

Classification of Ricci solitons

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Abstract. There are two important aspects of Ricci solitons. One looking at the influence on the topology by the Ricci soliton structure of the Riemannian manifold, and the other looking at its influence in its geometry. In this paper, we are interested in summarizing some new results about the classification of Ricci solitons and its rigidity.

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1 Introduction

Under the leadership of the famous Chinese mathematician, Shing-Tung Yau, the use of analytical and differential equations to study differential geometry has become a very important trend, called geometric analysis. One of its representative work is that Yau used the method of geometric analysis to prove the Calabi conjecture and the positive quality conjecture. On this basis, the geometric analysis has developed a lot of research results. To what extent can the geometry of a differential manifold reflect its topology, how its topology affects its geometry, and how to analyze important differential equations through geometric invariants, geometric estimation, geometric differential equations, and geometric research conditions. It is one of the central research topics of differential geometry.

The fundamental problem of capturing the topological properties of a manifold by its metric structure opened, in the last decades, extremely fruitful areas of mathematics. From this perspective, there has been an increasing interest in the study of Riemannian manifolds endowed with metrics satisfying special structural equation, possibility involving the curvature and vector fields. One of the most important example is represented by Ricci flow and Ricci solitons, that have become the subject of rapidly increasing investigation since the appearance of the seminal works of Hamilton and Perelman. The Ricci flow plays a key role in Perelman's proof of the Poincaré conjecture, and has been widely used to study the topology, geometry and complex structure of manifolds. It also features prominently in the proof of the differentiable sphere theorem for point-wise pinched manifolds. The Ricci flow equation is of own

interest as a geometric partial differential equation, it gives a canonical way of a critical metric. It has been remarkably successful program over years.

The concept of Ricci solitons was introduced by Hamilton [41] in mid 80's. They are natural generalization of Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton's Ricci flow [37] and often arise as limits of dilations of singularities in the Ricci flow [39, 28, 11, 64]. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the spaces of Riemannian metrics mod diffeomorphisms and scaling. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in physics literature. In this paper, we summary some of recent progress on Ricci solitons as well as the role they play in the study of the rigidity.

2 Ricci solitons

2.1 Ricci solitons

Recall that [13] a Riemannian metric g_{ij} is Einstein if its Ricci tensor

$$(2.1) \quad R_{ij} = \rho g_{ij}$$

for some constant ρ . A smooth n -dimensional manifold M^n with an Einstein metric g_{ij} is an Einstein manifolds. Ricci solitons, introduced by Hamilton, are natural generalizations of Einstein metrics.

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *Ricci soliton* if there exists a smooth vector field $V = (V^i)$ such that the Ricci tensor of metric g_{ij} satisfies the equation

$$(2.2) \quad R_{ij} + \frac{1}{2}(\nabla_i V_j + \nabla_j V_i) = \rho g_{ij}$$

for some constant ρ . Moreover, if V is a gradient vector field, then we have a *gradient Ricci soliton*, satisfying the equation

$$(2.3) \quad R_{ij} + \nabla_i V_j f = \rho g_{ij}$$

for some smooth function f on M^n . The function f is called a *potential function* of the Ricci soliton. For $\rho = 0$, the Ricci soliton is *steady*, for $\rho > 0$ it is *shrinking* and for $\rho < 0$ it is *expanding*.

Since $\nabla_i V_j + \nabla_j V_i$ is the Lie derivative of the metric g_{ij} in the direction of V , we also write the Ricci soliton equations (2.2) and (2.3) as

$$(2.4) \quad Ric + \frac{1}{2}L_V g = \rho g, Ric + \nabla^2 f = \rho g$$

respectively.

When the underlying manifold is a complex manifold, we have the corresponding notion of Kähler-Ricci solitons. A complete Kähler metric $g_{\alpha\bar{\beta}}$ on a complex manifold X^n of complex dimension n is called a *Kähler-Ricci soliton* if there exists a holomorphic vector field $V = (V^\alpha)$ on X such that the Ricci tensor $R_{\alpha\bar{\beta}}$ of the metric $g_{\alpha\bar{\beta}}$ satisfies the equation

$$(2.5) \quad R_{\alpha\bar{\beta}} + \frac{1}{2}(\nabla_{\bar{\beta}} V_\alpha + \nabla_\alpha V_{\bar{\beta}}) = \rho g_{\alpha\bar{\beta}}$$

for some constant ρ . It is called a *gradient Kähler-Ricci soliton* if the holomorphic vector field V comes from the gradient vector field of a real-valued function f on X^n so that

$$(2.6) \quad R_{\alpha\bar{\beta}} + \nabla_\alpha V_{\bar{\beta}} f = \rho g_{\alpha\bar{\beta}},$$

and

$$(2.7) \quad \nabla_\alpha V_\beta f = 0.$$

Note that the case $V = 0$ (i.e., f being a constant function) is an Einstein (or Kähler-Einstein) metric. Thus Ricci solitons are natural extensions of Einstein metrics. Also, by a suitable scale of the metric g , we can normalize $\rho = 0, +\frac{1}{2}, -\frac{1}{2}$.

2.2 Examples of Ricci solitons

When $n \geq 4$, there exist non-trivial compact gradient shrinking solitons. Also, there exist complete non-compact Ricci solitons (steady, shrinking and expanding) that are not Einstein. Below we list a number of such examples.

Example 2.1 (The cigar soliton). In dimension two, Hamilton [41] discovered the first example of a complete non-compact steady soliton on \mathbb{R}^2 , called the cigar soliton, where the metric is given by $ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$ with potential function

$$f = -\log(1+x^2+y^2).$$

The cigar has positive Gaussian curvature $R = 4e^f$ and linear volume growth, and is asymptotic to a cylinder of finite circumference at infinity.

Example 2.2 (The Bryant soliton). In the Riemannian case, higher dimensional examples of non-compact gradient steady solitons were found by Bryant on \mathbb{R}^n ($n \geq 3$), they are rotationally symmetric and have positive sectional curvature. Furthermore, the geodesic sphere \mathbb{S}^{n-1} of radius r has the diameter on the order \sqrt{r} . Thus the volume of geodesic balls $B_r(0)$ grows on the order of $r^{\frac{(n+1)}{2}}$.

Example 2.3 (Warped products). Using doubly warped product and multiple warped product constructions, Ivey [44] produced non-compact gradient steady solitons, which generalize the construction of Bryant's soliton. Also, Gastel-Kronz [35] produced a two-parameter family (doubly warped product metrics) of gradient expanding solitons on $\mathbb{R}^{m+1} \times \mathbb{N}$, where \mathbb{N}^n , ($n \geq 2$) is an Einstein manifold with positive scalar curvature.

Example 2.4 (Gaussian solitons). (\mathbb{R}^n, g_0) with flat Euclidean metric can be also equipped with both shrinking and expanding gradient Ricci solitons, called the Gaussian shrinker or expander.

(a) $(\mathbb{R}^n, g_0, \frac{|x|^2}{4})$ is a gradient shrinker with potential function $f = \frac{|x|^2}{4}$,

$$Ric + \nabla^2 f = \frac{1}{2}g_0$$

(b) $(\mathbb{R}^n, g_0, -\frac{|x|^2}{4})$ is a gradient shrinker with potential function $f = -\frac{|x|^2}{4}$,

$$Ric + \nabla^2 f = -\frac{1}{2}g_0$$

Example 2.5 (Compact gradient Kähler shrinkers). For real dimension 4, the first example of a compact shrinking soliton was constructed in early 90's by Koiso [47] and Cao [11] on compact complex surface $\mathbb{C}\mathbb{P}^2 \# (-\mathbb{C}\mathbb{P}^2)$, where $(-\mathbb{C}\mathbb{P}^2)$ denotes the complex projective space with the opposite orientation. This is a gradient Kähler-Ricci soliton, has $U(2)$ symmetry and positive Ricci curvature. More generally, they found $U(n)$ -invariant Kähler-Ricci soliton on twisted projective line bundle over $\mathbb{C}\mathbb{P}^{n-1}$ for $n \geq 2$.

Example 2.6 (Noncompact gradient Kähler shrinkers). Feldman-Ilmanen-Knopf [33] found the first complete noncompact $U(n)$ -invariant shrinking gradient Kähler-Ricci solitons, which are cone-like at infinity. It has positive scalar curvature but the Ricci curvature does not have a fixed sign.

