Naturally harmonic maps between tangent bundles

H. El Hendi and L. Belarbi

Abstract. In this paper, we investigate the harmonicity of a tangent map $\phi : (TM, \tilde{g}) \longrightarrow (TN, \tilde{h})$, in the case when the tangent bundles TM and TN are endowed with natural Riemannian metrics $\tilde{g}$, $\tilde{h}$. In this work we generalize previous results related to the article of A. Sanini [24].

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1 Introduction

Let $\varphi : M \longrightarrow N$ be a smooth map between the smooth manifolds M, N. The map φ induces the tangent map $\phi = d\varphi : TM \longrightarrow TN$ between the tangent bundles of M and N. In the case where M, N are Riemannian manifolds, their tangent bundles TM, TN may be endowed with the corresponding naturel metrics so that they become Riemannian manifolds. The motivation of this paper is to study harmonicity of the tangent map $\phi : (TM, \tilde{g}) \longrightarrow (TN, \tilde{h})$.
In this paper we deal with these problems in the case where the tangent bundles TM, TN are endowed with naturel Riemannian metrics $\tilde{g}, \tilde{h}$. We show that the map ϕ is harmonic if φ is totally geodesic. Further, if TM be a compact tangent bundle ϕ is harmonic if and only if φ is totally geodesic.
In section 4, we determine the conditions for a immersion isometric φ to induce a harmonic map $\phi_1 : (T_1M, \tilde{g}) \longrightarrow (T_1N, \tilde{h})$. If N is a space of constant curvature, we prove that F_1 is a harmonic map if and only if $f(M)$ is a minimal Einstein submanifold of N.

1.1 Harmonic maps

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

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(or over any compact subset $K \subset M$ A map is called harmonic if it is a critical point of the energy functional E (or $E(k)$ for all compact subsets $K \subset M$. For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V)dv_g, \tag{1.1}$$

where

$$\tau(\phi) = trace_g \nabla d\phi$$

is the tension field of ϕ. Then ϕ is harmonic if and only $\tau(\phi) = 0$. One can refer to [15], [22] for the background on harmonic maps.

The existence and explicit construction of harmonic mappings between two given Riemannian manifolds (M, g) and (N, h) are two of the most fundamental problems of the theory of harmonic mappings. If M is compact and N has nonpositive sectional curvature, then any smooth map from M to N can be deformed into a harmonic map using the heat flow method [Eells and Sampson 1964]. However, there is no general existence theory of harmonic mappings if the target manifold does not satisfy the nonpositivity curvature condition. This fact makes it interesting to find harmonic maps defined by vector fields as a map from Riemannian manifold (M, g) to its tangent bundleTM. Problems of this kind have been studied when TM is endowed with the Riemannian Sasaki metric see ([16], [17], [20], [22]) and with the Riemannian Cheeger-Gromoll metric (see [17]).

2 Some results on TM

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\begin{aligned}
\mathcal{V}_{(x,u)} &= \ker(d\pi_{(x,u)}) \\
&= \{a^i \frac{\partial}{\partial y^i}_{|(x,u)} ; a^i \in \mathbb{R}\} \\
\mathcal{H}_{(x,u)} &= \left\{\frac{\partial}{\partial x^i}_{|(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}_{|(x,u)} ; a^i \in \mathbb{R}\right\},
\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left(\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\right). \tag{2.2}$$

As consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, and hence $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1..n}$ is a local adapted frame in TTM.

Remark 2.1. 1. If $w = w^i \frac{\partial}{\partial x^i} + \bar{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)TM}$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - \bar{w}^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{\bar{w}^k + w^i u^j \Gamma_{ij}^k\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

2. If $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ then its vertical and horizontal lifts are defined by

$$u^V = u^i \frac{\partial}{\partial y^i} \in \mathcal{V}_{(x,u)} \in \mathcal{H}_{(x,u)}$$

$$u^H = u^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\}.$$

Proposition 2.1. *([17]). Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ and any point (p, u) in T^2M we have*

1. $[X^H, Y^H]_{(p,u)} = [X, Y]^H_{(p,u)} - (R_p(X, Y)u)^V$,
2. $[X^H, Y^V]_{(p,u)} = (\nabla_X Y)^V_{(p,u)}$,
3. $[X^V, Y^V]_{(p,u)} = 0$.

3 β- metric on TM

Definition 3.1. Let (M, g) be a Riemannian manifold and $\beta \in \mathbb{R}_+$. On the tangent bundle TM, we define a β-metric noted $\tilde{g}$ by

1. $\tilde{g}(X^H, Y^H) = g(X, Y) \circ \pi$
2. $\tilde{g}(X^H, Y^V) = 0$
3. $\tilde{g}_{(x,u)}(X^V, Y^V) = \frac{1}{\alpha}\Big(g_x(X, Y) + \beta g_x(X, u) g_x(Y, u)\Big)$,

where $X, Y \in \Gamma(TM), (x, u) \in TM, r = g(u, u), \alpha = 1 + \beta r$.

