Critical point equation on K-paracontact manifolds

Avijit Sarkar and Gour Gopal Biswas

Abstract. A. Besse posed a conjecture that a solution of a critical point equation is Einstein. The aim of our paper is to prove the conjecture for K-paracontact metrics.

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1 Introduction

Let M be a n-dimensional compact oriented manifold and $\mathcal{M}$ be the set of all Riemannian metrics of unit volume on M. The scalar curvature r_g is a non-linear function of the matric g. The differential at the point g in the direction of a (0,2) tensor field h is given by [2]

$$r'_g(h) = -\Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - g(S_g, h), \tag{1.1}$$

where Δ_g is the negative Laplacian operator, δ_g is the divergence operator and S_g is Ricci tensor of g. The L^2-adjoint $(r'_g)^*$ of r'_g is given by the formula

$$(r'_g)^*\gamma = -(\Delta_g\gamma)g + \nabla_g^2\gamma - \gamma S \tag{1.2}$$

for any C^∞-function γ on M, where ∇_g^2 is the Hessian operator of g.

Definition 1.1. Let (M^n, g), $n > 2$ be a compact Riemannian manifold with boundary ∂M. Then g is called a critical metric if there exists a smooth function λ on M^n such that

$$(r'_g)^*\lambda = g \tag{1.3}$$

on M and $\lambda = 0$ on ∂M. The function λ is known as the potential function.

The metrics which satisfy (1.3) are known as Miao-Tam critical metrics and we refer equation (1.3) as Miao-Tam equation. In [4], Miao-Tam equation has been studied

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on almost Kenmotsu manifolds. Miao and Tam[6] themselves have classified Einstein and conformally flat Riemannian manifolds satisfying Miao-Tam equation. In [5], the authors studied certain contact metric manifolds satisfying Miao-Tam equation.
The total scalar curvature functional $\Gamma : \mathcal{M} \to \mathbb{R}$ is defined by

$$\Gamma(g) = \int_M r_g dv_g$$

where r_g is the scalar curvature and dv_g the volume form determined by the metric and orientation. The Euler-Lagrange equation of the functional Γ restricted over $\{g \in \mathcal{M} : r_g = \text{constant}\}$ on a compact orientable manifold (M, g) can be written as critical point equation

$$(r'_g)^* \tilde{\lambda} = z_g \tag{1.4}$$

where z_g denotes the traceless Ricci tensor of M and $\tilde{\lambda}$ is a C^∞-function on M. If $\tilde{\lambda}$ is constant then from (1.4) we see that the metric g is Einstein. In this paper we consider $\tilde{\lambda}$ is a non-constant function. The equation $(r'_g)^* \tilde{\lambda} = 0$ is known as Fischer-Marsden equation.
In [2], A. Besse posed a conjecture that the solution of critical point equation is Einstein. In the paper [1], the authors proved that the conjecture is true for half conformally flat case. In [3], the authors proved that a K-contact metric satisfying critical point equation is Einstein and isometric to a unit sphere. They also proved that a (κ, μ)-contact metric satisfying critical point equation is flat and isometric to $E^{n+1} \times S^n(4)$.
In this paper we would like to study K-paracontact manifolds satisfying Miao-Tam equation and critical point equation. After the introduction we give required preliminaries in Section 2. Section 3 contains the study of K-paracontact manifolds satisfying Miao-Tam equation. In Section 4, we study K-paracontact manifolds satisfying Euler-Lagrange equation of total scalar curvature. The last section contains supporting example.

2 Preliminaries

Let M be a manifold of dimension $(2n+1)$. Let φ be a $(1,1)$ tensor field, ξ a vector field and η a 1-form on M. Then the triple (φ, ξ, η) is called an almost paracontact structure on M, if the following conditions are satisfied :

i) $\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1,$

ii) $\varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$

iii) the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of φ corresponding to the eigenvalues 1 and -1, respectively have equal dimension n.

If an almost paracontact manifold admits a pseudo-Riemannian metric such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, the set of all smooth vector fields on M, then we say that $(M, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold. Form (2.1) we have

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for all $X \in \chi(M)$.

The fundamental 2-form of an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by $F(X, Y) = g(X, \varphi Y)$. If $d\eta = F$, then the manifold $(M, \varphi, \xi, \eta, g)$ is said to be paracontact metric manifold.