Example 2.7 (Noncompact gradient steady Kähler solitons). In the Kähler case, Cao [12] found two examples of complete rotationally noncompact gradient steady Kähler-Ricci solitons:

(a) On \mathbb{C}^n (for $n = 1$ it is just the cigar soliton). These examples are $U(n)$ -invariant and have positive sectional curvature. It is interesting to point out that the geodesic sphere \mathbb{S}^{2n-1} of radius s is an \mathbb{S}^1 -bundle over $\mathbb{C}\mathbb{P}^{n-1}$ where the diameter of \mathbb{S}^1 is on the order 1, while the diameter of $\mathbb{C}\mathbb{P}^{n-1}$ is on order \sqrt{s} . Thus the volume of geodesic balls $B_r(0)$ grow on the order of r^n , n being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

(b) On the below-up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. This is the same space on which Eguchi-Hansen ($n = 2$) and Calabi ($n \geq 2$) constructed examples of Hyper-Kähler metrics. For $n = 2$, the underlying space is the canonical line bundle over $\mathbb{C}\mathbb{P}^1$.

Example 2.8 (Noncompact gradient expanding Kähler solitons). Cao [11] constructed a one-parameter family of complete noncompact expanding solitons on \mathbb{C}^n . These expanding Kähler-Ricci solitons all have $U(n)$ symmetry and positive sectional curvature, and are cone-like infinity.

3 Classification of gradient shrinking Ricci solitons

Gradient Ricci solitons play a fundamental role in Hamilton's Ricci flow as they correspond to self-similar solutions, and also arise as singularity models. From the seminal work of Hamilton and Perelman that any compact Ricci soliton is necessarily a gradient soliton, it is to see that any compact steady or expanding Ricci soliton must be Einstein. Therefore, it is crucial to classify gradient Ricci solitons and understand their geometry. Some results about the classification of solitons were obtained in the last decades. These results were derived under conformally flat, constant scalar curvature, nonnegative Ricci curvature or bounded compact nonnegative curvature operator. In dimension 2, Hamilton [41] proved that any 2-dimensional complete non-flat ancient solution of bounded curvature must be \mathbb{S}^2 , $\mathbb{R}\mathbb{P}^2$ or the cigar soliton. The two dimensional case is well understand and all complete Ricci solitons have been classified, see for instance the very recent [2] and references therein.

First we will focus our attention on complete gradient shrinking Ricci solitons, which are possible Type I singularity models in the Ricci flow. From the seminal work

gradient solitons, it is to see that any compact steady or expanding Ricci solitons must be Einstein.

Indeed, the below-up around Type I singularity point always converge to nontrivial gradient shrinking Ricci solitons. And a theorem of Perelman states that given any non-flat k -non-collapsed ancient soliton to Ricci flow with bounded and nonnegative curvature operator, the limit of some suitable below-back of the solution converges to a non-flat gradient shrinking soliton. Thus knowing the geometry of gradient shrinking solitons also helps us to understand the asymptotic behavior of ancient solitons.

In dimension 2, Hamilton completely classified shrinking gradient Ricci solitons with bounded curvature and proved that they are the sphere, the projecture space and the Euclidean space with constant curvature. In dimension 3, due to the efforts of Ivey [43], Perelman [57], Ni-Wallach [55], and Cao-Chen-Zhu [21], shrinking solitons have been completely classified: they are quotients of either the round sphere \mathbb{S}^3 , the round cylinder $\mathbb{R} \times \mathbb{S}^2$ or the shrinking Gaussian soliton \mathbb{R}^3 .

Theorem 3.1 (Perelman [58]). *There is no three-dimensional complete non-compact, k -non-collapsed gradient shrinking soliton with bounded and positive sectional curvature.*

Based on the investigation of the shrinking soliton equation $R_{ij} + f_{ij} + \frac{g_{ij}}{2t} = 0$ where $t < 0$ and applying Hamilton's strong maximum principle, Perelman proved:

Theorem 3.2. *Let (M^3, g_{ij}, f) be a non-flat gradient shrinking soliton to the Ricci flow on a three-manifold. Suppose (M^3, g_{ij}, f) has bounded and nonnegative sectional curvature and is k -non-collapsed on all scales for some $k > 0$. Then (M^3, g_{ij}, f) is one of the followings:*

- (a) *The round three-sphere \mathbb{S}^3 , or its metric quotients;*
- (b) *The round infinite cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its \mathbb{Z}_2 quotients.*

Under the assumption on k -non-collapsing and nonnegative sectional curvature condition, Cao generalize the results of Perelman.

Corollary 3.3 (Cao [14]). *The only three-dimensional complete non-compact k -non-collapsed gradient shrinking soliton with bounded and nonnegative sectional curvature are either \mathbb{R}^3 or quotients of $\mathbb{S}^2 \times \mathbb{R}$.*

The above Perelman's result has been improved by Ni-Wallach [55] and Naber [54], in which they dropped the assumption on k -non-collapsing condition and replaced nonnegative sectional curvature by nonnegative Ricci curvature.

Theorem 3.4 (Ni-Wallach [55]). *Let (M^n, g_{ij}, f) be a gradient shrinking soliton whose Ricci curvature is positive and satisfying $|R_{ijkl}(x)| \leq \exp(a(r(x) + 1))$ for some $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. Then M must be compact.*

Corollary 3.5. *Any three-dimensional complete non-compact gradient shrinking soliton with nonnegative Ricci curvature $Ric \geq 0$ and curvature bound $|Rm|(x) \leq Ce^{ar(x)}$ is a quotient of the round sphere \mathbb{S}^3 or round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

Ni-Wallach proved a more general result about the classification of three-dimensional gradient shrinking soliton since he assumed neither that gradient shrinking soliton is k -non-collapsed nor that the curvature is uniformly bounded.

Corollary 3.6. *Let (M^3, g_{ij}, f) be a complete gradient shrinking soliton with the positive sectional curvature and the Ricci curvature satisfies $|Ric|(y, t) \leq \exp(\varepsilon r^2(x) + \beta(\varepsilon))$ where any $\varepsilon > 0$, $\beta(\varepsilon) > 0$, for all $y \in B_{g(-\frac{1}{2})}(x, \frac{r(x)}{2})$ and $t \in [-\frac{1}{2}, 0]$. Then M must be the quotient of \mathbb{S}^3 .*

Under the additional assumption of being gradient, though not k -non-collapsed, the following was proved by using techniques more in line with maximum principles.

Corollary 3.7 (Naber [54]). *Let (M^3, g_{ij}, f) be a 3-dimensional shrinking gradient soliton with bounded curvature and $Ric \geq 0$. Then (M^3, g_{ij}) is isometric to \mathbb{R}^3 or to a finite quotient of the round sphere \mathbb{S}^3 or round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

Subsequently, Cao-Chen-Zhu [21] observed that one can remove all the curvature bound assumption.

Corollary 3.8. *Let (M^3, g_{ij}, f) be a 3-dimensional complete non-flat shrinking gradient soliton. Then (M^3, g_{ij}) is a quotient of the round sphere \mathbb{S}^3 or round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

Corollary 3.9. *Let (M^3, g_{ij}, f) be a 3-dimensional complete non-compact non-flat shrinking gradient soliton. Then (M^3, g_{ij}) is a quotient of the round neck $\mathbb{S}^2 \times \mathbb{R}$.*

Extending to the non-gradient case the previous of Perelman, Catino-Mastrolia-Monticelli-Rigoli got a new result.

Corollary 3.10 (Catino-Mastrolia-Monticelli-Rigoli [25]). *Let (M^3, g_{ij}, f) be a 3-dimensional complete generic shrinking Ricci soliton. Furthermore, if M is non-compact, assume that the curvature is bounded and $|\nabla X| = o(|X|)$ as $r \rightarrow \infty$. Then (M^3, g_{ij}) is isometric to a finite quotient of either \mathbb{S}^3 , \mathbb{R}^3 or $\mathbb{S}^2 \times \mathbb{R}$.*

The first classification theorem with $n \geq 4$ given by Gu-Zhu [36] that any non-flat, k -non-collapsing, rotationally symmetric gradient shrinking soliton with bounded and nonnegative sectional curvature must be the finite quotients of $\mathbb{S}^n \times \mathbb{R}$ or \mathbb{S}^{n+1} . Later, Kotschwar [48] improved this result showed that any complete rotationally symmetric gradient shrinking is the finite quotients of \mathbb{R}^{n+1} , $\mathbb{S}^n \times \mathbb{R}$ or \mathbb{S}^{n+1} .

The combination of the Hamilton's sphere theorem and Hamilton's strong maximum principle gives a complete classification of 3-dimensional compact manifolds with nonnegative Ricci curvature. By using his advanced maximum principle in a similar way, Hamilton [38] also proved a 4-dimensional differentiable sphere theorem.

Theorem 3.11 (Hamilton [38]). *A compact 4-manifold with positive curvature operator is diffeomorphic to the sphere \mathbb{S}^4 or the real projective space $\mathbb{R}P^4$.*

Hamilton also obtained the following classification theorem for four-manifolds with nonnegative curvature operators.

