On CW-translations of homogeneous Finsler spaces with (α, β)-metrics

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Abstract. In the present paper, we consider homogeneous Finsler spaces with two (α, β)-metrics. Taking a Killing vector field X on one of these spaces, we find necessary and sufficient conditions for X to be Killing on the other space. Further, by taking a Killing vector field X of constant length on one of these spaces, we find the condition under which the Killing vector filed X has constant length w.r.t. other space also. Taking Killing vector fields of constant length, we find CW translations on these spaces.

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1 Introduction

According to S. S. Chern [5], Finsler geometry is just Riemannian geometry without quadratic restriction. Finsler geometry is an interesting and active area of research for both pure and applied reasons [2, 1, 10, 13]. One of the current topics of research in Riemann-Finsler geometry is Clifford-Wolf translation which was first introduced by J. A. Wolf [25] for Riemannian manifolds in 1964. If d is the distance function on a Riemannian manifold M, then (M, d) becomes a metric space. An isometry ρ of M is called a Clifford-Wolf translation, or in short, CW-translation if the distance between a point $x \in M$ and its image, i.e., $d(x, \rho(x))$ is same for all $x \in M$. A Riemannian manifold M is called Clifford-Wolf homogeneous, or CW-homogeneous if for any two points x, y of M, there is a CW-translation taking x to y. Wolf proved that only bounded isometries of a Riemannian manifold are CW-translations. V. Ozols [19] worked on CW-translations in Riemannian symmetric spaces in 1974. Almost after three decades, in 2009, V. N. Berestovskii and Y. G. Nikonorov [4] worked on CW-translations on homogeneous Riemannian manifolds and they found necessary and sufficient condition for a simply connected (connected) Riemannian manifolds to be CW-homogeneous.

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The concept of CW-translations in Finsler spaces is similar to as in Riemannian spaces. Let (M, F) be a Finsler space, where F is positively homogeneous function of degree 1 but not absolute homogeneous in general. If d is the distance function defined on M which is not symmetric in general, then a CW-translation on M is an isometry ρ on M that moves each point of M, the same distance, i.e., $d(x, \rho(x))$ is constant. If for every two points $a, b \in M$, there exists a CW-translation ρ such that $\rho(a) = b$, then the Finsler space (M, F) is called CW-homogeneous. Recently, S. Deng and M. Xu ([8, 26, 9]) have worked on CW-translations in homogeneous Finsler spaces and have found necessary and sufficient condition for an isometry on a homogeneous Randers space to be a CW-translation.

CW-translations are closely related to Killing vector fields of constant length. If the manifold (Riemannian or Finsler) is compact, then CW-translation is generated by a Killing vector field of constant length.

The paper is organized as follows:
In Section 2, we discuss some basic definitions and results to be used in subsequent sections. In Section 3, we consider Finsler spaces with two well known (α, β)-metrics: Matsumoto metric and infinite series metric. Taking a Killing vector field X on one of these spaces, we find necessary and sufficient conditions for X to be Killing on the other space. Further, in Section 4, by taking a Killing vector field X of constant length on one of these spaces, we find the condition under which the Killing vector filed X has constant length w.r.t. other space also. Finally, we characterize CW-translations on these spaces with the help of Killing vector fields of constant length.

2 Preliminaries

First, we discuss some basic definitions and results required to study afore said spaces. We refer [3, 6, 7, 14] for notations and further details.

Definition 2.1. Let X be a vector field on a smooth manifold M and I be any open interval containing 0. The differentiable curve $\gamma : I \longrightarrow M$ on M is said to be an integral curve if and only if

$$\gamma'(t) = X_{\gamma(t)} \text{ or } X(\gamma(t)) \; \forall \; t \in I, \tag{2.1}$$

i.e., an integral curve is the solution curve of ODE (2.1). The point $\gamma(0) = p$ is called the starting point of the curve.

Suppose for each $p \in M$, and $X \in \chi(M)$, X has a unique integral curve $\psi^{(p)}$ starting at p and is defined all over $\mathbb{R}$,
i.e., $\psi^{(p)} : \mathbb{R} \longrightarrow M$ such that

$$(\psi^{(p)})'(t) = X(\psi^{(p)}(t)) \; \forall \; t \in \mathbb{R}, \text{ and } \psi^{(p)}(0) = p.$$

For each $t \in \mathbb{R}$, define $\psi_t : M \longrightarrow M$ as

$$\psi_t(p) = \psi^{(p)}(t),$$

i.e., $t \longmapsto \psi_t(p)$ is an integral curve starting at p.
For the additive group $\mathbb{R}$, the continuous $\mathbb{R}$-action $\psi : \mathbb{R} \times M \longrightarrow M$ on M satisfying

$$\psi(t_1, \psi(t_2, p)) = \psi(t_1 + t_2, p), \;\; \psi(0, p) = p$$

is called global flow (one-parameter group action) of X, where $\psi(t, p) = \psi_t(p)$.

