# On 2-plectic Lie groups

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**Abstract.** A 2-plectic Lie group is a Lie group endowed with a 2-plectic structure which is left invariant. In this paper we provide some interesting examples of 2-plectic Lie groups. Also we study the structure of the set of Hamiltonian covectors and vectors of a 2-plectic Lie algebra. Moreover, the existence of $i$-isotropic and $i$-Lagrangian subgroups are investigated. At last we obtain some results about the reduction of some 2-plectic structures.

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## 1 Introduction

A $k$-plectic manifold $(M, \omega)$ is a smooth manifold $M$ endowed with a $(k+1)$-form $\omega$ which is closed and nondegenerate in the sense that $\iota_X \omega = 0$, $X \in TM$, implies that $X = 0$. In this case $\omega$ is called a $k$-plectic structure. These structures are general version of symplectic structures and they naturally appear in the Hamiltonian formulation of classical fields (see [4]and references therein ). Mathematically, in spite of general case, symplectic structures are very interesting and so they have been studied extensively. However, in recent years, 2-plectic structures also have been considered and these structures also are studied extensively ([1], [8]). An important class of 2-plectic manifolds are 2-plectic Lie groups. Similar to symplectic case, a 2-plectic Lie group is defined as follows.

**Definition 1.1.** A 2-plectic Lie group $(G, \omega)$ is a Lie group $G$ endowed with a 2-plectic structure $\omega$ which is left invariant.

In contrast to the symplectic case, there are a lot of canonical 2-plectic structures which are induced by quadratic forms and bialgebra structures (see below). However, symplectic Lie groups have been studied extensively ([2] and references therein). In this article we study 2-plectic Lie groups and since for simply connected Lie groups, the study of 2-plectic Lie groups reduces to the study of 2-plectic Lie algebras, we will study 2-plectic Lie algebras.

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**Definition 1.2.** A 2-plectic Lie algebra $(\mathfrak{g}, \omega)$ is a Lie algebra $\mathfrak{g}$ equipped with a 2-plectic structure $\omega$ which is closed, in the sense that $\omega : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying

$$\begin{aligned} \delta\omega(x, y, u, v) &= \omega([x, y], u, v) - \omega([x, u], y, v) + \omega([x, v], y, u) \\ &\quad -\omega([y, u], x, v) + \omega([y, v], x, u) - \omega(x, y, [u, v]) = 0, \end{aligned}$$

for all $x, y, u, v \in \mathfrak{g}$.

The structure of the paper is as follows: In section 2 we provide some important examples. In section 3 we introduce Hamiltonian covectors and vectors and study their structures. In section 4 we study the existence of isotropic and Lagrangian ideals and subalgebras. In the last section we consider the reduction of some 2-plectic structures.

# 2 Examples

In this section we provide some important examples of 2-plectic Lie groups.

**Definition 2.1.** Let $(\mathfrak{g}, [\, , \,])$ be a Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is called
i) symmetric if $B(x, y) = B(y, x)$, for all $x, y \in \mathfrak{g}$,
ii) nondegenerate if $B(x, y) = 0$, for all $y \in \mathfrak{g}$ implies $x = 0$,
iii) invariant if $B([x, y], z) = -B(y, [x, z])$, for all $x, y, z \in \mathfrak{g}$.

If $B$ is a bilinear form on $\mathfrak{g}$, which is symmetric, nondegenerate and invariant, then it is called a quadratic form and the pair $(\mathfrak{g}, B)$ is called a quadratic Lie algebra. Quadratic form is a generalization of the Killing form. In general, a Lie algebra may not have a quadratic form. However, the existence and classifying quadratic forms on low dimensional Lie algebras has been studied extensively ([7], [6], [3]).
Let $(\mathfrak{g}, B)$ be a quadratic non Abelian Lie algebra and define the 3-form $\omega$ on $\mathfrak{g}$ by

$$\omega(x, y, z) = B([x, y], z), \quad x, y, z \in \mathfrak{g}. \tag{2.1}$$

Since $B$ is symmetric and invariant, then $\omega$ is totally skew symmetric and closed. But, in general, $\omega$ is not nondegenerate. Indeed, if $\omega(x, y, z) = 0$, for all $y, z \in \mathfrak{g}$, then $[x, y] = 0$, for all $y \in \mathfrak{g}$. So

$$Ker\, \omega = \{x \in \mathfrak{g} : \iota_x\omega = 0\} = \mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$. However, $\omega$ induces a 2-plectic structure on $\overline{\mathfrak{g}} = \frac{\mathfrak{g}}{\mathfrak{z}(\mathfrak{g})}$.

**Theorem 2.1.** *The 3-form $\bar{\omega}$ on $\overline{\mathfrak{g}}$, defined by*

$$\bar{\omega}(\overline{x}, \overline{y}, \overline{z}) = \omega(x, y, z), \qquad x, y, z \in \mathfrak{g} \tag{2.2}$$

*is a 2-plectic structure, where $\overline{x} = x + \mathfrak{z}(\mathfrak{g})$.*

*Proof.* Let $x_2 = x_1 + x_0, y_2 = y_1 + y_0, z_2 = z_1 + z_0$, where $x_0, y_0, z_0 \in \mathfrak{z}(\mathfrak{g})$. Then

$$\begin{aligned}
\omega(x_2, y_2, z_2) &= \omega(x_1 + x_0, y_1 + y_0, z_1 + z_0) \\
&= \omega(x_1, y_1, z_1) + \omega(x_1, y_1, z_0) + \omega(x_1, y_0, z_1) \\
&\quad + \omega(x_1, y_0, z_0) + \omega(x_0, y_1, z_1) + \omega(x_0, y_1, z_0) \\
&\quad + \omega(x_0, y_0, z_1) + \omega(x_0, y_0, z_0).
\end{aligned}$$

Since $x_0, y_0, z_0 \in \mathfrak{z}(\mathfrak{g})$ and $\omega(x, y, z) = B([x, y], z)$, then all terms of the last equation, except the first one, are zero. Hence, $\omega(x_2, y_2, z_2) = \omega(x_1, y_1, z_1)$. This shows that $\bar{\omega}$ is well-defined. Moreover, if $\bar{\omega}(\overline{x}, \overline{y}, \overline{z}) = 0$, for all $\overline{y}, \overline{z} \in \overline{\mathfrak{g}}$, then $B([x, y], z) = \omega(x, y, z) = 0$, for all $y, z \in \mathfrak{g}$. Thus $x \in \mathfrak{z}(\mathfrak{g})$ and hence $\overline{x} = 0$. At last, since $\omega$ is closed then $\bar{\omega}$ is closed. □

In the following, we will denote this 2-plectic Lie algebra by $(\overline{\mathfrak{g}}, \bar{\omega}, B)$. Using this theorem, we provide some important examples.