Theorem 3.12. *A compact 4-manifold with nonnegative curvature operator is diffeomorphic to one of the sphere \mathbb{S}^4 or $\mathbb{C}P^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$ or a quotient of one of the spaces \mathbb{S}^4 or $\mathbb{C}P^2$ or $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{S}^2 \times \mathbb{R}^2$ or \mathbb{R}^4 by a group of fixed point free isometries in the standard metrics.*

Naturally, one would ask if a compact Riemannian manifold M^n , with $n \geq 5$, of positive curvature operator (or 2-positive curvature operator) is diffeomorphic to a space form. This was in fact conjectured so by Hamilton, and proved by Böhm-Wilking [9], they developed a powerful new method to construct closed convex sets, which are invariant under the Ricci flow, in the space of curvature operator.

Corollary 3.13. *A compact Riemannian manifold of dimension $n \geq 5$ with positive curvature operator is diffeomorphic to a spherical space form.*

We remark that in 1988, by using minimal surface theory, Micallef-Moore [51] proved that any compact simply connected n -dimensional manifold with positive isotropic curvature is homeomorphic to the n -sphere \mathbb{S}^n , and the condition of positive isotropic curvature is weaker than both positive curvature operator and 1/4-pinched. Very recently, Brendle-Schoen [7] showed that when the initial metric has 1/4-pinched sectional curvature (in fact, under the weaker curvature condition that $M \times \mathbb{R}^2$ has positive isotropic curvature), the Ricci flow will converge to a spherical space form. As a corollary, they proved the long-standing Differential Sphere Theorem.

Theorem 3.14 (Brendle-Schoen [7]). *Let (M^n, g_{ij}, f) be a compact manifold with (point-wise) 1/4-pinched sectional curvature. Then M is diffeomorphic to \mathbb{S}^n or a quotient of \mathbb{S}^n by a group of fixed point free isometries in the standard metrics.*

By using the strong maximum principle to a powerful version, Brendle-Schoen[6] even obtained the following rigidity result.

Corollary 3.15. *Let M be a compact manifold with (point-wise) weakly 1/4-pinched sectional curvature in the sense that $0 \leq \text{sect}(P_1) \leq 4\text{sect}(P_2)$ for all two-planes $P_1, P_2 \in T_p M$. If M is not diffeomorphic to a spherical space form, then it is isometric to a locally symmetric space.*

Very recently, there are many new results about the classification of gradient shrinking solitons with nonnegative curvature operator, bounded nonnegative sectional curvature or some additional conditions. For $n = 4$, Ni-Wallach [56] showed that any 4-dimensional complete gradient shrinking soliton with nonnegative curvature operator and positive isotropic curvature, satisfying certain additional assumptions, is either a quotient of \mathbb{S}^4 or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. Based on this result, Naber [54] proved the following result.

Corollary 3.16. *Any 4-manifold complete non-compact shrinking Ricci soliton with bounded nonnegative curvature operator is isometric to either \mathbb{R}^4 or a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$ or $\mathbb{S}^3 \times \mathbb{R}$.*

Corollary 3.17. *A 4-manifold non-flat complete non-compact shrinking Ricci soliton with bounded nonnegative curvature operator is isometric to a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$ or $\mathbb{S}^3 \times \mathbb{R}$.*

For higher dimension, Gu-Zhu [36] proved that any complete, rotationally symmetric, non-flat, n -dimensional ($n \geq 3$) shrinking Ricci soliton with k -non-collapsing on all scales and with bounded and nonnegative sectional curvature must be the round sphere \mathbb{S}^n or the round cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$.

Theorem 3.18 (Petersen-Wylie [61]). *If (M^n, g_{ij}, f) is a shrinking gradient Ricci soliton with nonnegative sectional curvature and $R \leq 2\rho$, then the universal cover of M is isometric to either \mathbb{R}^n or $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.*

In the complete non-compact case, the identity

$$\int_M |\nabla Ric|^2 e^{-f} d\mu = \int_M |\operatorname{div} Rm|^2 e^{-f} d\mu$$

yields a classification of locally conformally flat gradient shrinking Ricci solitons with Ricci curvature bounded from below.

Theorem 3.19 (Cao-Wang-Zhang [26]). *Let (M^n, g_{ij}, f) , $n \geq 3$, be a complete non-compact gradient shrinking soliton whose Ricci curvature is bounded $|R_{ij}|(x) \leq \exp(a(r(x) + 1))$. Assume that it is locally conformally flat. Then its universal cover is either \mathbb{R}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Applying a theorem about Riemannian curvature tensor growing, Munteanu-Wang [53] proved a gap result for gradient shrinking solitons.

Corollary 3.20. *Let (M^n, g_{ij}, f) be a shrinking gradient Ricci soliton. If $|Rc| \leq \frac{1}{100n^2}$ on M , then M is isometric to the Gaussian soliton.*

Under the assumption that $DRic$ decays polynomially with a degree depending on other geometric quantities, Cai [10] proved:

Corollary 3.21. *Let (M^n, g_{ij}, f) be a complete non-compact gradient shrinking Ricci soliton with bounded nonnegative sectional curvature. Assume that there exist $\delta > 0$ such that $\int_M e^{\delta f} |DRic| d\operatorname{vol}_g < \infty$. Then (M^n, g_{ij}) is isometric to $\mathbb{N} \times \mathbb{R}^m$, where \mathbb{N} is a compact Einstein manifold.*

Note that this is the first rigidity result in high dimensions without assumptions on the Weyl tensor. The potential function is known to grow quadratically with respect to the distance from a fixed point, so the condition on $DRic$ says that it decays exponentially. The Cheeger-Gromoll soul theorem states that an open manifold with nonnegative sectional curvature is diffeomorphic to a vector bundle over a compact sub-manifold called a soul. The pull-back metric on the bundle can be highly twisted. However, if there exists a gradient soliton structure on such a bundle, then the metric has to be locally trivial, provided that the decay condition is satisfied. The decay condition on $DRic$ is imposed in the region where f is large. And the next corollary deals with the rigidity under a condition on $DRic$ imposed in the region where f is small.

Corollary 3.22. *Let (M^n, g_{ij}, f) be a complete gradient shrinking Ricci soliton with bounded nonnegative sectional curvature. Assume that the minima of f is a smooth compact non-degenerate critical sub-manifold, $DRic$ and $D^2 Ric$ vanish on the minima, then (M^n, g_{ij}) is non-compact and isometric to $\mathbb{N} \times \mathbb{R}^m$, where \mathbb{N} is a compact Einstein manifold.*

Theorem 3.23 (Yang-Zhang [68]). *Let (M^4, g_{ij}, f) be a 4-dimensional gradient shrinking Ricci soliton. If $\operatorname{div}^4 Rm = 0$ or $\operatorname{div}^3 Rm(\nabla f) = 0$ or $\operatorname{div}^3 W(\nabla f) = 0$, then*

(M^4, g_{ij}) is either

(a) Einstein, or

(b) a finite quotient of the Gaussian shrinking soliton \mathbb{R}^4 , $\mathbb{S}^2 \times \mathbb{R}^2$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.

As a generalization,

Corollary 3.24. *Let (M^4, g_{ij}, f) be a 4-dimensional rigid gradient shrinking Ricci soliton, then (M^4, g_{ij}) is either*

(a) Einstein, or

(b) a finite quotient of the Gaussian shrinking soliton \mathbb{R}^4 , $\mathbb{S}^2 \times \mathbb{R}^2$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.

The Ricci soliton can be interpreted as a prescribing condition on the Ricci tensor of g , that is on the trace part of the Riemannian tensor. Thus, we can except classification results for these structures only assuming further conditions on the traceless part of the Riemannian tensor, i.e., on the Weyl tensor W , if $n \geq 4$. For higher dimensions, it has been proven by several authors under curvature conditions on the Weyl tensor that complete locally conformally flat gradient shrinking Ricci solitons are finite quotients of either the round sphere \mathbb{S}^n , or the Gaussian shrinking soliton \mathbb{R}^n , or the around cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Under the weaker condition of harmonic Weyl tensor, n -dimensional complete gradient shrinking solitons are rigid in the sense that they are either Einstein, or finite quotient of the product $\mathbb{N}^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$, where \mathbb{N}^k is a k -dimensional Einstein manifold of positive scalar curvature.

The so called Weyl tensor is defined by the following decomposition formula in dimension $n \geq 3$,

$$W_{ijkl} = R_{ijkl} + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il})$$

The Cotton tensor is defined as

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R)$$

The Schouten tensor is defined as

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij}$$

The Weyl tensor satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero. Recall that a Riemannian manifold is locally conformally flat if the Weyl tensor vanishes.

Theorem 3.25 (Catino-Mantegazza [22]). *Any compact n -dimensional, locally conformally flat Ricci soliton is quotient of \mathbb{R}^n , \mathbb{S}^n and \mathbb{H}^n with their canonical metrics, for every $n \in \mathbb{N}$.*

The analysis of Ktoschwar [48] of rotationally invariant shrinking gradient Ricci solitons gives the following classification where the Gaussian soliton is defined as the flat \mathbb{R}^n with a potential function $f = \frac{\alpha|x|^2}{2n}$, for a constant α .