Note that, if $\beta = 0$ (resp $\beta = 1$), then $\tilde{g}$ is the Sasaki metric [16] (resp the Cheeger-Gromoll metric [17]).

Theorem 3.1. *([10]). Let (M, g) be a Riemannian manifold and $\tilde{g}$ be a β-metric relative to g on TM. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M, g) (resp*

$(TM, \tilde{g})$, then we have

$$
\begin{aligned}
(\widetilde{\nabla}_{X^H} Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)u)^V \\
(\widetilde{\nabla}_{X^H} Y^V)_p &= (\nabla_X Y)^V_p + \frac{1}{2\alpha}(R_x(u,Y)X)^H \\
(\widetilde{\nabla}_{X^V} Y^H)_p &= \frac{1}{2\alpha}(R_x(u,X)Y)^H \\
(\widetilde{\nabla}_{X^V} Y^V)_p &= \Big[\frac{-\beta}{\alpha}(g(X,u)Y^V + g(Y,u)X^V) - \frac{\beta^2}{\alpha}g(X,u)g(Y,u)u^V \\
&+ \beta\frac{(1+\alpha)}{\alpha}g(X,Y)u^V\Big]_p \\
(\widetilde{\nabla}_{X^H} u^V)_p &= 0 = (\widetilde{\nabla}_{u^V} X^H)_p \\
(\widetilde{\nabla}_{X^V} u^V)_p &= \frac{1}{\alpha}\Big(X^V + \beta g(X,u)u^V\Big)_p \\
(\widetilde{\nabla}_{u^V} X^V)_p &= \frac{\beta}{\alpha}\Big(g(X,u) - r^2 X^V\Big)_p \\
(\widetilde{\nabla}_{u^V} u^V)_p &= u^V,
\end{aligned}
$$

for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x,u) \in TM$, where R denote the curvature tensor of (M, g).

4 Harmonic tangent maps

Lemma 4.1. . *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. The map φ induces the tangent map*

$$
\begin{aligned}
\phi = d\varphi : (TM^m, g^s) &\longrightarrow (TN^n, h^s) \\
(x,y) &\longmapsto (\varphi(x), d\varphi(y))/.
\end{aligned}
$$

For all vector field $X \in \Gamma(TM)$, we have

$$
\begin{aligned}
d\phi\Big((X)^V\Big) &= \Big(d\varphi(X)\Big)^V \\
d\phi\Big((X)^H\Big) &= \Big(d\varphi(X)\Big)^H + \Big(\nabla d\varphi(y,X)\Big)^V.
\end{aligned}
$$

Proof. Let x^i and (x^i, y^i) be a local coordinates on M and TM respectively. The local frames of vector fields on M respectively TM is given by

$$
\Big\{\frac{\partial}{\partial x^i} : i = 1, ..., m\Big\}, \Big\{\Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\partial}{\partial x^i} - \overset{M}{\Gamma}{}^k_{ij}\, y^j \frac{\partial}{\partial y^k}, \quad \Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}, i, j, k = 1, .., m\Big\}.
$$

If u^α and (u^α, v^β) be a local coordinates on N and TN respectively. The local frames of vector fields on N respectively TN is given by

$$
\Big\{\frac{\partial}{\partial u^\alpha} : \alpha = 1, ..., n\Big\}, \Big\{\Big(\frac{\partial}{\partial u^\alpha}\Big)^H = \frac{\partial}{\partial u^\alpha} - v^\beta\, \overset{N}{\Gamma}{}^\gamma_{\alpha\beta} \frac{\partial}{\partial v^\gamma}, \Big(\frac{\partial}{\partial u^\alpha}\Big)^V = \frac{\partial}{\partial v^\alpha}, \alpha, \beta, \gamma = 1, .., n\Big\},
$$

where $v^\beta = y^j \dfrac{\partial \varphi^\beta}{\partial x^j} = y^j \phi^\beta_j$

$$
(4.1) \qquad \begin{aligned}
d\phi\Big((\frac{\partial}{\partial x^i})^H\Big) &= \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha} + y^j \frac{\partial \phi^\alpha_j}{\partial x^i} - \overset{M}{\Gamma}{}^k_{ij}\, y^j \phi^\alpha_k \frac{\partial}{\partial v^\alpha} \\
d\phi\Big((\frac{\partial}{\partial x^i})^V\Big) &= \phi^\alpha_i \frac{\partial}{\partial v^\alpha}.
\end{aligned}
$$