Example 2.2. Consider a vector field $X = \dfrac{\partial}{\partial x}$ or $(1, 0)$ on $\mathbb{R}^2$ with standard coordinate system (x, y). Then the integral curve of X is

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 \text{ such that } \gamma'(t) = X(\gamma(t)) \forall\; t \in I$$

which starts at $\gamma(0)$.
We have $\gamma(t) = (x(t), y(t))$. The condition $\gamma'(t) = X(\gamma(t))$ reduces to

$$x'(t) \frac{\partial}{\partial x} \mid_{\gamma(t)} + y'(t) \frac{\partial}{\partial y} \mid_{\gamma(t)} = \frac{\partial}{\partial x} \mid_{\gamma(t)}$$

which implies

$$x'(t) = 1, \;\; y'(t) = 0,$$

i.e.,

$$x(t) = t + a, \;\; y(t) = b \text{ for some constants } \; a \text{ and } \; b.$$

Therefore, the integral curve of X is given by

$$\gamma(t) = (t + a, b)$$

whose starting point is $\gamma(0) = (a, b)$.

The flow of X is the map $\psi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\psi(t, p) = \psi_t(p)$, where $p = (x, y) \in \mathbb{R}^2$ satisfies

$$\psi_t(x, y) = \psi^{(x,y)}(t) = \text{integral curve of } X \text{ starting at } (x, y),$$

i.e.,

$$\psi_t(x, y) = (t + x, y).$$

Definition 2.3. For a Riemannian space (M, g), a vector field X is said to be Killing if flow of X acts by isometries of g, i.e., g is invariant under the flow of X.

Equivalently, we say that X is Killing if and only if $L_X g = 0$, where L_X denotes the Lie derivative with respect to X. With the help of properties of Lie derivative, one can easily prove that in smooth local coordinate chart, a smooth vector field X is Killing if and only if

$$X^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk} \frac{\partial X^k}{\partial x^i} + g_{ik} \frac{\partial X^k}{\partial x^j} = 0$$

or

$$X g_{ij} + g_{jk} \frac{\partial X^k}{\partial x^i} + g_{ik} \frac{\partial X^k}{\partial x^j} = 0. \tag{2.2}$$

If ∇ is the Riemannian connection on M, then the condition $(L_X g)(Y, Z) = 0$ is equivalent to $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for any smooth vector fields Y and Z on M.

Example 2.4. Let g be the Euclidean metric on $\mathbb{R}^n$. The necessary and sufficient condition (2.2) for a vector field X to be Killing, in standard coordinates, on $(\mathbb{R}^n, g)$ becomes

$$\frac{\partial X^j}{\partial x^i} + \frac{\partial X^i}{\partial x^j} = 0. \tag{2.3}$$

One can easily check that the vector fields $y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ and $y^2 x\dfrac{\partial}{\partial x} - x^2 y\dfrac{\partial}{\partial x}$ are Killing on the Euclidean space $(\mathbb{R}^2, g)$.

Definition 2.5. A Killing vector field X on a Riemannian space (M, g) has constant length if every integral curve of X is a geodesic on (M, g).

Definition 2.6. An n-dimensional real vector space V is said to be a **Minkowski space** if there exists a real valued function $F : V \longrightarrow [0, \infty)$, called Minkowski norm, satisfying the following conditions:

- F is smooth on $V\backslash\{0\}$,
- F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v) \;\; \forall\ \lambda > 0$,
- For any basis $\{u_1,\ u_2,\ ...,\ u_n\}$ of V and $y = y^i u_i \in V$, the Hessian matrix $\left(g_{ij}\right) = \left(\frac{1}{2}F^2_{y^i y^j}\right)$ is positive-definite at every point of $V\backslash\{0\}$.

Definition 2.7. Let M be a connected smooth manifold. If there exists a function $F : TM \longrightarrow [0, \infty)$ such that F is smooth on the slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any $T_x M,\ x \in M$, is a Minkowski norm, then M is called a Finsler space and F is called a Finsler metric.