**Remark 2.2.** In this paper all Lie algebras are real. However, if a Lie algebra $\mathfrak{g}$ is a complex Lie algebra and $B$ is a complex valued quadratic form on $\mathfrak{g}$, then $Im\, B$ ( imaginary part of $B$) is a quadratic form on $\mathfrak{g}$, when it is considered as a real Lie algebra. So, in this case we can define $\omega$ and $\bar{\omega}$ by $Im\, B$ on $\mathfrak{g}$ (as a real Lie algebra).

**Example 2.3.** Suppose $\mathfrak{g}$ is a semisimple Lie algebra and $B = K$ is its Killing form. Since $\mathfrak{z}(\mathfrak{g}) = 0$, then $K$ induces a 2-plectic structure on $\mathfrak{g}$ itself, defined by (2.1). We will show this 2-plectic Lie algebra by $(\mathfrak{g}, \omega, K)$.

**Example 2.4.** Let $(\mathfrak{g}, [\,,\,])$ be a Lie bialgebra, i.e, a Lie algebra $\mathfrak{g}$ whose dual $\mathfrak{g}^*$ also has a Lie algebra structure $\{\,,\,\}$ which satisfies in a compatibility condition (see [5]). Then $\mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra structure $[\,,\,]_{\mathfrak{d}}$ defined by

$$[x + \alpha, y + \beta]_{\mathfrak{d}} = [x, y] + \{\alpha, \beta\} + ad^*_{\beta} x - ad^*_{x} \beta - ad^*_{\alpha} y + ad^*_{y} \alpha,$$

for all $x + \alpha, y + \beta$, where $ad_{\alpha} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint operator. The Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}^*, [\,,\,]_{\mathfrak{d}})$ is called the double of $\mathfrak{g}$ and it is denoted by $(\mathfrak{d}, [\,,\,]_{\mathfrak{d}})$. There is a natural symmetric and nondegenerate bilinear form $B$ on $\mathfrak{d}$ defined by

$$B(x + \alpha, y + \beta) = \alpha(y) + \beta(x), \qquad \forall x + \alpha, y + \beta.$$

It is well known that $B$ is also invariant (see [5]). Therefor, $(\mathfrak{d}, B)$ is a quadratic Lie algebra and hence $(\overline{\mathfrak{d}}, \bar{\omega}, B)$ is a 2-plectic Lie algebra. When $\mathfrak{g}$ is semisimple, $\mathfrak{d}$ itself is a 2-plectic Lie algebra ([9]).

**Example 2.5.** Let $(\mathfrak{g}, [\,,\,])$ be a simple Lie algebra and denote its Killing form by $K$. For $n > 1$, define $[\,,\,]_n$ on $\mathfrak{g}^n = \mathfrak{g} \oplus ... \oplus \mathfrak{g}$ by

$$[x, y]_n = (z_1, ..., z_n), \qquad z_k = \sum_{j=1}^{k-1} [x_j, y_{k-j}],$$

where $x = (x_1, ....x_n), y = (y_1, ..., y_n)$ are in $\mathfrak{g}^n$. Then $(\mathfrak{g}^n, [\,,\,]_n)$ is a nilpotent Lie algebra ([7]). Define the bilinear form $B$ on $\mathfrak{g}^n$ by

$$B(x, y) = \sum_{j=1}^{n} K(x_j, y_{n-j+1}), \qquad x, y \in \mathfrak{g}^n.$$

The bilinear form $B$ is a quadratic form on $\mathfrak{g}^n$ ([7]). Thus $(\overline{\mathfrak{g}}^n, \bar{\omega}, B)$ is a 2-plectic Lie algebra.

**Example 2.6.** Assume that $(\mathfrak{g}_i, B_i)$, $i = 1, ..., n$, is a quadratic Lie algebra. On $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$ consider the bilinear form $B = B_1 \oplus ... \oplus B_n$ defined by

$$B(x_1 + ... + x_n, y_1 + ... + y_n) = B_1(x_1, y_1) + ... + B_n(x_n, y_n).$$

Obviously, $(\mathfrak{g}, B)$ is a quadratic Lie algebra. Suppose that $\mathfrak{g}_i$ contains a central isotropic element $z_i$, i.e, $B_i(z_i, z_i) = 0$. Let $\mathfrak{j}$ be an ideal in $\mathfrak{g}$ spanned by the elements: $z_1 - z_n, ..., z_{n-1} - z_n$, and $\mathfrak{j}^{\perp}$ be the orthogonal complement of $\mathfrak{j}$ with respect to $B$. Put $\mathfrak{g}' = \frac{\mathfrak{j}^{\perp}}{\mathfrak{j}}$ and define the bilinear form $B'$ on $\mathfrak{g}'$ by

$$B'(\overline{x}, \overline{y}) = B(x, y), \qquad x, y \in \mathfrak{j}^{\perp}.$$

Then $(\mathfrak{g}', B')$ is a quadratic Lie algebra ([6]) which is called amalgamated product of quadratic Lie algebras. So, $(\overline{\mathfrak{g}'}, \bar{\omega}, B')$ is a 2-plectic Lie algebra.

**Example 2.7.** Consider the set $\mathcal{E} = \{e_j = -\frac{1}{2} i\lambda_j : j = 1, ..., 8\}$ as a basis for the Lie algebra $su(3)$ where,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If $\{e^1, ..., e^8\}$ is the corresponding dual basis, then its easy to see that $\omega = \delta(e^1 \wedge e^8)$ is a 2-plectic structure. Since $su(3)$ is simple, then every quadratic form is a scaler multiple of the Killing form. This fact implies that $\omega$ is not induced by a quadratic form. $\omega$ is exact, whereas the 2-plectic structure induced by the Killing form is not exact.