We have

$$
d\phi\Big((\frac{\partial}{\partial x^i})^V\Big) = \phi^\alpha_i (\frac{\partial}{\partial v^\alpha})^V = \Big(d\varphi(\frac{\partial}{\partial x^i})\Big)^V
$$

$$
\frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha} = \frac{\partial \varphi^\alpha}{\partial x^i}(\frac{\partial}{\partial u^\alpha})^H + \overset{N}{\Gamma}{}^\gamma_{\alpha\beta}\, v^\beta \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\gamma}.
$$

Substituting in equation 4.1, we obtain

$$
\begin{aligned}
d\phi\Big((\frac{\partial}{\partial x^i})^H\Big) &= \frac{\partial \varphi^\alpha}{\partial x^i}(\frac{\partial}{\partial u^\alpha})^H + \overset{N}{\Gamma}{}^\gamma_{\alpha\beta}\, v^\beta \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\gamma} + y^j \frac{\partial \phi^\alpha_j}{\partial x^i} - \overset{M}{\Gamma}{}^k_{ij}\, y^j \phi^\alpha_k \frac{\partial}{\partial v^\alpha} \\
&= (d\varphi(\frac{\partial}{\partial x^i}))^H + y^j \Big(\frac{\partial \phi^\gamma_j}{\partial x^i} + \phi^\beta_j \frac{\partial \varphi^\alpha}{\partial x^i}\,\overset{N}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{M}{\Gamma}{}^k_{ij}\,\phi^\gamma_k\Big)(\frac{\partial}{\partial u^\gamma})^V.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\nabla\phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= \nabla^\varphi_{\frac{\partial}{\partial x^i}} \phi(\frac{\partial}{\partial x^j}) - \phi(\nabla^M_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}) \\
&= \nabla^\varphi_{\frac{\partial}{\partial x^i}} \phi^\beta_j \frac{\partial}{\partial v^\beta} - \overset{M}{\Gamma}{}^k_{ij}\,\phi^\gamma_k \frac{\partial}{\partial v^\gamma} \\
&= \frac{\partial \phi^\beta_j}{\partial x^i}\frac{\partial}{\partial v^\beta} + \phi^\beta_j \nabla^\varphi_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial v^\beta} - \overset{M}{\Gamma}{}^k_{ij}\,\phi^\gamma_k \frac{\partial}{\partial v^\gamma} \\
&= \frac{\partial \phi^\beta_j}{\partial x^i}\frac{\partial}{\partial v^\alpha} + \phi^\beta_j \frac{\partial \varphi^\alpha}{\partial x^i}\,\overset{N}{\Gamma}{}^\gamma_{\alpha\beta}\,\frac{\partial}{\partial v^\gamma} - \overset{M}{\Gamma}{}^k_{ij}\,\phi^\gamma_k \frac{\partial}{\partial v^\gamma} \\
&= \Big(\frac{\partial \phi^\gamma_j}{\partial x^i} + \phi^\beta_j \frac{\partial \varphi^\alpha}{\partial x^i}\,\overset{N}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{M}{\Gamma}{}^k_{ij}\,\phi^\gamma_k\Big)(\frac{\partial}{\partial u^\gamma})^V \\
&= \Big(\nabla\phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\Big)^V.
\end{aligned}
$$

Then, we obtain

$$
d\phi\Big(\frac{\partial}{\partial x^i}\Big)^H = \Big(d\varphi(\frac{\partial}{\partial x^i})\Big)^H + \Big(\nabla\phi(\frac{\partial}{\partial x^i}, y)\Big)^V.
$$

□

Lemma 4.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a isometric map between Riemannian manifolds, then the energy density associated of the tangent map $\phi : (TM^m, \tilde{g}) \longrightarrow (TN^n, \tilde{h})$ is given by*

$$
e(\phi) \quad = \quad m + \frac{1-\alpha}{2\alpha} + \frac{1}{2\alpha} trace_g\Big(|\nabla d\varphi(y,*)|^2| + \beta g(\nabla d\varphi(y,*), d\varphi(y)\Big).
$$

Proof. We distinguish two cases.
First case: $y=0$. Let $\{e_1,...e_n\}$ be an orthonormal basis of T_xM. Then, we obtain n unit vectors $\{e_1^H,...e_n^H\}$ of $T_{(x,0)}TM$ which are orthogonal because $\tilde{g}(e_i^H,e_j^H)=g(e_i,e_j)=\delta_{ij}$. Moreover, $\{e_1^V,...e_n^V\}$ are also orthonormal vectors of $T_{(x,0)TM}$ since $\tilde{g}(e_i^V,e_i^V)=g(e_i,e_i)=\delta_{ij}$, when $y=0$ So, $\{e_1^H,...e_n^H,e_1^V,...e_n^V\}$ is an orthonormal basis of $T_{(x,0)}TM$.

$$2e(\phi)_{(\varphi(x),d\varphi(y))} = \sum_{i=1}^{m}\Big(\tilde{h}_{(\varphi(x),d\varphi(y))}(d\phi(e_i^H),d\phi(e_i^H))+\tilde{h}_{(\varphi(x),d\varphi(y))}(d\phi(e_i^V),d\phi(e_i^V))\Big).$$