In 1972, M. Matsumoto [16] introduced the concept of (α, β)-metrics. An (α, β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and a one-form $\beta = b_i(x) y^i$ on M and is of the form $F = \alpha\phi\left(\dfrac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M.
Basically, (α, β)-metrics are the generalizations of Randers metric introduced by G. Randers [20]. The simplest non-Riemannian metrics are the Randers metrics given by $F = \alpha + \beta$ with $\|\beta\|_\alpha < 1$. Besides Randers metrics, other interesting kind of non-Riemannian metrics are Matsumoto metric, infinite series metric, square metric etc.
Matsumoto metric (slope of mountain metric) was introduced by M. Matsumoto [17] and is of the form $\dfrac{\alpha^2}{\alpha - \beta}$. There is one more interesting (α, β)-metric, called r^{th} series (α, β)-metric [15] defined as follows:

$$F(\alpha, \beta) = \beta \sum_{i=0}^{r} \left(\frac{\alpha}{\beta}\right)^i; \quad |\alpha| < |\beta|. \tag{2.4}$$

$r = 1$ gives us well known Randers metric $F(\alpha, \beta) = \alpha + \beta$.
$r = \infty$ gives us the metric $F(\alpha, \beta) = \dfrac{\beta^2}{\beta - \alpha}$ which is called infinite series metric. Being

the difference of Randers metric and Matsumoto metric, this metric is remarkable. Many authors [11, 12, 21, 22, 23, 24, 27] have worked on (α, β)-metrics. Let us recall Shen's lemma [6] which provides necessary and sufficient condition for a function of α and β to be a Finsler metric.

Lemma 2.1. *Let $F = \alpha\phi(s),\ s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0 \ \ \forall \ \ |s| \leq b < b_0.$$

Let (M, F) be a Finsler space and x be any point in M. Let $\sigma_y(t)$ be a unit speed curve passing through x at $t = 0$ having initial velocity y. Let $F(x, y) = 1$, i.e., $y \in S_xM$ (indicatrix at x). The cut value i_y of y is defined as follows:

$$i_y = sup\ \{r : \text{ the segment } \sigma_y\ |_{[0,r]} \text{ is globally minimized}\}.$$

Definition 2.8. The injectivity radius of M at x is defined as follows:

$$i_x = \inf_{y \in S_xM} i_y,$$

and the injectivity radius i_M or $i(M)$ of M is defined as follows:

$$i_M = min\{i_x : x \in M\}.$$

Definition 2.9. Let (M, F) be a Finsler space. An isometry ϕ is a diffeomorphism on M satisfying:

$$F(x, y) = F\left(\phi(x), d\phi_x(y)\right) \ \ \forall\ x \in M \text{ and } \ \ y \in T_xM.$$

Definition 2.10. Let (M, F) be a Finsler space and d the distance function on M which is not symmetric in general as F is not absolute homogeneous in general. A Clifford-Wolf translation (CW-translation) on M is an isometry ρ on M such that $d(x, \rho(x))$ is constant for all $x \in M$.

Definition 2.11. A Finsler space (M, F) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any $a, b \in M$, there exists a CW-translation ρ such that $\rho(a) = b$.

Before defining homogeneous Finsler spaces, below we discuss some basic concepts required.

Definition 2.12. Let G be a smooth manifold having the structure of an abstract group. G is called a Lie group, if the maps $i : G \longrightarrow G$ and $\mu : G \times G \longrightarrow G$ defined as $i(g) = g^{-1}$, and $\mu(g, h) = gh$ respectively, are smooth.

Let G be a Lie group and M, a smooth manifold. Then a smooth map $f : G \times M \longrightarrow M$ satisfying

$$f(g_2, f(g_1, x)) = f(g_2g_1, x), \ \text{ for all } \ \ g_1, g_2 \in G, \ \text{ and } \ \ x \in M$$

is called a smooth action of G on M.

Definition 2.13. Let M be a smooth manifold and G, a Lie group. If G acts smoothly on M, then G is called a **Lie transformation group** of M.

The following theorem gives us a differentiable structure on the coset space of a Lie group.

Theorem 2.2. *Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H.*

Definition 2.14. Let (M, F) be a connected Finsler space and $G = I(M, F)$ the group of isometries of (M, F). If G acts transitively on M, then (M, F) is said to be a homogeneous Finsler space.

Let G be a Lie group acting transitively on a smooth manifold M. Then for $a \in M$, the isotropy subgroup G_a of G is a closed subgroup and by Theorem 2.2, G is a Lie transformation group of G/G_a. Further, G/G_a is diffeomorphic to M.