# 3 Hamiltonian covectors and vectors

Let $(\mathfrak{g}, \omega)$ be a 2-plectic Lie algebra. A covector $\alpha \in \mathfrak{g}^*$ is called Hamiltonian if there is a vector $x \in \mathfrak{g}$ such that

$$\delta(\alpha) = \iota_x \omega. \tag{3.1}$$

The vector $x$ satisfying in (3.1) is also called Hamiltonian and this vector is unique if it exists. We denote by $Ham(\mathfrak{g}, \omega)$ the space of all Hamiltonian vectors and by

$Ham^*(\mathfrak{g},\omega)$ the space of all Hamiltonian covectors. Let $\alpha \in \wedge^p\mathfrak{g}^*$ and $x \in \mathfrak{g}$, we recall that $d_x\alpha$ is defined by

$$d_x\alpha(y_1,...,y_p) = \alpha([x,y_1],y_2,...,y_p) + ... + \alpha(y_1,...,y_{p-1},[x,y_p]), \quad y_i \in \mathfrak{g}.$$

Suppose $\alpha,\beta \in \mathfrak{g}^*$ are Hamiltonian covectors and $x,y$ are Hamiltonian vectors related to $\alpha,\beta$ respectively. Define

$$\{\alpha,\beta\} = \iota_y\iota_x\omega. \tag{3.2}$$

**Lemma 3.1.** *Suppose $(\mathfrak{g},\omega)$ is a 2-plectic Lie algebra and $x,y$ are Hamiltonian vectors. Then*

$$d_x\omega(y,u,v) = 0, \qquad \forall u,v \in \mathfrak{g}. \tag{3.3}$$

*Proof.* Choose Hamiltonian covectors $\alpha_x,\alpha_y$ related to $x,y$, respectively. Using (3.2), it is easy to see that $\{\alpha_x,\alpha_y\} = d_y\alpha_x$. So

$$\begin{aligned}\delta(\{\alpha_x,\alpha_y\})(u,v) &= \{\alpha_x,\alpha_y\}([u,v]) \\ &= d_y\alpha_x([u,v]) \\ &= \omega(x,y,[u,v]).\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned}\delta(\{\alpha_x,\alpha_y\})(u,v) &= \delta\circ(d_y\alpha_x)(u,v) \\ &= d_y\circ\delta\alpha_x(u,v) \\ &= d_y\circ\iota_x\omega(u,v) \\ &= \omega(x,[y,u],v) + \omega(x,u,[y,v]).\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\omega(x,y,[u,v]) = -\omega([y,u],x,v) + \omega([y,v],x,u). \tag{3.6}$$

Since $\delta\omega(x,y,u,v) = 0$, then (3.6) proves the statement. □

**Theorem 3.2.** *Suppose $x,y$ are Hamiltonian vectors in $(\mathfrak{g},\omega)$ and $\alpha_x,\alpha_y$ are Hamiltonian covectors related to $x,y$ respectively. Then $[x,y]$ is a Hamiltonian vector related to Hamiltonian covector $\{\alpha_x,\alpha_y\}$. In particular, $Ham(\mathfrak{g},\omega)$ is a Lie subalgebra of $\mathfrak{g}$.*

*Proof.* Since $x,y$ are Hamiltonian, then by (3.3) we have

$$d_x\omega(y,u,v) = 0 = d_y\omega(x,u,v), \qquad \forall u,v \in \mathfrak{g}.$$

So

$$\omega([x,y[,u,v) = -\omega(y,[x,u],v) - \omega(y,u,[x,v]),$$
$$\omega([y,x],u,v) = -\omega(x,[y,u],v) - \omega(x,u,[y,v]).$$

Hence

$$\begin{aligned}0 = \omega([x,y],u,v) + \omega([y,x],u,v) &= -\omega(y,[x,u],v) - \omega(y,u,[x,v]) \\ &\quad -\omega(x,[y,u],v) - \omega(x,u,[y,v]) \\ &= \omega([x,u],y,v) - \omega([x,v],y,u) \\ &\quad +\omega([y,u],x,v) - \omega([y,v],x,u).\end{aligned} \tag{3.7}$$

Since $\delta\omega = 0$, then

$$\begin{aligned} 0 = \delta\omega(x, y, u, v) &= \omega([x, y], u, v) - \omega([x, u], y, v) + \omega([x, v], y, u) \\ &\quad -\omega([y, u], x, v) + \omega([y, v], x, u) - \omega(x, y, [u, v]) \\ &= \omega([x, y], u, v) - \omega(x, y, [u, v]) \qquad (by \quad (3.7)) \end{aligned}$$

Thus $\omega([x, y], u, v) = \omega(x, y, [u, v])$. Now

$$\begin{aligned} \delta(\{\alpha_x, \alpha_y\})(u, v) &= \delta(\iota_y \circ \iota_x\omega)(u, v) \\ &= \omega(x, y, [u, v]) \\ &= \omega([x, y], u, v) \\ &= \iota_{[x,y]}\omega(u, v). \end{aligned}$$

□

**Definition 3.1.** Let $(\mathfrak{g}, \omega)$ be a 2-plectic Lie algebra. A vector $x$ in $\mathfrak{g}$ is called 2-plectic if $\iota_x\omega$ is closed.

**Proposition 3.3.** *If $x$ and $y$ are 2-plectic, then $[x, y]$ is Hamiltonian.*

*Proof.* Since $\delta(\iota_x\omega)(y, u, v) = 0$, then

$$\omega(x, [y, u], v) - \omega(x, [y, v], u) + \omega(x, [u, v], y) = 0. \tag{3.8}$$

Similarly, since $\delta(\iota_y\omega)(x, u, v) = 0$, then

$$\omega(y, [x, u], v) - \omega(y, [x, v], u) + \omega(y, [u, v], y) = 0. \tag{3.9}$$

Adding equations (3.8) and (3.9) implies that

$$-\omega([y, u], x, v) + \omega([y, v], x, u) - \omega([x, u], y, v) + \omega([x, v], y, u) = 0.$$

Thus $\delta\omega(x, y, u, v) = 0$ implies that $\omega(x, y, [u, v]) = \omega([x, y], u, v)$. Now, an argument similar to Theorem 2 proves the statement. □

Let $Symp_2(\mathfrak{g}, \omega)$ denote the set of all 2-plectic vectors in $\mathfrak{g}$.