If ξ is a Killing vector field i.e. $h = \frac{1}{2}\pounds_\xi \varphi = 0$, where $\pounds$ is the Lie derivative, then $(M, \varphi, \xi, \eta, g)$ is called K-paracontact manifold. In a K-paracontact manifold the following relations hold :

$$\nabla_X \xi = -\varphi X, \tag{2.3}$$

$$R(X, \xi)\xi = -X + \eta(X)\xi, \tag{2.4}$$

$$R(\xi, X)Y = (\nabla_X \varphi)Y, \tag{2.5}$$

$$(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X \varphi)Y = 2g(X, Y)\xi - (X + \eta(X)\xi)\eta(Y) \tag{2.6}$$

for all $X, Y, Z \in \chi(M)$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric and R is the Riemannian curvature tensor. For details see [7].

Lemma 2.1 In a K-paracontact manifold $(M, \varphi, \xi, \eta, g)$,

$$Q\xi = -2n\xi \tag{2.7}$$

where Q is the Ricci operator.

Proof : From Proposition 2.4 of [7], we have

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= -g(N^{(1)}(Y, Z), \varphi X) - 2d\eta(\varphi Z, X)\eta(Y) \\ &+ 2d\eta(\varphi Y, X)\eta(Z) \end{aligned}$$

for all $X, Y, Z \in \chi(M)$, where $N^{(1)}(Y, Z) = \varphi^2[Y, Z] + [\varphi Y, \varphi Z] - \varphi[\varphi Y, Z] - \varphi[Y, \varphi Z] - 2d\eta(Y, Z)\xi$.
Using (2.5) in the above equation and noting that $d\eta(X, Y) = g(X, \varphi Y)$, we obtain

$$g(R(X, \xi)Y, Z) = \frac{1}{2}g(N^{(1)}(Y, Z), \varphi X) - g(X, Z)\eta(Y) + g(X, Y)\eta(Z). \tag{2.8}$$

Let $\{e_1, e_2, \cdots, e_n, e'_1, e'_2, \cdots, e'_n, \xi\}$ be a local orthogonal φ-basis with $g(e_i, e_j) = \delta_{ij},\ g(e'_i, e'_j) = -\delta_{ij},\ e'_i = \varphi e_i$ where $i, j \in \{1, 2, \cdots, n\}$. Contracting (2.8) over X and Z with respect to this φ-basis we get (2.7).

Lemma 2.2. [4] Let a Riemannian manifold (M^n, g) satisfies the Miao-Tam equation. Then the curvature tensor R can be expressed as

$$\begin{aligned} R(X,Y)D\lambda &= (X\lambda)QY - (Y\lambda)QX + \lambda((\nabla_X Q)Y - (\nabla_Y Q)X) \\ &+ (Xf)Y - (Yf)X, \end{aligned} \tag{2.9}$$

for any vector fields X, Y on M, where $f = -\frac{r\lambda+1}{n-1}$ and D is the gradient operator. Moreover,

$$\nabla_X D\lambda = \lambda QX + fX, \tag{2.10}$$

for all vector fields X on M.

Lemma 2.3. [3] Let $(g, \tilde{\lambda})$ be a non-trivial solution of the critical point equation (1.4) on an n-dimensional Riemannian manifold M. Then the curvature tensor R can be written as

$$\begin{aligned} R(X,Y)D\tilde{\lambda} &= (X\tilde{\lambda})QY - (Y\tilde{\lambda})QX + (\tilde{\lambda}+1)(\nabla_X Q)Y \\ &- (\tilde{\lambda}+1)(\nabla_Y Q)X + (X\tilde{f})Y - (Y\tilde{f})X \end{aligned} \tag{2.11}$$

for all vector field X and Y on M, $\tilde{f} = -r\left(\frac{\tilde{\lambda}}{n-1} + \frac{1}{n}\right)$ and r is the scalar curvature of g. Also

$$\nabla_X D\tilde{\lambda} = (\tilde{\lambda}+1)QX + \tilde{f}X. \tag{2.12}$$

for all vector fields X on M.