Using lemma 4.1, we obtain

$$\begin{aligned}&2e(\phi)_{(\varphi(x),d\varphi(y))}\\ &=\sum_{i=1}^{m}\Big(\tilde{h}_{(\varphi(x),d\varphi(y))}((d\varphi(e_i))^H,(d\varphi(e_i))^H\Big)\\ &+\sum_{i=1}^{m}\Big(\tilde{h}_{(\varphi(x),d\varphi(y))}((d\varphi(e_i))^V,(d\varphi(e_i))^V\Big)\\ &+\sum_{i=1}^{m}\Big(\tilde{h}_{(\varphi(x),d\varphi(y))}((\nabla d\varphi(y,e_i))^V,(\nabla d\varphi(y,e_i))^V\Big).\end{aligned}$$

Using Definition 3.1, we have

$$\begin{aligned}&2e(\phi)_{(\varphi(x),d\varphi(y))}\\ &=\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(e_i)),(d\varphi(e_i))\Big)\\ &+\frac{1}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((\nabla d\varphi(y,e_i)),(\nabla d\varphi(y,e_i))\Big)\\ &+\frac{\beta}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((\nabla d\varphi(y,e_i)),d\varphi(y))^2\Big)\\ &+\frac{1}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(e_i)),(d\varphi(e_i))\\ &+\frac{\beta}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(e_i)),d\varphi(y))^2\Big).\end{aligned}$$

From the isometry property of the map φ and at $y=0$, we have

$$e(\phi)_{(\varphi(x),d\varphi(0))} \quad = \quad m.$$

Second case: $y\neq 0$ The first step consists of constructing an orthonormal basis of $T_{(x,y)}TM$. So, from an orthonormal basis $\{e_1=\frac{y}{|y|},e_2,...e_n\}$ of T_xM, we obtain

n unit vectors $\{e_1^H,...e_n^H\}$ which are orthonormal because $\tilde{g}(e_i^H,e_j^H)=g(e_i,e_j)=\delta_{ij}$. To obtain a basis, we complete with $\{f_1^V,...,f_n^V\}$ where:

$$\begin{cases} f_1^V & = e_1^V \\ f_j^V & = \sqrt{\alpha}e_j^V, j=2..n \end{cases}$$

So $\{e_1^H,...e_n^H,f_1^V,...,f_n^V\}$ is an orthonormal basis of $T_{(x,y)}TM$. Analogously, we have

$$\begin{aligned} & 2e(\phi)_{(\varphi(x),d\varphi(y))} \\ & = \sum_{i=1}^{m}\Big(\tilde{h}_{(\varphi(x),d\varphi(y))}(d\phi(e_i^H),d\phi(e_i^H))+\tilde{h}_{(\varphi(x),d\varphi(y))}(d\phi(f_i^V),d\phi(f_i^V))\Big). \end{aligned}$$

Using lemma 4.1 and Definition 3.1, we have

$$\begin{aligned} & 2e(\phi)_{(\varphi(x),d\varphi(y))} \\ & = \sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(e_i)),(d\varphi(e_i))\Big) \\ & +\frac{1}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((\nabla d\varphi(y,e_i)),(\nabla d\varphi(y,e_i))\Big) \\ & +\frac{\beta}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((\nabla d\varphi(y,e_i)),d\varphi(y))^2\Big) \\ & +\frac{1}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(f_i)),(d\varphi(f_i)) \\ & +\frac{\beta}{\alpha}\sum_{i=1}^{m}\Big(h_{(\varphi(x)}((d\varphi(f_i)),d\varphi(y))^2\Big); \end{aligned}$$

by isometry of the map φ, we have

$$e(\phi) \quad = \quad m+\frac{1-\alpha}{2\alpha}+\frac{1}{2\alpha}\sum_{i=1}^{m}\Big(|\nabla d\varphi(y,e_i)|^2|+\beta g(\nabla d\varphi(y,e_i),d\varphi(y)\Big).$$

□

Theorem 4.3. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a isometric map between Riemannian manifolds, then the tension field associated of the tangent map $\phi:(TM^m,\tilde{g})\longrightarrow(TN^n,\tilde{h})$ is given by*

$$\begin{aligned} \tau(\phi) \quad = \quad & \Big[\frac{1}{2\alpha}tr_g R^N(d\varphi(y),\nabla d\varphi(y,*))d\varphi(*)\Big]^H+\Big[trace_g\Big(div(\nabla d\varphi))(y) \\ & -2\frac{\beta}{\alpha}h(\nabla d\varphi(y,*),d\varphi(y))\nabla d\varphi(y,*)-\frac{\beta^2}{\alpha}h(\nabla d\varphi(y,*),d\varphi(y))^2 d\varphi(y) \\ & +\frac{\beta(1+\alpha)}{\alpha}h(\nabla d\varphi(y,*),\nabla d\varphi(y,*))d\varphi(y)\Big)\Big]^V. \end{aligned}$$