**Corollary 3.4.** *$Ham(\mathfrak{g}, \omega)$ is an ideal in $Symp_2(\mathfrak{g}, \omega)$.*

**Theorem 3.5.** *If the first cohomology group $H^1(\mathfrak{g})$ is trivial and $Ham(\mathfrak{g}, \omega) = \mathfrak{g}$, then $\omega$ is induced by a symmetric, invariant and nondegenerate bilinear form.*

*Proof.* Since $Ham(\mathfrak{g}, \omega) = \mathfrak{g}$ and $H^1(\mathfrak{g}) = 0$, then for every $x \in \mathfrak{g}$ there is a unique $\alpha_x \in \mathfrak{g}^*$ with $\delta\alpha_x = \iota_x\omega$. Thus we define the bilinear form $B$ as follows

$$B(x, y) = \frac{1}{2}(\alpha_x(y) + \alpha_y(x)), \qquad x, y \in \mathfrak{g}.$$

Obviously, $B$ is symmetric and for $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned} B([x, y], z) &= \tfrac{1}{2}(\alpha_{[x,y]}(z) + \alpha_z([x, y])) \\ &= \tfrac{1}{2}(\{\alpha_x, \alpha_y\}(z) + \delta\alpha_z((x, y))) \\ &= \tfrac{1}{2}(\omega(x, y, z) + \omega(x, y, z)) \\ &= \omega(x, y, z). \end{aligned} \tag{3.10}$$

Similarly, $B(y,[x,z]) = -\omega(x,y,z)$. Then

$$B([x,y],z) + B(x,[y,z]) = 0,$$

and hence $B$ is invariant. Now, let $B(x,y) = 0$ for all $y \in \mathfrak{g}$. Thus $\alpha_x([y,z]) = -\alpha_{[y,z]}(x)$ and hence $\omega(x,y,z) = -\omega(y,z,x)$, for all $y,z \in \mathfrak{g}$. Since $\omega$ is nondegenerate, then $x = 0$. At last (3.10) shows that $\omega(x,y,z) = B([x,y],z)$.

□

**Corollary 3.6.** *If* $(\mathfrak{g},\omega)$ *is a* 2*-plectic Lie algebra with* $Ham(\mathfrak{g},\omega) = \mathfrak{g}$ *and* $H^1(\mathfrak{g}) = 0$*, then* $\mathfrak{g}$ *has trivial centre.*

*Proof.* Since $H^1(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is perfect. Therefore, if $\mathfrak{g}$ has nontrivial centre, it can not admit a quadratic form. □

The following result and Theorem 3.1 show that $Ham^*(\mathfrak{g},\omega)$ is also a Lie algebra.

**Proposition 3.7.** *suppose* $\alpha,\beta,\gamma$ *are Hamiltonian covectors. Then*

$$\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\} + \{\gamma,\{\alpha,\beta\}\} = 0.$$

*Proof.* Assume that $x,y,z$ are Hamiltonian vectors related to $\alpha,\beta,\gamma$ respectively. Then

$$\begin{aligned}\{\alpha,\{\beta,\gamma\}\}(u) &= -d_x(\{\beta,\gamma\})(u)\\ &= \{\beta,\gamma\}([u,x])\\ &= \delta(\{\beta,\gamma\})(u,x)\\ &= \omega([y,z],u,x).\end{aligned}$$

So

$$\begin{aligned}(\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\} + \{\gamma,\{\alpha,\beta\}\})(u) &= \omega([y,z],u,x)\\ &\quad +\omega([z,x],u,y)\\ &\quad +\omega([x,y],u,z)\\ &= \delta\omega(x,y,z,u)\\ &= 0.\end{aligned}$$

In the last equation we use (3.3). □

We end this section by some examples.

**Example 3.2.** Let $\mathfrak{g}$ be a semisimple Lie algebra and $\omega$ be the 2-plectic structure on $\mathfrak{g}$ induced by the Killing form $K$. Since $K$ induces an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$, defined by $x \to K(x,.)$, then $Ham(\mathfrak{g},\omega) = \mathfrak{g}$ and $Ham^*(\mathfrak{g},\omega) = \mathfrak{g}^*$.

**Example 3.3.** Let $(\mathfrak{g},B)$ be a non-Abelian quadratic Lie algebra with centre $\mathfrak{z}$ and $\omega,\bar{\omega}$ be as in (2.1) and (2.2) respectively. For $x \in \mathfrak{g}$, assume that $\alpha_x : \mathfrak{g} \to \mathbb{R}$ is defined by $\alpha_x(y) = B(x,y)$ and Let

$$ann(\mathfrak{z}) = \{x \in \mathfrak{g} : \alpha_x|_{\mathfrak{z}} = 0\}.$$

If $x, y \in ann(\mathfrak{z})$, then

$$\alpha_{[x,y]}(z) = B([x,y],z) = -B(y,[x,z]) = -\alpha_y([x,z]) = 0,$$

for all $z \in \mathfrak{z}$. Therefore, $ann(\mathfrak{z})$ is a sub Lie algebra of $\mathfrak{g}$. For $x \in ann(\mathfrak{z})$ define $\widetilde{\alpha}_x : \overline{\mathfrak{g}} \to \mathbb{R}$ by $\widetilde{\alpha}_x(\overline{y}) = \alpha_x(y)$. If $y_2 - y_1 = y_0 \in \mathfrak{z}$, then

$$\begin{aligned} \alpha_x(y_2) &= \alpha_x(y_1 + y_0) \\ &= \alpha_x(y_1) + \alpha_x(y_0) \\ &= \alpha_x(y_1). \end{aligned}$$