3 K-paracontact manifolds satisfying Miao-Tam equations.

In this section, we study K-paracontact manifolds satisfying Miao-Tam equation. Here we prove the following:

Theorem 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a K-paracontact manifold of dimension $(2n+1)$. If there is a function $\lambda : M \to \mathbb{R}$ such that (g, λ) satisfies the Miao-Tam equation, then it is Einstein.*

Proof : Since ξ is Killing vector field, $\pounds_\xi Q = 0$. By (2.3) this equation gives

$$(\nabla_\xi Q)X = Q\varphi X - \varphi QX \tag{3.1}$$

for all $X \in \chi(M)$. Taking covariant derivative of (2.7) along an arbitrary vector field X, we get

$$(\nabla_X Q)\xi = Q\varphi X + 2n\varphi X. \tag{3.2}$$

Putting $X = \xi$ and replacing Y by X in (2.9) and using (3.1) and (3.2), we have

$$\begin{aligned} R(\xi, X)D\lambda &= (\xi\lambda)QX + 2n(X\lambda)\xi - \lambda\varphi QX - 2n\lambda\varphi X \\ &+ (\xi f)X - (Xf)\xi. \end{aligned} \tag{3.3}$$

Taking inner product of (3.3) with an arbitrary vector field Y and using (2.5), we get

$$\begin{aligned} & g((\nabla_X\varphi)Y, D\lambda) + (\xi\lambda)g(QX,Y) + 2n(X\lambda)\eta(Y) \\ - \ & \lambda g(\varphi QX,Y) - 2n\lambda g(\varphi X,Y) + (\xi f)g(X,Y) - (Xf)\eta(Y) = 0. \end{aligned} \tag{3.4}$$

Replacing X by φX and Y by φY in (3.4) and using (2.7), we get

$$\begin{aligned} & g((\nabla_{\varphi X}\varphi)\varphi Y, D\lambda) + (\xi\lambda)g(Q\varphi X,\varphi Y) \\ + \ & \lambda g(Q\varphi X,Y) + 2n\lambda g(\varphi X,Y) - (\xi f)g(X,Y) + (\xi f)\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.5}$$

Subtracting (3.5) from (3.4) and using (2.6), we obtain

$$\begin{aligned} & 2\xi(f-\lambda)g(X,Y) + X\{(2n+1)\lambda - f\}\eta(Y) \\ + \ & \xi(\lambda - f)\eta(X)\eta(Y) + (\xi\lambda)g(QX,Y) - (\xi\lambda)g(Q\varphi X,\varphi Y) \\ - \ & \lambda g(\varphi QX,Y) - \lambda g(Q\varphi X,Y) - 4n\lambda g(\varphi X,Y) = 0. \end{aligned}$$

By antisymmetrization with respect to X and Y in the above equation, we have

$$\begin{aligned} & X\{(2n+1)\lambda - f\}\eta(Y) - Y\{(2n+1)\lambda - f\}\eta(X) \\ - \ & 2\lambda g(Q\varphi X,Y) - 2\lambda g(\varphi QX,Y) - 8n\lambda g(\varphi X,Y) = 0. \end{aligned}$$

Substituting X by φX and Y by φY in the above equation and using (2.7), we get

$$\lambda[g(Q\varphi X,Y) + g(\varphi QX,Y)] = -4n\lambda g(\varphi X,Y). \tag{3.6}$$

Since λ does not vanish in the interior of M, the last equation gives

$$Q\varphi X + \varphi QX = -4n\varphi X. \tag{3.7}$$

Let $\{e_1, e_2, \cdots, e_n, e'_1, e'_2, \cdots, e'_n, \xi\}$ be a local orthogonal φ-basis with $g(e_i, e_j) = \delta_{ij}$, $g(e'_i, e'_j) = -\delta_{ij}$, $e'_i = \varphi e_i$ where $i, j \in \{1, 2, \cdots, n\}$. Using equation (2.1), $g(Qe_i, e_i) = -g(\varphi Qe_i, \varphi e_i)$. Using this φ-basis, (2.7) and (3.7) , we compute the scalar curvature

$$\begin{aligned} r &= \sum_{i=1}^{n} g(Qe_i, e_i) - \sum_{i=1}^{n} g(Q\varphi e_i, \varphi e_i) + g(Q\xi, \xi) \\ &= -\sum_{i=1}^{n} g(\varphi Qe_i + Q\varphi e_i, \varphi e_i) - 2n \\ &= -2n(2n+1). \end{aligned}$$