Proof. Let $(\varphi(x), d\varphi(y)) \in TN$, and $\{e_i^H, f_i^V\}_{i=1}^m$ be a local orthonormal frame on TM such that $(\nabla_{e_i} e_i)_x = 0$ then by summing over i, we have

$$\begin{aligned}
\tau(\phi) &= \nabla^{TN}_{d\phi(e_i^H)} d\phi(e_i^H) - d\phi(\nabla^{TM}_{e_i^H} e_i^H) \\
&+ \nabla^{TN}_{d\phi(f_i^V)} d\phi(f_i^V) - d\phi(\nabla^{TM}_{f_i^V} f_i^V) \\
&= \nabla^{TN}_{(d\varphi(e_i)^H} (d\varphi(e_i)^H + \nabla^{TN}_{(d\varphi(e_i)^H} (\nabla d\varphi(y, e_i))^V \\
&+ \nabla^{TN}_{(\nabla d\varphi(y,e_i))^V} (d\varphi(e_i)^H + \nabla^{TN}_{(\nabla d\varphi(y,e_i))^V} (\nabla d\varphi(y, e_i))^V \\
&+ \nabla^{TN}_{(d\varphi(f_i))^V} d\varphi(f_i))^V - d\phi(\nabla^{TM}_{e_i^H} e_i^H) - d\phi(\nabla^{TM}_{f_i^V} f_i^V);
\end{aligned}$$

by Theorem 3.1, we have

$$\begin{aligned}
\nabla^{TN}_{d\phi(e_i^H)} d\phi(e_i^H) &= (\nabla^{\varphi}_{e_i} d\varphi(e_i))^H \\
\nabla^{TN}_{(d\varphi(e_i)^H} (\nabla d\varphi(y, e_i))^V &= (\nabla^{\varphi}_{e_i} \nabla d\varphi(y, e_i))^V + (\frac{1}{2\alpha} R(d\varphi(y), \nabla d\varphi(y, e_i)) d\varphi(e_i))^H \\
\nabla^{TN}_{(\nabla d\varphi(y,e_i))^V} (d\varphi(e_i)^H &= \frac{1}{2\alpha} (R(d\varphi(y) \nabla d\varphi(y, e_i)) d\varphi(e_i))^H \\
\nabla^{TN}_{(\nabla d\varphi(y,e_i))^V} (\nabla d\varphi(y, e_i))^V &= \frac{-2\beta}{\alpha} h(\nabla d\varphi(y, e_i), d\varphi(y)) \nabla d\varphi(y, e_i) \\
&- \frac{\beta^2}{\alpha} h(\nabla d\varphi(y, e_i), d\varphi(y))^2 d\varphi(y) \\
&+ \frac{\beta(1+\alpha)}{\alpha} h(\nabla d\varphi(y, e_i), \nabla d\varphi(y, e_i)) d\varphi(y)\Big)^V \\
\nabla^{TN}_{(d\varphi(f_i))^V} d\varphi(f_i))^V &= \frac{-2\beta}{\alpha} h(d\varphi(f_i), d\varphi(y)) d\varphi(f_i) \\
&- \frac{\beta^2}{\alpha} h(d\varphi(f_i), d\varphi(y))^2 d\varphi(y) \\
&+ \frac{\beta(1+\alpha)}{\alpha} h(d\varphi(f_i), d\varphi(f_i) d\varphi(y)\Big)^V \\
-d\phi(\nabla^{TM}_{e_i^H} e_i^H) &= -d\phi((\nabla_{e_i} e_i)^H) = -(d\varphi(\nabla^M_{e_i} e_i))^H \\
&- \nabla d\varphi(y, \nabla^M_{e_i} e_i))^V \\
-d\phi(\nabla^{TM}_{f_i^V} f_i^V) &= \frac{2\beta}{\alpha} g(f_i, y) d\varphi(f_i) \\
&+ \frac{\beta^2}{\alpha} h(f_i, y)^2 d\varphi(y) \\
&- \frac{\beta(1+\alpha)}{\alpha} h(f_i, f_i) d\varphi(y)\Big)^V;
\end{aligned}$$

by isometry of the map φ, the proof of the Theorem 4.3 is completed. □

From Theorem 4.3, we obtain

Theorem 4.4. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a isometric map between Riemannian manifolds, then the tangent map $\phi : (TM^m, \tilde{g}) \longrightarrow (TN^n, \tilde{h})$ is harmonic if and only* *if the following conditions are verified*

$$\begin{cases} R^N(d\varphi(y), \nabla d\varphi(y,*))d\varphi(*) = 0 \\ \\ div(\nabla d\varphi)(y) - \frac{2\beta}{\alpha}\alpha h(\nabla d\varphi(y,*), d\varphi(y))\nabla d\varphi(y,*) - \frac{\beta^2}{\alpha}h(\nabla d\varphi(y,*), d\varphi(y))^2 d\varphi(y) \\ +\frac{\beta(1+\alpha)}{\alpha}h(\nabla d\varphi(y,*), \nabla d\varphi(y,*))d\varphi(y) = 0. \end{cases}$$