This shows that $\widetilde{\alpha}_x$ is well-defined. Moreover,

$$\begin{aligned} \delta\widetilde{\alpha}_x(\overline{y}, \overline{z}) &= \widetilde{\alpha}_x([\overline{y}, \overline{z}]) \\ &= \alpha_x([y,z]) \\ &= \delta\alpha_x(y,z) \\ &= \omega(x,y,z) \\ &= \bar{\omega}(\overline{x}, \overline{y}, \overline{z}). \end{aligned}$$

Thus $\delta\widetilde{\alpha}_x = \iota_{\overline{x}}\bar{\omega}$. Hence, for $x \in ann(\mathfrak{z})$, $\overline{x}$ is a Hamiltonian vector in $\overline{\mathfrak{g}}$. So, we can define the map $F : ann(\mathfrak{z}) \to Ham(\overline{\mathfrak{g}}, \bar{\omega})$ by $F(x) = \overline{x}$. This map is a Lie homomorphism. We claim that it is also onto. To prove this, let $\overline{x} \in Ham(\overline{\mathfrak{g}}, \bar{\omega})$ and $\widetilde{\alpha}_{\overline{x}}$ be a Hamiltonian covector related to $\overline{x}$. Define the covector $\alpha$ by $\alpha(y) = \widetilde{\alpha}_{\overline{x}}(\overline{y})$. It is trivial that $\alpha|_{\mathfrak{z}} = 0$. Furthermore, there is a vector $x_0$ in $\mathfrak{g}$ such that $\alpha(y) = B(x_0, y)$ and $F(x_0) = \overline{x}$.

**Example 3.4.** Suppose $\omega$ is the 2-plectic structure of Example 2.5 on $su(3)$. Then $Ham(su(3), \omega) = span\{e_1, e_8\}$.

# 4 Isotropic and Lagrangian ideals and Lie subalgebras

Let $(V, \omega)$ be a 2-plectic vector space and $U \subseteq V$ be a subspace. Put

$$\begin{aligned} U^{\perp,1} &= \{v \in V : \omega(v,u,.) = 0, \forall u \in U\}, \\ U^{\perp,2} &= \{v \in V : \omega(v,u,w) = 0, \forall u, w \in U\}. \end{aligned}$$

For $i = 1, 2$, the subspace $U$ is called
1) $i$-isotropic if $U \subseteq U^{\perp,i}$,
2) $i$-coisotropic if $U^{\perp,i} \subseteq U$,
3) $i$-Lagrangian if $U^{\perp,i} = U$.
In this section we study existence of ideals and Lie subalgebras, which are $i$-isotropic, $i$-coisotropic or $i$-Lagrangian. We start by the following easy result.

**Proposition 4.1.** *Suppose $(\mathfrak{g}, \omega)$ is a 2-plectic Lie algebra and $\mathfrak{j}$ is an ideal.*
*1) If $\mathfrak{j}$ is 1-isotropic, then $\mathfrak{j}$ is Abelian.*

*2)* $\mathfrak{j}^{\perp,1}$ *is a subalgebra of* $\mathfrak{g}$.
*3) If* $\mathfrak{j}^{\perp,1}$ *is an ideal, then* $[\mathfrak{j}^{\perp,1}, \mathfrak{j}] = 0$.
*4) If* $\mathfrak{j}$ *is* 2*-coisotropic, then* $\mathfrak{j}^{\perp,2}$ *is a sub Lie algebra. In this case, the quotient* $\frac{\mathfrak{j}^{\perp,2}}{\mathfrak{j}}$ *carries a natural* 2*-plectic structure.*

*Proof.* These results are easy consequences of the fact that $\omega$ is exact. □

**Theorem 4.2.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $\omega$ *be the* 2*-plectic structure induced by the Killing form.*
*1)* $\mathfrak{g}$ *has no* 2*-coisotropic ideal and Lie subalgebra.*
*2)* $\mathfrak{g}$ *has a* 1*-lagrangian Lie subalgebra.*

*Proof.* 1) Consider the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_n$, where $\mathfrak{g}_i$, $i = 1, ..., n$ is a simple Lie algebra. Then every ideal $\mathfrak{j}$ of $\mathfrak{g}$ is a direct sum of $\mathfrak{g}_i$s. Suppose $A \subseteq \{1, 2, ..., n\}$ and $\mathfrak{j} = \oplus_{i\in A}\mathfrak{g}_i$ is an ideal. Since $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, then $\mathfrak{j}^{\perp,2} = \oplus_{i\in A^c}\mathfrak{g}_i$, where $A^c$ is the complement of $A$. This shows that $\mathfrak{j}$ is not 2-coisotropic. Similarly, every Lie subalgebra $L$ of $\mathfrak{g}$ is a direct sum of Lie subalgebras $L_i$ of $\mathfrak{g}_i$. So, Let $A$ be as above and $L = \oplus_{i\in A}L_i$ be a Lie subalgebra. Since $L \subseteq \oplus_{i\in A}\mathfrak{g}_i$, then $L^{\perp,2} \supseteq (\oplus_{i\in A}\mathfrak{g}_i)^{\perp,2} = \oplus_{i\in A^c}\mathfrak{g}_i$. So $L^{\perp,2}$ is not contained in $L$.
2) If $L$ is a Lie subalgebra of $\mathfrak{g}$, then its easy to see that $L^{\perp,1}$ is the centralizer of $L$. So, if $L$ is 1-Lagrangian, then it is Abelian. In particular, every maximal Abelian Lie subalgebra is 1-Lagrangian. Hence the maximal Abelian Lie subalgebra containing the Cartan subalgebra is 1-Lagrangian. □

**Example 4.1.** Suppose $L$ is the Lie subalgebra of $su(n)$ generated by all diagonal traceless matrices. Then $L$ is a maximal Abelian Lie subalgebra and hence $L$ is 1-Lagrangian. Similarly, Lie subalgebra $L$ of $sl(2n, \mathcal{C})$ consisting of all matrices of the form $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, with $A$ any $n \times n$ matrix, is a maximal Abelian Lie subalgebra. So, $L$ is a 1-Lagrangian Lie subalgebra with respect to the 2-plectic structure on $sl(2n, \mathcal{C})$ induced by the imaginary part of the Killing form.

