychagin 1-form and the Legendre submanifolds. We obtained the same result in [1] for symplectomorphisms.
We also found new formulas linking the contact flux homomorphism to the Lychagin chart. We suspect that these formulas can be used to show that the contact flux homomorphism kernel is arcwise connected. In our future work, we intend to explore the Weinstein-Lychagin chart in cosymplectic geometry setting.

Acknowledgment

We are grateful to the referees who read this article and to our student Lyaki Ndukukwa who helped us to tape this text in latex.

References

[1] F. Balibuno Luganda, L. Todjihounde, *Symplectomorphisms and Weinstein 1-form*, BJGA, 26, 2 (2021), 13-22.

[2] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, Kluwer Academic Publisher, Volume 400, 41 (6), 1997.

[3] A. Banyaga, A. Mc Inerney, *On Isomorphic Classical Diffeomorphism Groups III*, Annals of Global Analysis and Geometry 13, Kluwer Academic Publishers 1995.

[4] T. Duchamp, *The Classification of Legendre Immersions*, Mathematics, 2015.

[5] V. Lychagin, *Contact Geometry and Non-linear Second-Order Differential Equations*, Russia Math., 1979.

[6] J. Moser, *A Fixed Point Theorem in Symplectic Geometry*, Courant Institute of Mathematical Sciences; New York University, USA, 1978.

[7] Öldrich Spăcil, *On the Chern-Weil theory for transformation groups of contact manifolds*, London, UK, 2014.

[8] J. Sniatyki, W. Tulczjew, *Geometry Forms of Lagrangian Submanifolds*, Indian University Mathematics Journal, 22(3) (1972), 267-275.

[9] A. Weinstein, *Lagrangian Submanifolds and Hamiltonian Systems*, The Annals of Mathematics, 98(3), 377, 1973.

[10] A. Weinstein, *Symplectic Manifolds and Their Lagrangian Submanifolds*, Advances in Mathematics, 6(3); 329-346, 1971.

Authors' address:

Fidele Balibuno Luganda, Alain Musesa Landa
Departement of Mathematic and Computer Science,
Faculty of Science, University of Kinshasa, Kinshasa, DRC.
E-mail: fidelelugandabalibuno@gmail.com , alain.musesa@unikin.ac.cd