Theorem 3.26. *The shrinking gradient locally conformally flat Ricci solitons of dimension $n \geq 4$ are given by the quotients of \mathbb{S}^n , the Gaussian solitons with $\alpha > 0$ and the quotients of $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Instead of assuming the uniform bound on curvature, we only need very mild growth control on the curvature. Maybe more importantly we do not assume that the gradient shrinking soliton is k -non-collapsed, as required by Perelman.

Theorem 3.27 (Ni-Wallach [55]). *Let (M^n, g_{ij}, f) be a gradient shrinking Ricci soliton whose Ricci curvature is nonnegative. If $n \geq 4$, we assume that (M^n, g_{ij}, f) is locally conformally flat. Assume further that $|R_{ijkl}|(x) \leq \exp(a(r(x) + 1))$ for some $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. Then its universal cover is either \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Corollary 3.28. *Let (M^n, g_{ij}, f) , $n \geq 4$, be a complete noncompact gradient shrinking Ricci soliton whose Ricci curvature satisfies $|R_{ijkl}|(x) \leq \exp(a(r(x) + 1))$ for some constant $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. Assume that (M^n, g_{ij}) is locally conformally flat, then its universal cover is either \mathbb{R}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

If we assume neither that gradient shrinking soliton is k -non-collapsed nor that the curvature is uniformly bounded.

Corollary 3.29. *Let (M^n, g_{ij}, f) be a locally conformally flat gradient shrinking Ricci soliton whose Ricci curvature is nonnegative satisfying $|\text{Ric}|(y, t) \leq \exp(\varepsilon r^2(x) + \beta(\varepsilon))$ where any $\varepsilon > 0$, $\beta(\varepsilon) > 0$, for all $y \in B_{g(-\frac{1}{2})}(x, \frac{r(x)}{2})$ and $t \in [-\frac{1}{2}, 0]$. Then its universal cover is either \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

By a maximality argument, passing to the universal covering of the manifold, Catino-Mantegazza [22] got the following conclusion.

Corollary 3.30. *If $n \geq 4$, any n -dimensional locally conformally flat Ricci soliton with constant scalar curvature is either a quotient of \mathbb{S}^n , \mathbb{R}^n and \mathbb{H}^n with their canonical metrics, or a quotient of the Riemannian products $\mathbb{S}^{n-1} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{H}^{n-1}$.*

Corollary 3.31. *If $n \geq 4$, any n -dimensional locally conformally flat Ricci soliton with nonnegative Ricci tensor is either a quotient of \mathbb{R}^n and \mathbb{S}^n with their canonical metrics, or a quotient of $\mathbb{S}^{n-1} \times \mathbb{R}$ or it is a warped product \mathbb{S}^{n-1} of on a proper interval of \mathbb{R} .*

Based on the Hamilton-Ivey type pinching estimate on higher dimension:

$$R \geq (-\nu) \left[\log(-\nu) + \log(1 + t) - \frac{n(n+1)}{2} \right]$$

at all points and all times $t \geq 0$, whenever $\nu < 0$. Zhang obtained the following theorem (without any curvature bound assumption).

Theorem 3.32 (Zhang [69]). *Any complete gradient shrinking soliton with vanishing Weyl tensor must be the finite quotients of \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Note that complete locally conformally flat gradient Ricci solitons, i.e. $W_{ikjl} = 0$. And any rationally symmetric metric has vanishing Weyl tensor.

By using a different set of formulas Petersen-Wylie[60] proved:

Corollary 3.33. *Let (M^n, g_{ij}, f) be a complete shrinking gradient soliton of dimension $n \geq 3$ such that $\int_M |Ric|^2 e^{-f} dvol_g < \infty$ and $W = 0$. Then (M^n, g_{ij}) is infinite of \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

If relax the Weyl curvature condition and instead assume that the scalar curvature is constant they also got a nice general classification.

Corollary 3.34. *Let (M^n, g_{ij}, f) be a complete shrinking gradient Ricci soliton with $n \geq 3$, constant scalar curvature, and $W(\nabla f, \cdot, \cdot, \nabla f) = o(|\nabla f|^2)$. Then M is a flat bundle of rank 0, 1 or n over an Einstein manifold.*

Then Catino [15] generalized the previous result concerning the classification of complete gradient shrinking Ricci solitons to the case when Ricci tensor is nonnegative and a very general pinching condition on the Weyl tensor is in force, without assume the soliton metric to be locally conformally flat.

Corollary 3.35. *Any k -dimensional complete gradient shrinking Ricci soliton with nonnegative Ricci curvature and satisfying*

$$|W|S \leq \sqrt{\frac{2(n-1)}{n-2}} \left(|T| - \frac{1}{\sqrt{n(n-1)}} S \right)^2$$

is a finite quotient of \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$. Where $T = Ric - \frac{1}{n}Sg$.

In higher dimensions,

Corollary 3.36 (Catino-Mastrolia-Monticelli-Rigoli [25]). *Let (M^n, g_{ij}, f) be a complete generic shrinking Ricci soliton of dimension $n > 3$. Furthermore, if M is non-compact, assume that the curvature is bounded and $|\nabla X| = o(|X|)$ as $r \rightarrow \infty$. If for some $a > 0$, $|Ric| \leq aS$, and*

$$|W|S \leq \sqrt{\frac{2(n-1)}{n-2}} \left(|T| - \frac{1}{\sqrt{n(n-1)}} S \right)^2$$

Then (M^n, g_{ij}) is isometric to a finite quotient of either \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.

With harmonic Weyl tensor, Menunteanu-Sesum [52] extended the results from above.

Theorem 3.37. *Any n -dimensional complete gradient shrinking Ricci soliton with harmonic Weyl tensor is a finite quotient of \mathbb{R}^n , \mathbb{S}^n , or $\mathbb{S}^{n-1} \times \mathbb{R}$.*

With the following definitions,

$$div^4(W) = \nabla_k \nabla_j \nabla_l \nabla_i W_{ikjl}$$

$$div^3(C) = \nabla_i \nabla_j \nabla_k W_{ijk}$$

Where W and C are the Weyl and Cotton tensors, respectively. Note that, in dimension $n \geq 4$, $\operatorname{div}^4(W) = 0$ if and only if $\operatorname{div}^3(C) = 0$. Then Catino used these equations to improve the results on gradient shrinking solitons with harmonic Weyl tensor in [69].

Corollary 3.38 (Catino-Mastrolia-Monticelli [24]). *Every complete gradient shrinking Ricci soliton of dimension $n \geq 4$ with $\operatorname{div}^4(W) = 0$ on M is either Einstein is isometric to a finite quotient of $\mathbb{N}^{n-k} \times \mathbb{R}^k$, ($k > 0$), the product of a Einstein manifold \mathbb{N}^{n-k} with the Gaussian shrinking soliton \mathbb{R}^k .*

Dimension four is the lowest dimension where there are interesting examples of shrinking gradient Ricci solitons. The first examples were constructed by Koiso [46] and Cao [11]. Note that all of the known interesting examples are Kähler. In dimension 4, the Hodge star splits the space of 2-forms into the self dual and anti-self dual parts and consequently the curvature tensor and Weyl tensor respect this splitting. It is thus natural to consider self dual and anti-self dual part of Weyl curvature W^\pm commonly called the half Weyl curvature. Chen-Wang [27] and Cao-Chen [18] proved that half conformally flat ($W^\pm = 0$) four dimensional gradient shrinking Ricci soliton is a finite quotient of \mathbb{S}^4 , $\mathbb{C}\mathbb{P}^2$, \mathbb{R}^4 , or $\mathbb{S}^3 \times \mathbb{R}$.

Half conformally flat metrics are also known as self-dual or anti-self-dual if $W^- = 0$ or $W^+ = 0$, respectively. For anti-self-dual soliton, Chen-Wang [27] proved:

Theorem 3.39. *Any 4-dimensional complete gradient shrinking Ricci soliton with bounded curvature and $W^+ = 0$ must be isometric to finite quotients of \mathbb{S}^4 , $\mathbb{C}\mathbb{P}^2$, \mathbb{R}^4 , or $\mathbb{S}^3 \times \mathbb{R}$.*

By a theorem of [1], a compact 4-dimensional half-conformally Einstein manifold (of positive scalar curvature) is \mathbb{S}^4 or $\mathbb{C}\mathbb{P}^2$. Combining Hitchin's theorem, Cao arrive the following classification of 4-dimensional compact half-conformally flat gradient shrinking Ricci solitons which was first obtained by Chen-Wang [27].