**Theorem 4.3.** *Let* $(\bar{\mathfrak{g}}, \bar{\omega}, B)$ *be a* 2*-plectic Lie algebra, where* $(\mathfrak{g}, B)$ *is a quadratic solvable (or nilpotent) non Abelian Lie algebra. Then* $\bar{\mathfrak{g}}$ *has a* 2*-coisotropic ideal* $\bar{\mathfrak{j}}$ *such that* $\bar{\mathfrak{j}}^{\perp,2}$ *is also an ideal.*

*Proof.* At first, let $\bar{\mathfrak{j}} \subseteq \bar{\mathfrak{g}}$ be an arbitrary ideal and $\mathfrak{j} = \pi^{-1}(\bar{\mathfrak{j}})$, where $\pi : \mathfrak{g} \to \bar{\mathfrak{g}}$ is the canonical projection. Then

$$\begin{aligned} \bar{\mathfrak{j}}^{\perp,2} &= \{\overline{x} \in \bar{\mathfrak{g}} : B(x, y, z) = 0, \forall y, z \in \mathfrak{j}\} \\ &= \{\overline{x} \in \mathfrak{g} : x \in [\mathfrak{j}, \mathfrak{j}]^{\perp}\} \\ &= \pi([\mathfrak{j}, \mathfrak{j}]^{\perp}), \end{aligned}$$

where, $[\mathfrak{j}, \mathfrak{j}]^{\perp}$ is the orthogonal complement of $[\mathfrak{j}, \mathfrak{j}]$ with respect to $B$. So, $\bar{\mathfrak{j}}^{\perp,2}$ is an ideal of $\bar{\mathfrak{g}}$. Now, since $\mathfrak{g}$ is solvable (res. nilpotent), then $\bar{\mathfrak{g}}$ is solvable (res. nilpotent), and hence it has an ideal with codimension 1. Suppose $\bar{\mathfrak{j}}$ is a such ideal. Then $\bar{\mathfrak{j}}$ is 2-coisotropic (see [4]). □

## 4.1 Invariant 1-Lagrangian subspaces for nilpotent endomorphisms

Let $V$ be a vector space and $\omega \in \bigwedge^3(V^*)$. (Here we do not assume that $\omega$ is nondegenerate). Consider the following equation on the set of endomorphisms of $V$,

$$\omega(\varphi^2 u, v, w) + \omega(u, \varphi^2 v, w) + \omega(u, v, \varphi^2 w) + \omega(\varphi u, \varphi v, w) = 0. \tag{4.1}$$

**Lemma 4.4.** *Let $\varphi$ be an endomorphism on $V$ which satisfies $\varphi^k = 0$, for some $k \geq 1$, and it is a solution of the equation* (4.1)*. Then*

$$\omega(u, \varphi v, \varphi^{k-1} w) = 0,$$

*for all $u, v$ and $w$ belong to $V$.*

*Proof.* Define $\tau_{ml} = \omega(\varphi^m u, \varphi^l v, \varphi^{k-(m+l)} w)$, $0 \leq m + l \leq k$. Since $\varphi$ is a solution of (4.1), the following equations are satisfied,

$$\tau_{21} + \tau_{03} + \tau_{01} + \tau_{12} = 0,$$

$$\tau_{30} + \tau_{12} + \tau_{10} + \tau_{21} = 0,$$

$$\tau_{21} + \tau_{01} + \tau_{03} + \tau_{11} = 0,$$

$$\tau_{12} + \tau_{10} + \tau_{30} + \tau_{11} = 0,$$

$$\tau_{30} + \tau_{10} + \tau_{12} + \tau_{20} = 0,$$

$$\tau_{03} + \tau_{01} + \tau_{21} + \tau_{02} = 0.$$

$$\tau_{20} + \tau_{02} + \tau_{01} = 0,$$

$$\tau_{20} + \tau_{02} + \tau_{10} = 0,$$

$$\tau_{20} + \tau_{02} + \tau_{11} = 0,$$

So $\tau_{01} = 0$. □

Now, define alternating 3-form $\alpha$ on $V$ by

$$\alpha(u, v, w) = \omega(\varphi u, v, w) + \omega(u, \varphi v, w) + \omega(u, v, \varphi w).$$

**Proposition 4.5.** *Let $\varphi$ be an endomorphism on $V$ which satisfies $\varphi^k = 0$, for some $k \geq 1$, and it is a solution of the equation* (4.1)*. Then $\alpha$ is degenerate.*

*Proof.* Let $z \in im(\varphi^{k-1})$, by Lemma 4.1 we have $\omega(u, \varphi v, z) = 0$, for all $u, v \in V$. So $\alpha(z, u, v) = 0$, for all $u, v \in V$. □

**Proposition 4.6.** *Let $\varphi$ be an endomorphism on $V$ which satisfies $\varphi^k = 0$, for some $k \geq 1$, and it is a solution of the equation* (4.1)*. Then the subspace $z^{\perp,1}$ is invariant by $\varphi$, for all $z \in im(\varphi^{k-1})$.*

*Proof.* By Lemma 4.1, it is obvious. □

**Theorem 4.7.** *Let $(V, \omega)$ be a vector space with alternating 3-form. Let $\varphi$ be an endomorphism on $V$ which satisfies $\varphi^k = 0$, for some $k \geq 1$, and it is a solution of the equation (4.1). Then there exists a $\varphi$-invariant 1-lagrangian subspace of $(V, \omega)$.*

*Proof.* We may assume that $\varphi^{k-1} \neq 0$. Now, the statement is proved by induction on dimension of $V$. Suppose $\dim(V) = n$ and assume that the result is true for all vector spaces with alternating 3-form and with dimension less than $n$. Choose $0 \neq z \in im(\varphi^{k-1})$. By Proposition 4.3, the subspace $z^{\perp,1}$ is invariant by $\varphi$. Let $W = \frac{z^{\perp,1}}{\langle z \rangle}$ and $\bar{\omega}$ be the alternating 3-form on $W$ induced by $\omega$. Let endomorphism $\overline{\varphi}$ on $W$ induced by $\varphi$. By induction hypothesis, $W$ has a subspace $U$ which is 1-lagrangian and $\overline{\varphi}$-invariant. The preimage of $U$ in $z^{\perp,1}$ is 1-lagrangian subspace of $V$, which by construction is invariant by $\varphi$. □