Theorem 2.3. *[7] Let (M, F) be a Finsler space. Then $G = I(M, F)$, the group of isometries of M is a Lie transformation group of M. Let $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at a. Then $I_a(M, F)$ is compact.*

Let (M, F) be a homogeneous Finsler space, i.e., $G = I(M, F)$ acts transitively on M. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.15. [18] Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, where $\mathfrak{k}$ is a subspace of $\mathfrak{g}$ such that $\mathrm{Ad}(h)(\mathfrak{k}) \subset \mathfrak{k} \ \ \forall\ h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Therefore, we can write, any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.
In case of reductive homogeneous manifold, we can identify the tangent space $T_H(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{k}$ through the map

$$Y \longmapsto \frac{d}{dt} exp(tX)H|_{t=0}, \quad Y \in \mathfrak{k},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as T_eG.

3 Killing vector fields on Finsler spaces

Let (M, F) be a Fisnler space and X be a vector field on M. Further, let $\{\psi_t\}$ be one-parameter group generated by X, i.e., ψ_t is flow of X on M, then $\tilde{\psi_t}$ is also flow on $TM_0 = TM \backslash \{0\}$ defined by

$$\tilde{\psi_t}(x, y) = (\psi_t(x)), \psi_{t*}(y), \text{ for } \ x \in M, \ y \in T_xM$$

which we call lift of ψ_t. So, there is a vector field $\tilde{X}$ (called complete lift of X) on TM induced by $\tilde{\psi}_t$, which is defined, in local coordinates, as follows:

$$\tilde{X} = X^i \frac{\partial}{\partial x^i} + y^k \frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial y^i} \text{ or } X^i \partial_i + y^k (\partial_k X^i) \dot{\partial}_i.$$

Definition 3.1. A vector field X on a Finsler space (M, F) is called an infinitesimal isometry or Killing vector field if the flow ψ_t generated by X is an isometry, i.e., the flow $\tilde{\psi}_t$ fixes F, i.e., $\tilde{\psi}_t(F) = F$.

Equivalently, X is said to be Killing if $L_{\tilde{X}} F = 0$, where $L_{\tilde{X}}$ denotes the Lie derivative with respect to $\tilde{X}$.

Theorem 3.1. *Let X be a Killing vector field on a Finsler space M with infinite series metric $\tilde{F} = \frac{\beta^2}{\beta - \alpha}$. Then X is a Killing vector field on M with Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$ if and only if X is a Killing vector field for α and $L_X \beta = 0$.*

Proof. Since X is a Killing vector field on a Finsler space M with infinite series metric $\bar{F} = \frac{\beta^2}{\beta - \alpha}$, the flow $\tilde{\psi}_t$ fixes $\bar{F}$, i.e.,

$$\tilde{\psi}_t \bar{F} = \bar{F}. \tag{3.1}$$

Also,

$$\bar{F} = \frac{\beta^2}{\beta - \alpha} = \frac{\beta^2 - \alpha^2}{\beta - \alpha} + \frac{\alpha^2}{\beta - \alpha} = \alpha + \beta - F.$$

Firstly suppose X is a Killing vector field on M with Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$, i.e.,

$$\tilde{\psi}_t F = F. \tag{3.2}$$

From equation (3.1), we have

$$\tilde{\psi}_t(\alpha + \beta - F) = \alpha + \beta - F$$

which implies

$$\tilde{\psi}_t \alpha + \tilde{\psi}_t \beta - \tilde{\psi}_t F = \alpha + \beta - F.$$

Using equation (3.2) in above relation, we get

$$\tilde{\psi}_t \alpha + \tilde{\psi}_t \beta = \alpha + \beta. \tag{3.3}$$

Replacing the pair (x, y) by $(x, -y)$ in equation (3.3), we have

$$\tilde{\psi}_t \alpha - \tilde{\psi}_t \beta = \alpha - \beta. \tag{3.4}$$

Addition of equations (3.3) and (3.4) gives us

$$\tilde{\psi}_t \alpha = \alpha,$$

and subtraction of equation (3.4) from the equation (3.3) gives us

$$\tilde{\psi}_t \beta = \beta \quad \Longrightarrow \quad L_X \beta = 0.$$

For the converse part, let us suppose that $L_X\beta = 0$, and X be Killing vector field for α, i.e., $\tilde{\psi}_t \alpha = \alpha$. Then

$$\begin{aligned} \tilde{\psi}_t F =& \tilde{\psi}_t(\alpha + \beta - \bar{F}) \\ =& \tilde{\psi}_t \alpha + \tilde{\psi}_t \beta - \tilde{\psi}_t \bar{F} \\ =& \alpha + \beta - \bar{F} \\ =& F, \end{aligned}$$

which shows that X is a Killing vector field on M with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$.
□

In similar manner, we can prove following:

Theorem 3.2. *Let X be a Killing vector field on a Finsler space M with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$. Then X is a Killing vector field on M with infinite series metric $\tilde{F} = \dfrac{\beta^2}{\beta - \alpha}$ if and only if X is a Killing vector field for α and $L_X\beta = 0$.*

Thus for above discussed two Finsler spaces with (α, β)-metrics, taking a Killing vector field X on one of these spaces, we have found necessary and sufficient conditions for X to be Killing on the other space.