From Lemma 2.2, we have $f = -\frac{r\lambda+1}{2n}$. Since $r = -2n(2n+1)$, it follows that

$$(2n+1)\lambda - f = \frac{1}{2n}. \tag{3.8}$$

Taking inner product of (3.3) with $D\lambda$ and using (3.8), we obtain

$$(\xi\lambda)(QD\lambda + 2nD\lambda) + \lambda(Q\varphi D\lambda + 2n\varphi D\lambda) = 0. \tag{3.9}$$

Putting $X = D\lambda$ in (3.7), we have

$$Q\varphi D\lambda = -\varphi QD\lambda - 4n\varphi D\lambda. \tag{3.10}$$

Using (3.10) in (3.9), we get

$$(\xi\lambda)(QD\lambda + 2nD\lambda) - \lambda(\varphi QD\lambda + 2n\varphi D\lambda) = 0. \tag{3.11}$$

Now operating φ on the above equation and using (2.7), we obtain

$$\lambda(QD\lambda + 2nD\lambda) - (\xi\lambda)(\varphi QD\lambda + 2n\varphi D\lambda) = 0. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$((\xi\lambda)^2 - \lambda^2)(QD\lambda + 2nD\lambda) = 0.$$

From the above equation we have either (i) $QD\lambda + 2nD\lambda = 0$, or (ii) $(\xi\lambda) = \pm\lambda$.
Case (i) : In this case $QD\lambda + 2nD\lambda = 0$.Taking covariant differentiation of this equation along an arbitrary vector field X and using (2.10), we obtain

$$(\nabla_X Q)D\lambda + \lambda Q^2 X + (f + 2n\lambda)QX + 2nfX = 0.$$

Contracting this equation over X with respect to an orthonormal basis $\{E_i\}$, we get

$$g((\nabla_{E_i} Q)D\lambda, E_i) + \lambda|Q|^2 - 4n^2(2n+1)\lambda = 0.$$

Using the formula $\operatorname{div} QX = \frac{1}{2}Xr$ in the above equation and noting that scalar curvature is constant, we have $\lambda|Q|^2 - 4n^2(2n+1)\lambda = 0$. Since λ does not vanish in interior of M, it follows that $|Q|^2 = 4n^2(2n+1)\lambda$.
Now using $r = -2n(2n+1)$,

$$\left|Q - \frac{r}{2n+1}I\right|^2 = |Q|^2 - \frac{2r^2}{2n+1} + \frac{r^2}{2n+1} = 0.$$

Since the length of the symmetric tensor $Q - \frac{r}{2n+1}I$ vanish, we must have $Q - \frac{r}{2n+1}I = 0$. Since $r = -2n(2n+1)$, we get $QX = -2nX$ for all $X \in \chi(M)$. This shows that M is Einstein.
Case (ii) : If $\xi\lambda = \lambda$, then $\xi(\xi\lambda) = \xi\lambda = \lambda$. Also if $\xi\lambda = -\lambda$, then $\xi(\xi\lambda) = -\xi\lambda = \lambda$. In either case $\xi(\xi\lambda) = \lambda$. Putting $X = \xi$ in (2.10), taking inner product with ξ and using (2.7), we have

$$\xi(\xi\lambda) = -2n\lambda + f.$$

Since $\xi(\xi\lambda) = \lambda$, using (3.8) the above equation implies $\frac{1}{2n} = 0$, a contradiction. Therefore, only Case (i) holds.

4 K-paracontact manifolds satisfying Euler-Lagrange equation of total scalar curvature.

In this section, we study K-paracontact manifolds satisfying Euler-Lagrange equation of total scalar curvature. Here, we prove the following:

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a K-paracontact manifold of dimension $(2n+1)$. If there is a function $\tilde{\lambda} : M \to \mathbb{R}$ such that $(g, \tilde{\lambda})$ satisfies the critical point equation, then it is Einstein and $(g, \tilde{\lambda})$ satisfies Fischer-Marsden equation.*

Proof : Putting $X = \xi$ and replacing Y by X in (2.11) and using (3.1) and (3.2), we have

$$\begin{aligned} R(\xi, X)D\tilde{\lambda} &= (\xi\tilde{\lambda})QX + 2n(X\tilde{\lambda})\xi - (\tilde{\lambda}+1)\varphi QX \\ &- 2n(\tilde{\lambda}+1)\varphi X + (\xi\tilde{f})X - (X\tilde{f})\xi. \end{aligned} \tag{4.1}$$