## 4.2 Existence of isotropic ideals

Let $(\mathfrak{g}, \omega)$ be a 2-plectic Lie algebra and define the ascending chain $C^i\mathfrak{g}$ and $C_i\mathfrak{g}$ of ideals in $\mathfrak{g}$ by

$$C^0\mathfrak{g} = \mathfrak{g}, C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}].$$

$$C_0\mathfrak{g} = \{0\}, C_{i+1}\mathfrak{g} = \{v | [\mathfrak{g}, v] \subseteq C_i\mathfrak{g}\}.$$

Then

$$[C^i\mathfrak{g}, C_j\mathfrak{g}] \subseteq C_{j-i-1}\mathfrak{g}. \tag{4.2}$$

A Lie algebra of $\mathfrak{g}$ is called $k$-step nilpotent if $C^k\mathfrak{g} = 0$, for some $k \geq 0$ and is called nilpotency class $k$ if $C^k\mathfrak{g} = \{0\}$ and $C^{k-1}\mathfrak{g} \neq \{0\}$. If $\mathfrak{g}$ is of nilpotency class $k$, we have

$$C^{k-i}\mathfrak{g} \subseteq C_i\mathfrak{g}. \tag{4.3}$$

**Lemma 4.8.** *Let $(\mathfrak{g}, \omega)$ be a 2-plectic Lie algebra. If $C^i\mathfrak{g} \subseteq (C_i\mathfrak{g})^{\perp,1}$, then $C^{i+1}\mathfrak{g} \subseteq (C_{i+1}\mathfrak{g})^{\perp,2}$, for all $i \geq 0$.*

*Proof.* Consider $z_1, z_2 \in C_{i+1}\mathfrak{g}$, and $w = [u, v] \in C^{i+1}\mathfrak{g}$, where $v \in C^i\mathfrak{g}$. Thus

$$\omega(w, z_1, z_2) = \omega([u, v], z_1, z_2) = \omega([u, z_1], v, z_2)$$

$$+\omega([z_2, u], v, z_1) + \omega([z_1, v], u, z_2) + \omega([v, z_2], u, z_1) + \omega([z_2, z_1], u, v).$$

Note that $[z_2, z_1], [u, z_1]$ and $[u, z_2] \in C_i\mathfrak{g}$, so

$$\omega([z_2, z_1], u, v) = \omega([u, z_1], v, z_2) = \omega([z_2, u], v, z_1) = 0.$$

On the other hand, by (4.2), $[z_1, v]$ and $[z_2, v] \in C_{i+1-i-1}\mathfrak{g} = C_0\mathfrak{g} = 0$. So $\omega([z_1, v], u, z_2) = \omega([v, z_2], u, z_1) = 0$. Therefor, $\omega(w, z_1, z_2) = 0$. □

**Lemma 4.9.**

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}(\mathfrak{g})^{\perp,2}.$$

*Proof.* Since $C^0\mathfrak{g} \subseteq (C_0\mathfrak{g})^{\perp,1}$, so by Lemma 4.2, we have $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}(\mathfrak{g})^{\perp,2}$. □

**Theorem 4.10.** *Let $(\mathfrak{g},\omega)$ be a two-step nilpotent 2-plectic Lie algebra. Then the following hold:*
*1- The ideal $[\mathfrak{g},\mathfrak{g}]$ is 2-isotropic.*
*2- $(\mathfrak{g},\omega)$ has a 2-lagrangian ideal.*

*Proof.* Since $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{z}(\mathfrak{g})$, by Lemma 4.3, we have $[\mathfrak{g},\mathfrak{g}]\subseteq[\mathfrak{g},\mathfrak{g}]^{\perp,2}$. Any maximally isotropic subspace $\mathfrak{a}$ contains $[\mathfrak{g},\mathfrak{g}]$ is a 2-lagrangian ideal. □

**Lemma 4.11.** *Let $(\overline{\mathfrak{g}},\bar{\omega},B)$ be a 2-plectic Lie algebra, where $(\mathfrak{g},B)$ is a quadratic non Abelian Lie algebra. Then $C^i\overline{\mathfrak{g}}\subseteq(C_i\overline{\mathfrak{g}})^{\perp,1}$ for all $i\geq 0$.*
*In particular $[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]\subseteq\mathfrak{z}(\overline{\mathfrak{g}})^{\perp,1}$.*

*Proof.* The proof is by induction. Assume that the statement is true for all $i\leq l-1$. For the induction steps, consider $z_1\in C_l\overline{\mathfrak{g}}$, $w=[u,v]\in C^l\overline{\mathfrak{g}}$, where $v\in C^{l-1}\overline{\mathfrak{g}}$, and where $z_2\in\overline{\mathfrak{g}}$.

Thus

$$\bar{\omega}(w,z_1,z_2)=\bar{\omega}([u,v],z_1,z_2)=\bar{\omega}([u,z_1],v,z_2)$$
$$+\bar{\omega}([z_2,u],v,z_1)+\bar{\omega}([z_1,v],u,z_2)+\bar{\omega}([v,z_2],u,z_1)+\bar{\omega}([z_2,z_1],u,v).$$

Note that $[u,z_1],[z_2,z_1]\in C_{l-1}\overline{\mathfrak{g}}$, so by induction

$$\bar{\omega}([u,z_1],v,z_2)=\bar{\omega}([z_2,z_1],u,v)=0.$$

On the other hand,

$$[z_1,v]\in C_{l-l+1-1}\overline{\mathfrak{g}}=C_0\overline{\mathfrak{g}}=0.$$

So

$$\bar{\omega}([z_1,v],u,z_2)=0.$$

Using Jacobi identity, we have

$$\bar{\omega}([z_2,u],v,z_1)=\bar{\omega}([u,z_1],z_2,v)+\bar{\omega}([z_1,z_2],u,v)=0,$$

and

$$\bar{\omega}([u,z_1],u,z_2)=\bar{\omega}([z_1,z_2],v,u)+\bar{\omega}([z_1,v],z_2,u)=0.$$