Corollary 3.40. *Let (M^4, g_{ij}, f) be a compact half-conformally flat gradient shrinking Ricci soliton, then (M^4, g_{ij}) is isometric to the standard \mathbb{S}^4 or $\mathbb{C}\mathbb{P}^2$.*

We know that a compact four-dimensional gradient shrinking Ricci soliton with $\delta W^\pm = 0$ and half two-nonnegative curvature operator (which is equivalent to half nonnegative isotropic curvature) is finite quotient of \mathbb{S}^4 for Kähler-Einstein. Then Wu-Wu-Wylie [67] complete the classification of four-dimensional gradient shrinking Ricci solitons with harmonic half Weyl curvature.

Corollary 3.41. *A four-dimensional gradient shrinking Ricci soliton with $\delta W^\pm = 0$ is either Einstein, or a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$, \mathbb{R}^4 , or $\mathbb{S}^3 \times \mathbb{R}$.*

Bach tensor was introduced by Bach in early 1920s' to study conformally relativity. On any n -dimensional manifold (M^n, g_{ij}) , ($n \geq 4$) the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ij}^{kl}$$

Here W_{ikjl} is the Weyl tensor. It is easy to see that if (M^n, g_{ij}) is either locally conformally flat (i.e. $W_{ikjl} = 0$) or Einstein, then (M^n, g_{ij}) is Bach-flat: $B_{ij} = 0$.

This can be seen as a vanishing condition involving second and zero order terms in Weyl, which a posteriori captures a more rigid class of solitons than in the harmonic Weyl case. In addition, in dimension $n = 4$, if a 4-manifold is half-conformally flat or locally conformal to an Einstein 4-manifold, then it is also Bach flat. Bach flat metrics are precisely the critical points of the conformally invariant functional metrics. Recent, Cao-Chen [18] shown that Bach-flat gradient shrinking Ricci solitons are either Einstein, or finite quotients of \mathbb{R}^n or $\mathbb{N}^{n-1} \times \mathbb{R}$, where \mathbb{N}^{n-1} is an $(n - 1)$ -dimensional Einstein manifold.

Theorem 3.42 (Cao-Chen [18]). *Let (M^n, g_{ij}, f) , $(n \geq 5)$ be a complete Bach-flat gradient shrinking Ricci soliton, then (M^n, g_{ij}) is either*

- (a) *Einstein, or*
- (b) *a finite quotient of the Gaussian shrinking soliton \mathbb{R}^n , or*
- (c) *a finite quotient of $\mathbb{N}^{n-1} \times \mathbb{R}$, where \mathbb{N}^{n-1} is an Einstein manifold of positive scalar curvature.*

Corollary 3.43. *Let (M^4, g_{ij}, f) be a 4-dimensional complete Bach-flat gradient shrinking Ricci soliton, then (M^4, g_{ij}) is either*

- (a) *Einstein, or*
- (b) *Locally conformally flat, hence a finite quotient of either the Gaussian shrinking soliton \mathbb{R}^4 or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.*

In their study of the geometry of locally conformally flat and Bach flat gradient solitons, Cao-Chen [19] introduced a three tensor D_{ijk} related to the geometry of the level surfaces of the potential function:

$$D_{ijk} = \frac{1}{n-2}(A_{jk} \nabla_i f - A_{ik} \nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk} E_{il} - g_{ik} E_{jl}) \nabla_l f$$

Where A_{ij} is the Schouten tensor and E_{ij} is the Einstein tensor. The vanishing of D , which is a consequence of the curvature assumption on Weyl, is crucial ingredient in their classification results. Then Cao-Chen [18] proved that:

Corollary 3.44. *Let (M^n, g_{ij}, f) , $(n \geq 4)$ be a complete gradient shrinking Ricci soliton with $D_{ijk} = 0$, then*

- (a) *(M^4, g_{ij}, f) is either Einstein, or a finite quotient of \mathbb{R}^4 or $\mathbb{S}^3 \times \mathbb{R}$;*
- (b) *For $n \geq 5$, (M^n, g_{ij}, f) is either Einstein, or a finite quotient of the Gaussian shrinking soliton \mathbb{R}^n , or a finite quotient of $\mathbb{N}^{n-1} \times \mathbb{R}$, where \mathbb{N}^{n-1} is an Einstein.*

4 Classification of gradient steady Ricci solitons

Next, we will study the gradient steady Ricci solitons, which are possible Type II singularity models and correspond to translating solutions in the Ricci flow.

Hamilton [41] showed that the only complete steady soliton on a 2-dimensional manifold with bounded scalar curvature which attains its maximum at a point or with positive curvature is the cigar soliton up to a scaling. For $n \geq 3$, Bryant showed that there exists an unique complete rotationally symmetric gradient Ricci soliton on \mathbb{R}^n up to scaling. In higher dimensions, Cao-Chen proved in [19] that complete n -dimensional ($n \geq 3$) locally conformally flat gradient steady Ricci solitons are isometric to either

a finite quotient of or the Bryant soliton. When $n = 4$, Chen-Wang [27] showed that any four dimensional complete half-conformally flat gradient steady Ricci soliton is either Ricci flat, or isometric to the Bryant soliton. Again, these are rigidity results under zero order conditions on Weyl.

In the steady three dimensional case the known examples are given by quotients of \mathbb{R}^3 , $\mathbb{R} \times \Sigma^2$ and the rotationally symmetric one constructed by Bryant. It is still an open problem to classify three dimensional steady solitons. But it is well-known that compact gradient steady solitons must be Ricci flat.

Provided that (M^n, g_{ij}, f) satisfies certain asymptotic conditions near infinity and also inspired in part by Robinson's proof of the uniqueness of the Schwarzschild black hole. Brendle proved the following result.

Theorem 4.1 (Brendle [4]). *Let (M^3, g_{ij}, f) be a three-dimensional steady Ricci soliton. Supposed that the scalar curvature is positive and approaches zero at infinity. Moreover, assume that there exist an exhaustion of M by bounded domains Ω_l such that $\lim_{l \rightarrow \infty} \int_{\partial\Omega_l} e^{u(R)} \langle \nabla R + \psi(R) \nabla f, \nu \rangle = 0$. Then (M^3, g_{ij}, f) is rotationally symmetric.*

Where the function $\psi : (0, 1) \rightarrow \mathbb{R}$ so that $\nabla R + \psi(R) \nabla f = 0$ on the Bryant soliton, and $u(s) = \log \psi(s) + \int_{\frac{1}{2}}^s \left(\frac{3}{2(1-t)} - \frac{1}{(1-t)\psi(t)} \right) dt$.

In the seminal paper by Brendle, it was shown that Bryant soliton is the only non-flat, k -collapsed, steady soliton, then Brendle proving a famous conjecture by Perelman [57].

Theorem 4.2 (Brendle [3]). *Let (M^3, g_{ij}, f) be a three-dimensional complete steady gradient Ricci soliton which is non-flat and k -non-collapsed. Then (M^3, g_{ij}, f) is rotationally symmetric, and is therefore isometric to the Bryant soliton up to scaling.*

In higher dimension, motivated in part by the work of Simon-Solomon [65], which deals with uniqueness question for minimal surfaces with prescribed tangent cones at infinity, Brendle [5] also proved:

Corollary 4.3. *Let (M^n, g_{ij}, f) be a steady gradient Ricci soliton of dimension $n \geq 4$ which has positive sectional curvature and is asymptotically cylindrical. Then (M^n, g_{ij}, f) is rotationally symmetric. In particular, (M^n, g_{ij}, f) is isometric to the n -dimensional Bryant soliton up to scaling.*

Where we say that (M^n, g_{ij}, f) is asymptotically cylindrical if the following holds:

(a) The scalar curvature satisfies $\frac{a_1}{d(p_0, p)} \leq R \leq \frac{a_2}{d(p_0, p)}$ at infinity, where a_1 and a_2 are positive constants.

(b) Let p_m be an arbitrary sequence of marked points going to infinity. Consider the rescaled metrics $\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m}^*(g)$, where $r_m R(p_m) = \frac{n-1}{2} + o(1)$. As $m \rightarrow \infty$, the flows $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(\mathbb{S}^{n-1} \times \mathbb{R}, \bar{g}(t))$, $t \in (0, 1)$. The metric $\bar{g}(t)$ is given by $\bar{g}(t) = (n-2)(2-2t)g_{\mathbb{S}^{n-1}} + dz \otimes dz$, where $g_{\mathbb{S}^{n-1}}$ denotes the standard metric on \mathbb{S}^{n-1} with constant sectional curvature 1.

Under integral assumptions on the scalar curvature, and using the hypothesis that the steady Ricci soliton has nonnegative sectional curvature, implies that the scalar curvature is nonnegative, bounded, and globally Lipschitz, and thus $R \rightarrow 0$ at infinity, Catino-Mastrolia-Monticelli [23] proved:

Theorem 4.4 (Catino-Mastrolia-Monticelli [23]). *Let (M^n, g_{ij}, f) be a complete gradient steady Ricci soliton of dimension $n \geq 3$ with nonnegative sectional curvature. Suppose that $\lim_{r \rightarrow \infty} \inf \frac{1}{r} \int_{B_r(o)} R = 0$. Then, (M^n, g_{ij}) is isometric to a quotient of \mathbb{R}^n or $\mathbb{R}^{n-2} \times \Sigma^2$, where Σ^2 is the cigar soliton.*

In the three dimensional case, they proved the analogous results under weaker assumptions.