Taking inner product in (4.1) with Y and using (2.5), we obtain

$$\begin{aligned} &g((\nabla_X\varphi)Y, D\tilde{\lambda}) + (\xi\tilde{\lambda})g(QX,Y) - 2n(\tilde{\lambda}+1)g(\varphi X, Y) \\ &+ \{2n(X\tilde{\lambda}) - X\tilde{f}\}\eta(Y) - (\tilde{\lambda}+1)g(\varphi QX, Y) + (\xi\tilde{f})g(X,Y) = 0. \end{aligned} \tag{4.2}$$

Substituting X by φX and Y by φY in (4.1), we get

$$\begin{aligned} &g((\nabla_{\varphi X}\varphi)\varphi Y, D\tilde{\lambda}) + (\xi\tilde{\lambda})g(Q\varphi X, \varphi Y) + 2n(\tilde{\lambda}+1)g(\varphi X, Y) \\ &+ (\tilde{\lambda}+1)g(Q\varphi X, Y) - (\xi\tilde{f})g(X,Y) + (\xi\tilde{f})\eta(X)\eta(Y) = 0. \end{aligned} \tag{4.3}$$

Subtracting (4.3) from (4.2) and using (2.6), we have

$$\begin{aligned} &2\xi(\tilde{f} - \tilde{\lambda})g(X,Y) + X\{(2n+1)\tilde{\lambda} - \tilde{f}\}\eta(Y) \\ &+ \xi(\tilde{\lambda} - \tilde{f})\eta(X)\eta(Y) + (\xi\tilde{\lambda})g(QX,Y) - (\xi\tilde{\lambda})g(Q\varphi X, \varphi Y) \\ &- (\tilde{\lambda}+1)\{g(\varphi QX, Y) + g(Q\varphi X, Y) + 4ng(\varphi X, Y)\} = 0. \end{aligned}$$

Antisymmetrizing the above equation, we get

$$\begin{aligned} &X\{(2n+1)\tilde{\lambda} - \tilde{f}\}\eta(Y) - Y\{(2n+1)\tilde{\lambda} - \tilde{f}\}\eta(X) \\ &- 2(\tilde{\lambda}+1)[g(Q\varphi X, Y) + g(\varphi QX, Y) + 4ng(\varphi X, Y)] = 0. \end{aligned}$$

Setting $X = \varphi X$ and $Y = \varphi Y$ in the above equation, we have

$$(\tilde{\lambda}+1)[g(Q\varphi X, Y) + g(\varphi QX, Y) + 4ng(\varphi X, Y)] = 0.$$

Since $\tilde{\lambda}$ is a non-constant function, the above equation gives

$$Q\varphi X + \varphi QX = -4n\varphi X. \tag{4.4}$$

Continuing the same processes as in the proof of Theorem 3.1, we have

$$r = -2n(2n+1)$$

From Lemma 2.3, we get $\tilde{f} = -r\left(\frac{\tilde{\lambda}}{2n} + \frac{1}{2n+1}\right)$. Since $r = -2n(2n+1)$, it follows that

$$(2n+1)\tilde{\lambda} - \tilde{f} = -2n \tag{4.5}$$

Proceeding in the same way as in proof of the Theorem 3.1, we obtain

$$\{(\xi\tilde{\lambda})^2 - (\tilde{\lambda}+1)^2\}(QD\tilde{\lambda} + 2nD\tilde{\lambda}) = 0.$$

From the above equation we have either (i) $QD\tilde{\lambda}+2nD\tilde{\lambda}=0$ or, (ii) $\xi\tilde{\lambda}=\pm(\tilde{\lambda}+1)$.

Case (i) : By similar argument as in the proof of Theorem 3.1, we get g is Einstein metric. Since g is Einstein, $z_g=0$. Therefore from (1.4) we have $(r'_g)^*\tilde{\lambda}=0$. This proves that $(g,\tilde{\lambda})$ satisfies the Fischer-Marsden equation.
Case (ii) : In this case $\xi\lambda=\pm(\tilde{\lambda}+1)$. Therefore $\xi(\xi\tilde{\lambda})=\pm(\xi\lambda)=\tilde{\lambda}+1$. Putting $X=\xi$ in (2.12), then taking inner product with ξ, we get

$$\xi(\xi\tilde{\lambda})=-2n(\tilde{\lambda}+1)+\tilde{f}.$$

As $\xi(\xi\tilde{\lambda})=\tilde{\lambda}+1$, we arrive in a contradiction by (4.5). So only Case (i) holds.