Corollary 4.5. *Let $\phi : (TM^m, \tilde{g}) \longrightarrow (TN^n, \tilde{h})$ be a the tangent map of isometric map $\varphi : (M^m, g) \longrightarrow (N^n, h)$, if φ is totally geodesic then ϕ is harmonic.*

Theorem 4.6. *Let TM be a compact tangent bundle and $\phi : (TM^m, \tilde{g}) \longrightarrow (TN^n, \tilde{h})$ be its the tangent map of a isometric map $\varphi : (M^m, g) \longrightarrow (N^n, h)$. Then ϕ is a harmonic if and only if φ is totally geodesic.*

Proof. If φ is totally geodesic, from Corollary 4.5, we deduce that ϕ is harmonic. Inversely.
Let $\omega : I \times M \longrightarrow N$ be a smooth map satisfying for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in M$

$$\omega(t, x) = \varphi_t(x) = (1 + t)\varphi(x),$$

and

$$\omega(0, x) = \varphi(x).$$

The variation vector field $v \in \Gamma(\varphi^{-1}TN)$ associated to the variation $\{\varphi_t\}_{t\in I}$ is given for all $x \in M$ by

$$v(x) = d_{(0,x)}\omega(\frac{d}{dt}),$$

From Lemma 4.2, we have

$$e(\phi_t) = m + \frac{1-\alpha}{2\alpha} + \frac{(1+t)^2}{2\alpha} trace_g\Big(|\nabla d\varphi(y,*)|^2| + \beta g(\nabla d\varphi(y,*), d\varphi(y)\Big).$$

If ϕ is a critical point of the energy functional, from equation 1.1 we have

$$\begin{aligned} \frac{d}{dt}E(\phi_t)_{t=0} &= 0 \\ &= \frac{1}{\alpha}\int_{TM} |\nabla d\varphi(*, u)|^2 + \beta g(\nabla d\varphi(y,*), d\varphi(y) dv_{h^s} = 0 \end{aligned}$$

then $\nabla d\varphi = 0$. □

5 Harmonicity of the map $\phi_1 : T_1M^m \longrightarrow T_1N^n$

Lemma 5.1. *Let $j : M_1^{m_1} \longrightarrow M^m$ be a isometric immersion of Riemannian manifold M. Then the tension field associated of the map $\tilde{\varphi} = \varphi \circ j : M_1^{m_1} \xrightarrow{j} M^m \xrightarrow{\varphi} N^n$ is given by*

$$\tau(\tilde{\varphi}) = \tau(\varphi)_{|M_1} + m_1 d\varphi(H_1) - \nabla d\varphi(e_\alpha, e_\alpha), \tag{5.1}$$

where e_α be a local orthonormal frame on TM_1 and H_1 the mean curvature of M_1.

Proof. Let j be a isometric immersion¿ Then the tension field associated of the map j is given by

$$\tau(j) = m_1 H_1,$$

and we have

$$\begin{aligned}\tau(\tilde{\varphi}) &= \tau(\varphi \circ j) = d\varphi \circ \tau(j) + tr_g \nabla d\varphi(dj(*), dj(*)) \\ &= d\varphi(m_1 H_1) + \nabla d\varphi(dj(e_i), dj(e_i)) \\ &= m_1 d\varphi(H_1) + \tau(\varphi)_{|M_1} - \nabla d\varphi(e_\alpha, e_\alpha).\end{aligned}$$

□

Proposition 5.2. *Let $\tilde{\phi} : T_1M^m \longrightarrow TN^m$ the restriction of the tangent map ϕ to unit tangent bundle T_1M^m. Then tension field associated of the map $\tilde{\phi}$ is given by*

$$\tau(\tilde{\phi}) = \tau(\phi)_{|T_1M} - (m-1)(d\varphi(y))^V. \tag{5.2}$$

Proof. From the Lemma 5.1, we have

$$\tau(\tilde{\phi}) = \tau(\phi)_{|T_1M} + d\phi((2m-1)H_1^{T_1M}) - \widetilde{\nabla} d\phi(Y, Y), \tag{5.3}$$

where Y the normal unit vector field on T_1M. Let $\{e_i^H, f_i^V\}_{i=1}^m$ be a local orthonormal frame on T_1M^m, the mean curvature of T_1M^m is given by

$$H_1^{T_1M} = -\tilde{g}(e_i^H, \widetilde{\nabla}_{e_i^H} Y)Y - \tilde{g}(f_i^V, \widetilde{\nabla}_{f_i^V} Y)Y;$$

by Theorem 3.1, we have

$$H_1^{T_1M} = -(m+1)Y$$

and

$$\widetilde{\nabla} d\phi(Y, Y) = 0.$$

Then

$$\begin{aligned}\tau(\tilde{\phi}) &= \tau(\phi)_{|T_1M} - (m-1)d\phi(Y) \\ &= \tau(\phi)_{|T_1M} - (m-1)(d\varphi(y))^V.\end{aligned}$$