4 Killing vector fields on homogeneous Finsler spaces

As we have already discussed that a Finsler space M with either of the afore said metrics is homogeneous if the group G of isometries of M acts transitively on M, the space M can be identified with G/H, where H is compact isotropy subgroup of G at some point x. The metrics under consideration can be determined from the data at tangent space at x which is isomorphic to $\mathfrak{g}/\mathfrak{h} = \mathfrak{k}$. Recall that [8] α can be determined by an inner product and V, the dual of β is a vector of $\mathfrak{k}$, both are invariant under Ad-action of H. The decomposition of any $X \in \mathfrak{g}$ is given as $X = X_{\mathfrak{h}} + X_{\mathfrak{k}}$, and consequently both the metrics are given by

$$\text{Matsumoto metric } F = \frac{\langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle}{\langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle^{1/2} - \langle (Ad_g X)_{\mathfrak{k}}, V \rangle},$$

$$\text{infinite series metric } \bar{F} = \frac{\langle (Ad_g X)_{\mathfrak{k}}, V \rangle^2}{\langle (Ad_g X)_{\mathfrak{k}}, V \rangle - \langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle^{1/2}}.$$

Theorem 4.1. *Let X be a Killing vector field of constant length on a connected homogeneous Finsler space G/H with infinite series metric $\bar{F} = \dfrac{\beta^2}{\beta - \alpha}$ as well as on*

Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$. *Then* X *satisfies*

$$\frac{\langle [Y, Ad_g X]_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle}{\langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle^{1/2}} + \langle [Y, Ad_g X]_{\mathfrak{k}}, V \rangle = 0, \ \text{for } Y \in \mathfrak{g} \ \text{ and } \ g \in G. \tag{4.1}$$

Proof. We have

$$F = \alpha + \beta - \bar{F},$$

i.e.,

$$F = \langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle^{1/2} + \langle (Ad_g X)_{\mathfrak{k}}, V \rangle - \bar{F}.$$

For a family $g_t = exp(tY).g, \quad Y \in \mathfrak{g}$, differentiating for t at 0 gives us

$$\frac{d}{dt} F|_{t=0} = \frac{\langle [Y, Ad_g X]_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle}{\langle (Ad_g X)_{\mathfrak{k}}, (Ad_g X)_{\mathfrak{k}} \rangle^{1/2}} + \langle [Y, Ad_g X]_{\mathfrak{k}}, V \rangle + \frac{d}{dt} \bar{F}|_{t=0},$$

which gives us required result. □

Theorem 4.2. *Let* X *be a Killing vector field of constant length on a connected homogeneous Finsler space* G/H *with infinite series metric* $\bar{F} = \dfrac{\beta^2}{\beta - \alpha}$. *If* H *is connected, then (4.1) is the sufficient condition for the Killing vector field* X *to be of constant length on* G/H *with respect to Matsumoto metric* $F = \dfrac{\alpha^2}{\alpha - \beta}$.

Theorem 4.3. *Let* X *be a Killing vector field of constant length on a connected homogeneous Finsler space* G/H *with Matsumoto metric* $F = \dfrac{\alpha^2}{\alpha - \beta}$. *If* H *is connected, then (4.1) is the sufficient condition for the Killing vector field* X *to be of constant length on* G/H *with respect to infinite series metric* $\bar{F} = \dfrac{\beta^2}{\beta - \alpha}$.

Recall [8] the following theorem:

Theorem 4.4. *Let* (M, F) *be a complete Finsler space with positive injectivity radius. Let* X *be a Killing vector field of constant length on* M, *and* ψ_t *be the flow generated by* X, *then* ψ_t *is a CW translation for all sufficiently small* $t > 0$.

From Theorem 4.4, we have the following theorem:

Theorem 4.5. *Let* G/H *be a complete homogeneous Finsler space with infinite series metric or Matsumoto metric having positive injectivity radius. Let* X *be a Killing vector field of constant length on* G/H. *If* ψ_t *is the flow generated by* X, *then* ψ_t *is a CW translation for all sufficiently small* $t > 0$.

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