5 Example

In this section, we construct an example of a K-paracontact manifold which satisfies Miao-Tam equation, critical point equation and Fischer-Marsden equation.

We consider the three dimensional manifold $M=\{(x,y,z):(x,y,z)\in\mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Define the almost paracontact structure (φ,ξ,η) on M by

$$\varphi(e_1)=e_2,\quad \varphi(e_2)=e_1,\quad \varphi(e_3)=0,\quad \xi=e_3,\quad \eta=-dz$$

where $e_1=e^z\frac{\partial}{\partial x}$, $e_2=e^z\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)$, $e_3=-\frac{\partial}{\partial z}$. e_1,e_2,e_3 are linearly independent at each point of M. we have

$$[e_1,e_2]=0,\quad [e_1,e_3]=e_1,\quad [e_2,e_3]=e_2.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_1)=-g(e_2,e_2)=g(e_3,e_3)=1,\quad g(e_i,e_j)=0, i\neq j$$

where $i,j=1,2,3$.
By the linearity property of g and φ, we have

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y).$$

It is easy to verify that, (M,ϕ,ξ,η,g) is a K-paracontact manifold. Let ∇ be the Levi-Civita connection with respect to g. Then by Koszul formula

$$\nabla_{e_1}e_1=-e_3,\qquad \nabla_{e_1}e_2=0,\qquad \nabla_{e_1}e_3=e_1,$$

$$\nabla_{e_2}e_1=0,\qquad \nabla_{e_2}e_2=e_3,\qquad \nabla_{e_2}e_3=e_2,$$

$$\nabla_{e_3}e_1=0,\qquad \nabla_{e_3}e_2=0,\qquad \nabla_{e_3}e_3=0.$$

The components of the curvature tensor $R(X,Y)Z$ are

$$R(e_1,e_2)e_1=e_2,\quad R(e_1,e_2)e_2=e_1,\quad R(e_1,e_2)e_3=0,$$

$$R(e_1, e_3)e_1 = e_3, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = -e_1,$$
$$R(e_2, e_3)e_1 = 0, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_2, e_3)e_3 = -e_2.$$

The Ricci tensor is given by

$$S(X, Y) = g(R(e_1, X)Y, e_1) - g(R(e_2, X)Y, e_2) + g(R(e_3, X)Y, e_3)$$

for all $X, Y \in \chi(M)$. Using the components of the curvature tensor in the above, we have

$$S(e_1, e_1) = -S(e_2, e_2) = S(e_3, e_3) = -2,$$
$$S(e_1, e_2) = S(e_2, e_3) = S(e_1, e_3) = 0.$$

In view of above relation,

$$S(X, Y) = -2g(X, Y), \text{ and } r = -6$$

for all $X, Y \in \chi(M)$. So the manifold is Einstein.
Let $\lambda = e^{-z} + \frac{1}{2}$. By direct computation we have

$$D\lambda = \left(\lambda - \frac{1}{2}\right) e_3 \text{ and } \Delta_g \lambda = 3\left(\lambda - \frac{1}{2}\right)$$

.
Also $\nabla_X D\lambda = \left(\lambda - \frac{1}{2}\right) X$, for all $X \in \chi(M)$. Hence

$$-(\Delta_g \lambda)g(X, Y) + g(\nabla_X D\lambda, Y) - \lambda S(X, Y) = g(X, Y)$$

for all $X, Y \in \chi(M)$. This implies that g satisfies Miao-Tam equation and the example varifies Theorem 3.1.
Again taking $\tilde{\lambda} = e^{-z}$, similarly it can be verified that

$$-(\Delta_g \tilde{\lambda})g(X, Y) + g(\nabla_X D\tilde{\lambda}, Y) - \tilde{\lambda} S(X, Y) = z_g = 0,$$

for all $X, Y \in \chi(M)$. This implies that g satisfies critical point equation. Also g satisfies Fischer-Marsden equation. Hence the example verifies Theorem 4.1.

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Authors' address:

Avijit Sarkar
Department of Mathematics,
University of Kalyani,
Kalyani, Pin-741235,
West Bengal, India.
Email:*avjaj@yahoo.co.in*

Gour Gopal Biswas
Department of Mathematics,
University of Kalyani,
Kalyani, Pin-741235,
West Bengal, India.
Email: *ggbiswas6@gmail.com*