Therefor, $\bar{\omega}(w,z_1,z_2)=0$. □

**Theorem 4.12.** *Let $(\overline{\mathfrak{g}},\bar{\omega},B)$ be a 2-plectic Lie algebra, where $(\mathfrak{g},B)$ is a quadratic non Abelian Lie algebra. Also let $\overline{\mathfrak{g}}$ be a two-step nilpotent. Then the following holds:*
*1- The ideal $[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]$ is 1-isotropic.*
*2- $(\overline{\mathfrak{g}},\bar{\omega})$ has a 1-lagrangian ideal.*

*Proof.* The proof is similar to Theorem 4.4. □

**Theorem 4.13.** *Let $(\overline{\mathfrak{g}},\bar{\omega},B)$ be a 2-plectic Lie algebra, where $(\mathfrak{g},B)$ is a quadratic non Abelian Lie algebra which is of nilpotency class $k$. Then*
*$C^i\overline{\mathfrak{g}}$ is an 1-isotropic ideal of $\overline{\mathfrak{g}}$ for all $i$, with $2i\geq k$.*

*Proof.* Recall from (4.3), that $C^i\overline{\mathfrak{g}}\subseteq C_{k-i}\overline{\mathfrak{g}}$ if $\overline{\mathfrak{g}}$ has nilpotency class $k$. Since $C_{k-i}\overline{\mathfrak{g}}\subseteq C_i\overline{\mathfrak{g}}$, so $C^i\overline{\mathfrak{g}}\subseteq C_i\overline{\mathfrak{g}}$. On other hand, by Lemma 4.4 , $[\overline{\mathfrak{g}},\overline{\mathfrak{g}}]\subseteq\mathfrak{z}(\overline{\mathfrak{g}})^{\perp,1}$. So $C^i\overline{\mathfrak{g}}$ is an 1-isotropic ideal. □

# 5 Reduction

Let $(\mathfrak{g},\omega)$ be a 2-plectic Lie algebra and $\mathfrak{j}$ be a 2-coisotropic ideal in $\mathfrak{g}$. Then, according to Proposition 3 part 4, the 2-plectic structure $\omega$ induces a 2-plectic structure $\widetilde{\omega}$ on the Lie algebra $\widetilde{\mathfrak{g}}=\frac{\mathfrak{j}^{\perp,2}}{\mathfrak{j}}$.

**Definition 5.1.** The 2-plectic Lie algebra $(\widetilde{\mathfrak{g}},\widetilde{\omega})$ is called the 2-plectic reduction of $(\mathfrak{g},\omega)$ with respect to the 2-coisotropic ideal $\mathfrak{j}$.

In this short section, we obtain some results about 2-plectic reduction.

**Theorem 5.1.** *If $(\mathfrak{g},\omega,K)$ is a semisimple 2-plectic Lie algebra, then it has no reduction.*

*Proof.* This is a consequence of Theorem 4.1. □

**Theorem 5.2.** *Let $(\bar{\mathfrak{g}},\bar{\omega},B)$ be a 2-plectic Lie algebra, where $(\mathfrak{g},B)$ is a quadratic solvable (or nilpotent) non Abelian Lie algebra. Then $\bar{\mathfrak{g}}$ has a 2-plectic reduction.*

*Proof.* This is a consequence of Theorem 4.2. □

Of course, in the above theorem we have to note that the dimension of the quotient space must be greater than 4. Since there is no 2-plectic structure in dimension 4.

**Theorem 5.3.** *Suppose $(\bar{\mathfrak{g}}_1,\bar{\omega}_1,B_1)$ is a 2-plectic Lie algebra. If $\varphi:\mathfrak{g}_1\to\mathfrak{g}_1$ is an anti-symmetric derivation such that $\mathfrak{z}(\mathfrak{g})\subseteq Ker\,\varphi$, then $(\bar{\mathfrak{g}}_1,\bar{\omega}_1,B_1)$ is a symplectic reduction of a 2-plectic Lie algebra.*

*Proof.* Since $(\mathfrak{g}_1,B_1)$ is a quadratic Lie algebra, the Lie algebra $(\bar{\mathfrak{g}}=\mathfrak{g}_1\oplus\langle e\rangle\oplus\langle f\rangle,\overline{[\,.\,]},\overline{B})$ is a quadratic Lie algebra, where

$$\overline{[x,y]}=[x,y]+B(\varphi(x),y)f,\qquad \forall x,y\in\mathfrak{g}_1,$$

$$\overline{[f,\bar{\mathfrak{g}}]}=0,\qquad \overline{[e,x]}=\varphi(x),\quad \forall x\in\mathfrak{g}_1,$$

$$\overline{B}(e,e)=\overline{B}(f,f)=\overline{B}(e,\mathfrak{g}_1)=\overline{B}(f,\mathfrak{g}_1)=0,$$

$$\overline{B}(e,f)=1,\qquad \overline{B}(x,y)=B_1(x,y),\qquad \forall x,y\in\mathfrak{g}_1.$$

If $\mathfrak{z}(\mathfrak{g}_1)$ is the centre of $\mathfrak{g}_1$, then $\mathfrak{z}(\bar{\mathfrak{g}})=\mathfrak{z}(\mathfrak{g}_1)\oplus\langle f\rangle$ is the centre of $\bar{\mathfrak{g}}$. Let $\bar{\omega}$ be the 2-plectic structure induced by $\overline{B}$ on $\frac{\bar{\mathfrak{g}}}{\mathfrak{z}(\bar{\mathfrak{g}})}$. Now, it is easy to see that the map $\psi:\frac{\bar{\mathfrak{g}}_1}{\mathfrak{z}(\bar{\mathfrak{g}}_1)}\to\frac{\bar{\mathfrak{g}}}{\mathfrak{z}(\bar{\mathfrak{g}})}$ defined by

$$x+\mathfrak{z}(\mathfrak{g}_1)\mapsto x+\mathfrak{z}(\bar{\mathfrak{g}})$$

is a monomorphism with $\psi^*(\bar{\omega})=\bar{\omega}_1$. □

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