Corollary 4.5. *Let (M^3, g_{ij}, f) be a three dimensional complete gradient steady Ricci soliton. Suppose that $\lim_{r \rightarrow \infty} \inf \frac{1}{r} \int_{B_r(o)} R = 0$. Then, (M^3, g_{ij}) is isometric to a quotient of \mathbb{R}^3 or $\mathbb{R} \times \Sigma^2$, where Σ^2 is the cigar soliton.*

As a consequence of the integral decay estimate in [31], it follows that the assumption g has less than quadratic volume growth, i.e., $\text{vol}(B_r(0)) = o(r^2)$ as $r \rightarrow \infty$. This implies the following.

Theorem 4.6 (Catino-Mastrolia-Monticelli [23]). *The only complete gradient steady Ricci solitons of dimension $n \geq 3$ with nonnegative sectional curvature and less than quadratic volume growth are quotients of $\mathbb{R}^{n-2} \times \Sigma^2$.*

In particular, in dimension three the nonnegativity assumption on the curvature is automatically satisfied [17].

Corollary 4.7. *The only three dimensional complete gradient steady Ricci solitons less than quadratic volume growth are quotients of $\mathbb{R} \times \Sigma^2$.*

For $n \geq 4$, it is natural to ask if the Bryant soliton is the only complete non-compact, positively curved, locally conformally flat gradient steady soliton.

Motivated in part by the works of physicists Israel [42] and Robinson [63] concerning the uniqueness of the Schwarzschild black hole among all static, asymptotically flat vacuum space-times. Cao-Chen [19] given an affirmative answer.

Theorem 4.8 (Cao-Chen [19]). *Let (M^n, g_{ij}, f) , $n \geq 3$, be a n -dimensional complete non-compact locally conformally flat gradient steady Ricci soliton with positive sectional curvature. Then (M^n, g_{ij}, f) is isometric to the Bryant soliton.*

Corollary 4.9. *Let (M^n, g_{ij}, f) , $n \geq 3$, be a n -dimensional complete non-compact locally conformally flat gradient steady Ricci soliton. Then (M^n, g_{ij}, f) is either flat or isometric to the Bryant soliton.*

By the analysis of Bryant in the steady case, [8] showing that there exists a unique non-flat steady gradient soliton which is a warped product of \mathbb{R}^{n-1} on a half line of \mathbb{R} , Catino-Mantegazza [22] got the following classification.

Corollary 4.10. *The steady gradient locally conformally flat Ricci solitons of dimension $n \geq 4$ are given by the quotients of \mathbb{R}^n and the Bryant soliton.*

With vanishing Weyl tensor,

Theorem 4.11 (Cao-Chen [19]). *Suppose $n \geq 4$, any complete n -dimensional gradient steady soliton with vanishing Weyl tensor must be either flat or isometric to the Bryant soliton.*

Without requiring the curvature to be bounded globally, but assume the soliton is anti-self-dual, Chen-Wang [27] proved:

Corollary 4.12. *Any 4-dimensional complete gradient steady Ricci soliton with $W^+ = 0$ must be isometric to one of the following two types:*

- (a) *The Bryant soliton up to a scaling.*
- (b) *A manifold which is anti-self-dual and Ricci flat.*

Very recently, Kim [46] get a new classification with harmonic Weyl curvature.

Corollary 4.13. *A 4-dimensional complete steady gradient Ricci soliton with $\delta W = 0$ is either Ricci flat, or isometric to the Bryant soliton.*

Classification results have been obtained for Bach flat steady solitons case in dimension $n \geq 4$. In particular, it follows that Bach flatness implies local conformally flatness. It is still an open question if similar results can be obtained under first order vanishing conditions on Weyl.

The Bach tensor is defined as

$$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ij}^{kl}$$

here W_{ikjl} is the Weyl tensor. In terms of Cotton tensor

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R)$$

we also have

$$B_{ij} = \frac{1}{n-2} (\nabla_k C_{kij} + R_{kl} W_{ij}^{kl})$$

when $n = 3$, the expression B_{ij} is defined as $B_{ij} = \nabla_k C_{kij}$. For Bach flat gradient Ricci solitons, there are some results concerning the classification.

Theorem 4.14 (Cao-Catino-Chen-Mantegazza-Mazzieri [20]). *Let (M^n, g_{ij}, f) , $n \geq 4$, be a complete steady gradient Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum at some interior point. If in addition (M^n, g_{ij}, f) is Bach flat, then it is isometric to the Bryant soliton up to a scaling factor.*

Corollary 4.15. *Let (M^3, g_{ij}, f) be a three-dimensional complete steady gradient Ricci soliton with divergence-free Bach tensor (i.e., $\text{div} B = 0$). Then (M^3, g_{ij}, f) is either Einstein or locally conformally flat.*

The assumption of Bach flat or divergence-free Bach is weaker than that of locally conformally flat. Then using the three-dimensional classification of locally conformally flat gradient steady Ricci solitons, they proved:

Corollary 4.16. *A complete three-dimensional gradient steady Ricci soliton with divergence-free Bach tensor is either flat or isometric to the Bryant soliton up to a scaling factor.*

Corollary 4.17. *Let (M^n, g_{ij}, f) , $(n \geq 4)$, be a complete gradient steady Ricci soliton with $D_{ijk} = 0$. If the Ricci curvature is positive and the scalar curvature R attains its maximum at some interior point, then (M^n, g_{ij}, f) is isometric to the Bryant soliton up to a scaling factor.*

Combining with the covariant 3-tensor D_{ijk} defined as before, Cao-Chen [18] improved the above result for four dimension with vanishing D_{ijk} .

Corollary 4.18. *Let (M^4, g_{ij}, f) be a complete gradient steady Ricci soliton with $D_{ijk} = 0$, then (M^4, g_{ij}, f) is either Ricci flat or isometric to the Bryant soliton.*

Theorem 4.19 (Catino-Mastrolia-Monticelli [24]). *Every three dimensional complete gradient steady Ricci soliton with $\text{div}^3(C) = 0$ on M is isometric to either a finite quotient of \mathbb{R}^3 or the Bryant soliton up to scaling.*

5 Classification of gradient expanding Ricci solitons

Expanding gradient solitons are self-similar solutions to the Ricci flow that flows by diffeomorphism and expanding homothety, they model Type III singularities in the Ricci flow and provide examples of equality in Hamilton's Harnack inequality [40]. The case of expanding solitons is clearly the less rigid. However, some properties and classification theorems have been proved in the recent years by various authors, and several interesting results under vanishing conditions on Weyl have been obtained.

Schulze-Simon [66] have constructed solitons to Ricci flow coming out of the asymptotic cone at infinity of manifolds with positive curvature operator and shown that such a solution to Ricci flow must be an expanding gradient soliton. The simplest example of non-Einstein expanding gradient soliton is the Gaussian soliton. Various authors have obtained uniqueness results concerning expanding gradient solitons. In [28], Chen-Zhu show that a non-compact expanding gradient soliton with positive sectional curvature and uniformly pinched Ricci curvature must be the flat expanding Gaussian soliton. In addition, Bryant has constructed non-flat expanding gradient solitons which are rotationally symmetric and are asymptotic to a cone at infinity.

Theorem 5.1 (Peterman-Wylie [60]). *The only 3-dimensional expanding gradient Ricci solitons with constant curvature are quotients of \mathbb{R}^3 , $\mathbb{H}^2 \times \mathbb{R}$, or \mathbb{H}^3 .*

It has been known for some time that compact expanding Ricci solitons are necessarily trivial [32], the next theorem below, extend this conclusion to the non-compact setting up to imposing suitable integral conditions on potential function under L^p conditions on the relevant quantities.

Theorem 5.2 (Pigolia-Rimoldi-Setti [59]). *A complete expanding gradient Ricci soliton (M^n, g_{ij}, f) is trivial provided $|\nabla f| \in L^p(M, e^{-f} d\text{vol})$, for some $1 \leq p \leq +\infty$.*

Corollary 5.3. *Let (M^n, g_{ij}, f) be a complete expanding gradient Ricci soliton. Let S be the scalar curvature of M . If $S \geq 0$ and $S \in L^1(M, e^{-f} d\text{vol})$, then M is isometric to the standard Euclidean space.*

Additionally, [20] have shown that an expanding gradient soliton with positive Ricci curvature must be rotationally symmetric under certain assumption on the Bach tensor.