□

Proposition 5.3. *([24]) Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$, $f : (N^n, h) \longrightarrow (P^d, l)$ are smooths maps. Then the map φ is harmonic if and only if $\tau(\varphi \circ f)$ is orthogonal to N.*

Using Propositions 5.2, 5.3, we obtain the Theorem

Theorem 5.4. *The differential of an isometric immersion $\varphi : M^m \longrightarrow N^n$ induces a harmonic map $\phi_1 : T_1M^m \longrightarrow T_1N^n$ if and only if the following conditions are verified*

$$\begin{cases} R^N(d\varphi(y), \nabla d\varphi(y,*))d\varphi(*) = 0 \\ \\ div(\nabla d\varphi)(y) - \frac{2\beta}{\alpha}\alpha h(\nabla d\varphi(y,*), d\varphi(y))\nabla d\varphi(y,*) - \frac{\beta^2}{\alpha} h(\nabla d\varphi(y,*), d\varphi(y))^2 d\varphi(y) \\ + \frac{\beta(1+\alpha)}{\alpha} h(\nabla d\varphi(y,*), \nabla d\varphi(y,*))d\varphi(y) = \lambda d\varphi(y). \end{cases}$$

In other words, identifying M with $f(M)$ and place $b = \nabla d\varphi$, if X an arbitrary unit vector field then we have

Theorem 5.5. *Let N^n be a smooth manifold with a constant sectional curvature k. Then the differential of an isometric immersion $\varphi : M^m \longrightarrow N^m$ induces a harmonic map $\phi_1 : T_1M^m \longrightarrow T_1N^n$ if and only if the following conditions are verified*

$$\begin{cases} kb(X,X) = 0, \qquad |X| = 1, \quad k \in \mathbb{R} \\ n\nabla^N_X H + Ricci^M(X) = \mu X, \qquad\qquad \mu = \lambda + \dfrac{\beta(1+\alpha)h(div(b)X,X))}{\alpha} + k(1-n). \end{cases}$$

Proof. By Theorem 5.4, the map $\phi_1 : T_1M^m \longrightarrow T_1N^n$ is harmonic if and only if

$$R^N(X, b(X,e_i))e_i = 0 \tag{5.4}$$

$$\begin{aligned} &(\nabla^N_{e_i} b)(e_i,X) - \frac{2\beta}{\alpha}\alpha h(b(X,e_i),X)b(X,e_i) - \frac{\beta^2}{\alpha}h(b(X,e_i),X)^2 X \\ &+\frac{\beta(1+\alpha)}{\alpha}h(b(X,e_i),b(X,e_i))X = \lambda X. \end{aligned} \tag{5.5}$$

Then, we have

$$R^N(X, b(X,e_i))e_i \;=\; kb(X,X) \tag{5.6}$$

$$\begin{aligned} (\nabla^N_{e_i} b)(e_i,X) &= \nabla^N_X(b(e_i,e_i)) - R^N(e_i,X)e_i + Ricci^M(X) \\ &= n\nabla^N_X H + (k(n-1))X + Ricci^M(X) \end{aligned} \tag{5.7}$$

$$\begin{aligned} div(b)(X) &= (\nabla^N_{e_i} b)(e_i,X) = \nabla^N_{e_i} b(e_i,X) - b(\nabla^M_{e_i} e_i, X) \\ &- b(e_i, \nabla^M_{e_i} X) \end{aligned}$$

$$\begin{aligned} h(div(b)(X),X) &= h(\nabla^N_{e_i} b(e_i,X),X) = -h(b(e^i,X),\nabla^N_{e_i} X) \\ &= -h(b(e_i,X),b(e_i,X)) \end{aligned} \tag{5.8}$$

$$h(b(X,e_i),X) = 0. \tag{5.9}$$

Replacing equations (5.6), (5.7), (5.8), (5.9) in equations (5.4) and (5.5) the proof of Theorem 5.5 is completed. □

Theorem 5.6. *Let N^n be a smooth manifold with a constant sectional curvature $k \neq 0$. Then the differential of an isometric immersion $\varphi : M^m \longrightarrow N^m$ induces a harmonic map $\phi_1 : T_1M^m \longrightarrow T_1N^n$ if and only if $f(M)$ is a minimal Einstein submanifold of N.*