Theorem 5.4 (Cao-Catino-Chen-Mantegazza-Mazzieri [20]). *Let (M^n, g_{ij}, f) , $n \geq 4$, be a complete Bach flat gradient expanding Ricci soliton with nonnegative Ricci curvature, then it is rotationally symmetric.*

Corollary 5.5. *Let (M^3, g_{ij}, f) be a three-dimensional complete expanding gradient Ricci soliton with divergence-free Bach tensor and nonnegative Ricci curvature. Then (M^3, g_{ij}, f) is rotationally symmetric.*

Based on the works of Brendle [3], [5] in which it is shown that a steady Ricci soliton with positive sectional curvature that parabolically blows down to a shrinking cylinder must be rotationally symmetric. Chodosh proved:

Theorem 5.6 (Chodosh [29]). *Supposed that (M^n, g_{ij}, f) is an expanding gradient soliton (for $n \geq 3$) that has positive sectional curvature and is asymptotically conical as a soliton, then (M^n, g_{ij}, f) is rotationally symmetric.*

A Riemannian manifold (M^n, g_{ij}) is Ricci pinched if there exists some $\varepsilon \in (0, 1]$ such that $Ric(g) \geq \frac{\varepsilon}{n} R_g g$. Deruelle [30] didn't assume a bound on the full curvature tensor, but derived such bounds by the sole assumption of being Ricci pinched. Then he proved:

Theorem 5.7 (Deruelle [30]). *Let (M^n, g_{ij}, f) be an expanding gradient Ricci soliton with nonnegative scalar curvature which is Ricci pinched. Then,*

- (a) *(M^n, g_{ij}, f) is asymptotically conical to a Ricci flat metric cone at exponential rate.*
- (b) *If $n = 4$, (M^4, g_{ij}, f) is isometric to the Gaussian soliton.*
- (c) *If g is sufficiently pinched, i.e., if $\varepsilon \in [\varepsilon(n), 1]$, then (M^n, g_{ij}, f) is isometric to the Gaussian soliton.*

Under integral assumptions on the scalar curvature, Catino-Mastrolia-Monticelli [23] proved:

Corollary 5.8. *Let (M^n, g_{ij}, f) be a complete expanding gradient Ricci soliton of $n \geq 3$ with nonnegative sectional curvature. If $R \in L^1(M^n)$, then M is isometric to a quotient of the Gaussian soliton \mathbb{R}^n .*

In the three dimensional case, they also proved the analogous results under weaker assumptions.

Corollary 5.9. *Let (M^3, g_{ij}, f) be a three dimensional complete expanding gradient Ricci soliton of with nonnegative Ricci curvature. If $R \in L^1(M^3)$, then M is isometric to a quotient of the Gaussian soliton \mathbb{R}^3 .*

Using neither geometric decay nor nonnegative curvedness, but with harmonic Weyl curvature $\delta W = 0$, Kim [46] proved:

Theorem 5.10 (Kim [46]). *A 4dimensional complete expanding gradient Ricci soliton with harmonic Weyl curvature is one of the following:*

- (a) *g_{ij} is an Einstein metric with f a constant function.*
- (b) *g_{ij} is isometric to a finite quotient of $\mathbb{R}^2 \times \mathbb{N}_\lambda$ where \mathbb{R}^2 has the Euclidean metric and \mathbb{N}_λ is a 2-dimensional Riemannian manifold of constant curvature $\lambda < 0$ and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor.*
- (c) *g_{ij} is locally conformally flat.*

Homogeneous solutions to the Ricci flow have long been studied for both their relative simplicity and their appearance as limits of the flow. Furthermore, a homogeneous Ricci soliton metric naturally arise as a preferred choice of metric in the absence of Einstein metrics. The classification of homogeneous Ricci soliton spaces has also been a central problem. In this respect, Jablonski [45] proved an important result:

Theorem 5.11. *Compact homogeneous Ricci solitons are necessarily Einstein.*

6 Rigidity of gradient Ricci solitons

In particular, a gradient soliton is said to be rigid if it is isometric to a quotient of $\mathbb{N} \times \mathbb{R}^k$ where \mathbb{N} is an Einstein manifold and $f = \frac{\rho}{2}|x|^2$ on the Euclidean factor. As generalizations of Einstein manifolds, Ricci solitons enjoy some rigidity properties, which can take the form of classification(metric rigidity), or alternatively, triviality of the soliton structure(soliton rigidity). It is known that not all gradient solitons are rigid. Here we summarize several natural conditions that characterize rigid gradient solitons. In dimension 2 [41] and 3 [43] all compact solitons are rigid. Building on the work of Ni-Wallach [56], Naber has shown that every 4-dimensional complete shrinking soliton with nonnegative curvature operator is rigid. The famous Bryant soliton show that there are non-rigid rotationally symmetric steady and expanding gradient solitons with positive curvature operator. Moreover, it is also not hard to see that, in any dimension, compact steady or expanding solitons are rigid.

For compact manifolds it is easy to see that they are rigid precisely when the scalar curvature is constant see [32]. More general, Petersen-Wylie [61] proved:

Theorem 6.1 (Petersen-Wylie [61]). *A compact gradient soliton is rigid with trivial f if $Ric(\nabla f, \nabla f) \leq 0$.*

In the non-compact case Perelman has shown that all 3-dimensional shrinking gradient solitons with nonnegative sectional curvature are rigid.

Theorem 6.2 (Perelman [58]). *Any (M^3, g_{ij}, f) be a complete gradient shrinking Ricci soliton with nonnegative sectional curvature is rigid.*

If a soliton is rigid, then the "radial" curvature vanishing, i.e., $R(\cdot, \nabla f) \nabla f = 0$ and the scalar curvature is constant. Conversely we just saw that constant scalar curvature and radial Ricci flatness: $Ric(\nabla f, \nabla f) = 0$ each imply rigidity on compact solitons. In the non-compact case Petersen-Wylie [61] showed:

Corollary 6.3. *A shrinking (expanding) gradient soliton is rigid if and only if it has constant scalar curvature and is radially flat, i.e., $sec(E, \nabla f) = 0$.*

Corollary 6.4. *All complete non-compact shrinking gradient solitons of co-homogeneity 1 with nonnegative Ricci curvature and $sec(E, \nabla f) \geq 0$ are rigid.*

A function u is rectifiable if it can be written as $u = h(r)$ where r is a distance function. For rectifiable shrinking solitons with nonnegative radial curvature, they proved:

Corollary 6.5. *A complete, non-compact, rectifiable, shrinking gradient soliton with nonnegative radial sectional curvature, and nonnegative Ricci curvature is rigid.*

Then they showed that all gradient solitons with maximal symmetry are rigid.

Theorem 6.6. *All homogeneous gradient Ricci solitons are rigid.*

Using the maximum principles for the Laplacian and the f -Laplacian and under the assumptions that the Ricci tensor is nonnegative and its sectional curvature has an upper bound.

Theorem 6.7 (López-Río [49]). *Let (M^n, g_{ij}, f) be a complete gradient shrinking Ricci soliton with bounded nonnegative Ricci tensor. Then (M^n, g_{ij}) is rigid if and only if the sectional curvature is bounded from above by $\frac{|Ric|^2}{2(R^2 - |Ric|^2)}$.*

With Weyl tensor, Petersen-Wylie [60] got the following result

Theorem 6.8. *Any gradient soliton with constant scalar curvature, $\rho \neq 0$ and $W(\nabla f, \cdot, \cdot, \nabla f) = o(|\nabla f|^2)$ is rigid.*

Then Fernández-Río [50] showed that an n -dimensional compact Ricci soliton is rigid if and only if it has harmonic Weyl tensor. Also, Munteanu-Sesum [52] proved that any n -dimensional complete non-compact gradient shrinking with harmonic Weyl tensor is rigid. Building on this, Catino-Mastrolia-Monticelli proved that gradient shrinking Ricci solitons are rigid if $div^4 W = 0$. More recently, Yang-Zang [68] obtained new results, as follows:

Theorem 6.9 (Yang-Zang [68]). *Let (M^n, g_{ij}, f) be a complete non-compact gradient shrinking Ricci soliton. If $div^4 W = 0$, then (M^n, g_{ij}) is rigid.*

Corollary 6.10. *Let (M^n, g_{ij}, f) be a complete non-compact gradient shrinking Ricci soliton. If $div^4 Rm = 0$ or $div^3 Rm(\nabla f) = 0$ or $div^3 W(\nabla f) = 0$, then (M^n, g_{ij}) is rigid.*

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References

- [1] A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987. MR867684.
- [2] J. Bernstein, T. Mettler, *Two-dimensional gradient Ricci solitons revised*, Int. Math. Res. Not. (2013).
- [3] S. Brendle, *Rotational symmetry of self-similar solutions to the Ricci flow*, Mathematics, (2002).
- [4] S. Brendle, *Uniqueness of gradient Ricci solitons*, arXiv: math. DG/1010.3684v2.
- [5] S. Brendle, *Rotational symmetry of Ricci solitons in higher dimensions*, J. Differential Geom. 97, 2 (2013), 191-214.

- [6] S. Brendle, R. Schoen, *Classification of manifolds with weakly $\frac{1}{4}$ -pinched curvature*, Acta. Math. 200, 1 (2008), 1-13.
- [7] S. Brendle, R. Schoen, *Manifolds with $\frac{1}{4}$ -pinched curvature are space forms*, Jour. Math. Soc. 22, 1 (2009), 287-307.
- [8] R. L. Bryant, *Locally existence of gradient Ricci solitons*, Unpublished work. 1987.
- [9] C. Böhm, B. Wilking, *Manifolds with positive curvature operator are space forms*, Ann. Math. 167, 3 (2008), 1079-1097.
- [10] M. L. Cai, *On shrinking gradient Ricci solitons with nonnegative sectional curvature*, arXiv: math. DG/1303. 2728v3
- [11] H. D. Cao, *Existence of gradient Kähler Ricci solitons*, Elli. Para. Meth. Geom. (1996), 1-16.
- [12] H. D. Cao, *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. 45, (1997), 257-272.
- [13] H. D. Cao, *Recent progress on Ricci solitons*, Mathematics. (2009).
- [14] H. D. Cao, *Geometry of complete gradient shrinking Ricci solitons*, Geom. Anal. Adv. Lect. Math. 17 (2011), 227-246.
- [15] G. Catino, *Complete gradient shrinking Ricci solitons with pinched curvature*, Math. Ann. 355, 2 (2013), 629-635.
- [16] G. Catino, *Integral pinched shrinking Ricci solitons*, Adv. Math. 303 (2016), 279-294.
- [17] B. L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. 82, 2 (2009), 363-382.
- [18] H. D. Cao, Q. Chen, *On Bach-flat gradient shrinking Ricci solitons*, J. Differential Geom. 85, 2 (2009), 175-185.
- [19] H. D. Cao, Q. Chen, *On locally conformally flat gradient steady Ricci solitons*, Trans. Amer. Math. Soc. 364, 5 (2012), 2377-2391.
- [20] H. D. Cao, G. Catino, Q. Chen, C. Mantegazza, L. Mazzieri, *Bach-flat gradient steady Ricci solitons*, Calc. Vari. Part. Diff. 49, 1-2 (2014), 125-138.
- [21] H. D. Cao, B. L. Chen, X. P. Zhu, *Recent developments on Hamilton's Ricci flow*, Surv. Diff. Geom. 12 (2008), 47-112.
- [22] G. Catino, C. Mantegazza, *The evolution of the Weyl tensor under the Ricci flow*, Ann. Ins. Four. 61, 4 (2013), 1407-1435.
- [23] G. Catino, P. Mastrolia, D. D. Monticelli, *Classification of expanding and steady Ricci solitons with integral curvature decay*, Submitted Paper. (2016).
- [24] G. Catino, P. Mastrolia, D. D. Monticelli, *Gradient Ricci solitons with vanishing conditions on Weyl*, Journal De Mathématiques Pures Et Appliquées. (2017).
- [25] G. Catino, P. Mastrolia, D.D. Monticelli and M. Rigoli, *Analytic and geometric properties of generic Ricci solitons*, Tra. Amer. Math. Soc. 368, 11 (2014).
- [26] X. D. Cao, B. A. Wang, Z. Zhang, *On locally conformally flat gradient shrinking Ricci solitons*, Comm. Cont. Math. 13, 2 (2011), 269-282.
- [27] X. X. Chen, Y. Q. Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, J. Geom. Anal. 25, 2 (2013), 1335-1343.
- [28] B. L. Chen, X. P. Zhu, *Complete Riemannian manifolds with point-wise pinched curvature*, Invent. Math. 140, 2 (2000), 423-452.

- [29] O. Chodosh, *Expanding Ricci solitons asymptotic to cones*, Calc. Vari. Part. Diff. 51, 1-2 (2014), 1-15.
- [30] O. Chodosh, *Asymptotic estimates and compactness of expanding gradient Ricci solitons*, Annali Della Scuola Normale Superiore Di Pisa. 17 (2014), 485-530.
- [31] A. Deruelle, *Steady gradient Ricci soliton with curvature in L^1* , Comm. Anal. Geom. 20, 1 (2012), 31-53.
- [32] M. Eminent, G. La. Nave, C. Mantegazza, *Ricci solitons: the equation point of view*, Manu. Math. 127, 2 (2008), 345-367.
- [33] M. Feldman, T. Ilmanen, D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Diff. Geom. 65, (2003), 169-209.
- [34] H. P. Fu, L.Q. Xiao, *Rigidity theorem for integral pinched shrinking Ricci solitons*, Monatshefte Für Mathematik. (2015), 1-8.
- [35] A. Gastel, M. Kronz, *A family of expanding Ricci solitons*, Progr. Nonlinear Differential Equations Appl. 59, (2004), 81-93.
- [36] H. L. Gu, X. P. Zhu, *The existence of type II singularities for the Ricci flow on S^{n+1}* , Comm. Anal. Geom. 16, 3 (2007), 0691-0691.
- [37] R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), 255-306.
- [38] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. 24 (1986), 153-179.
- [39] R. S. Hamilton, *Eternal solutions to the Ricci flow*, J. Differential Geom. 38 (1993), 1-11.
- [40] R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geom. 37, 1 (1993), 225-243.
- [41] R. S. Hamilton, *The Ricci flow on surfaces*, Cont. Math. 71 (1998), 237-261.
- [42] W. Israel, *Event horizons in static vacuum space-times*, Phys. Rev. 164, 5 (1967), 1776-1779.
- [43] T. Ivey, *Ricci solitons on compact three-manifolds*, Diff. Geom. Appl. 4, 3 (1993), 301-307.
- [44] T. Ivey, *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. 122, (1994), 241-245.
- [45] M. Jablonski, *Homogeneous Ricci solitons*, J. Reine Angew. Math. 699 (2015), 159-182.
- [46] J. Kim, *On a classification of 4-d gradient Ricci solitons with harmonic Weyl curvature*, arXiv: math. DG/1604. 02827v1.
- [47] N. Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Diff. Anal. Geom. 18, (1990), 327-337.
- [48] B. Kotschwar, *On rotationally invariant shrinking Ricci solitons*, Pacific J. Math. 236, 1 (2008), 73-88.
- [49] M. F. López, E. G. Río, *Maximum principles and gradient Ricci solitons*, J. Differential Equation. 251, (2011), 73-81.
- [50] M. F. López, E. G. Río, *Rigidity of shrinking Ricci solitons*, Math. Zeit. 269, (2011), 461-466.
- [51] M. Micallef, J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. Math. 127, 2 (1998), 199-227.

- [52] O. Munteanu, N. Sesum, *On gradient Ricci solitons*, J. Geom. Anal. 23, 2 (2013), 539-561.
- [53] O. Munteanu, M. T. Wang, *The curvature of gradient Ricci solitons*, Math. Res. Lett. 18, 6 (2011), 1051-1070.
- [54] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Rein. Ange. Math. 645, 2 (2010), 125-153.
- [55] L. Ni, N. Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. 15, 5 (2008), 941-955.
- [56] L.Ni, N. Wallach, *On four-dimensional gradient shrinking solitons*, Int. Math. Res. Not. 4 (2008), 13.
- [57] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Mathematics. (2002).
- [58] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv: math.DG/0303109.
- [59] S. Pigola, M. Rimoldi, A. G. Setti, *Remarks on non-compact gradient Ricci solitons*, Mathematische Zeitschrift. 268, 3-4 (2011), 777-790.
- [60] P. Petersen, W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topo. 14, 4 (2010), 2277-2300.
- [61] P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. 241, 2 (2012), 329-345.
- [62] P. Petersen, W. Wylie, *On gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. 137, 6 (2015), 2085-2092.
- [63] D. C. Robnsono, *A simple proof of the generalization of Isarel's theorem*, Gene. Rel. Grav. 8, 8 (1997), 695-698.
- [64] N. Sesum, *Limiting behaviour of the Ricci flow*, arXiv: math.DG/0402194
- [65] L. Simon, B. Solomon, *Minimal hyper-spaces asymptotic to quadratic cones in R^{n+1}* , Invent. Math. (1986), 535-551.
- [66] F. Schulze, M.Simon, *Expanding solitons with nonnegative curvature operator coming out of cones*, Mathematische Zeitschrift. 275, 1-2 (2013), 625-639.
- [67] J. Y. Wu, P. Wu, W. Wylie, *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, Mathematics. (2014).
- [68] F. Yang, L. D. Zhang, *Rigidity of gradient shrinking Ricci solitons*, Math. Diff. Geom. 53C44.
- [69] Z. H. Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. 242, 1 (2008), 189-200.

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