Proof. By theorem 5.5, $\phi_1 : T_1M^m \longrightarrow T_1N^n$ is harmonic if and only if

$$\begin{cases} kb(X,X) = 0, \qquad |X| = 1, \quad k \in \mathbb{R} \\ \\ n\nabla^N_X H + Ricci^M(X) = \mu X, \qquad\qquad \mu = \lambda + \dfrac{\beta(1+\alpha)h(div(b)X,X))}{\alpha} + k(1-n). \end{cases}$$

If $k \neq 0$ then $b(X,X) = 0$, for each unit vector X tangent to M and therefore

$$b(X,Y) = h(X,Y)H, \quad \forall X,Y \in \Gamma(TM)$$

we have $b(X,X) = H = 0$. Then M is a minimal submanifold of N.
$n\nabla^N_X H + Ricci^M(X) = \mu X$ is equivalent to the condition of Einstein, because $\nabla^N_X H = -|H|^2 X$. □

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References

[1] L. Belarbi, M. Belarbi and H. El Hendi, *Legendre curves on Lorentzian Heisenberg space*, Bull. Transilv. Univ. Brasov Ser. III, 13 (62) (1) (2020), 41–50.

[2] L. Belarbi, H. El Hendi and F. Latti, *On the geometry of the tangent bundle with vertical rescaled generalized Cheeger-Gromoll metric*, Bull. Transilv. Univ. Brasov SER. III. 12(61)(2)(2019), 247–264.

[3] L. Belarbi and H. El Hendi, *Harmonic and biharmonic maps between tangent bundles*, Acta Math. Univ. Comenianae Vol. 88(2)(2019), 187-199.

[4] L. Belarbi and H. El Hendi, *Geometry of twisted Sasaki metric*, J. Geom. Symmetry Phys. 53(2019), 1–19.

[5] C.L. Bejan, M. Benyounes, *Harmonic φ morphisms*, Beitrge zur Algebra und Geometry, 44(2)(2003), 309–321.

[6] N. Cengiz, A. A. Salimov, *Diagonal lift in the tensor bundle and its applications*, Appl. Math. Comput. 142(2-3)(2003), 309-319.

[7] J. Cheeger, D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. 96 (2) (1972) 413-443.

[8] M. Djaa, H. EL Hendi and S. Ouakkas, *Biharmonic vector field*, Turkish J. Math. 36(2012), 463-474.

[9] M. Djaa, J. Gancarzewicz, *The geometry of tangent bundles of order r*, Boletin Academia, Galega de Ciencias, Espagne. 4(1985), 147-165.

[10] M. Djaa, N. E. H. Djaa and R. Nasri, *Natural metrics on T^2M and harmonicity*, International Electronic Journal of Geometry, 6 (2013), 100-111.

[11] N. E. H. Djaa, S. Ouakkas and M. Djaa, *Harmonic sections on the tangent bundle of order two*, Ann. Math. Inform. 38(2011), 15-25.

[12] H. El Hendi and L. Belarbi, *Deformed diagonal metrics on tangent bundle of order two and harmonicity*, Panamer. Math. J. 27(2)(2017), 90-106.

[13] H. El Hendi and L. Belarbi,*On paraquaternionic submersions of tangent bundle of order two*, Nonlinear Studies. Vol. 25(3)(2018),653-664.

[14] H. El Hendi, M. Terbeche and M. Djaa, *Tangent bundle of order two and biharmonicity*, Acta Math. Univ. Comenianae . Vol. 83(2)(2014),165-179.

[15] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86(1964), 109-60.

[16] S. Gudmundsson, E. Kappos, *On the geometry of the tangent bundles*, Expo. Math. 20(1)(2002), 1-41.

[17] S. Gudmundsson, E. Kappos,*On the geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math. 25(1) (2002). 75-83.

[18] T. Ishihara,*Harmonic sections of tangent bundles*, J. Math. Univ. Tokushima. 13(1979) ,23-27.

[19] G. Y. Jiang, *Harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A. 7(1986), 389-402.

[20] J. J. Konderak,*On harmonic vector fields*, Publicationes Mathematiques, 36(1992), 217-288.

[21] H. Mazouzi, H. El hendi and L. Belarbi, *On the generalized bi-f-harmonic map equations on singly warped product manifolds*, Comm. Appl. Nonlinear Anal. **25**(3)(2018), 52-76.

[22] V. Oproiu,*On Harmonic Maps Between Tangent Bundles*, Rend. Sem. Mat. 47(1989), 47-55.

[23] A.A. Salimov, A. Gezer and K. Akbulut, *Geodesics of Sasakian metrics on tensor bundles*, Mediterr. J. Math. 6(2)(2009), 135-147.

[24] A. Sanini, *Applicazioni armoniche tra i fibrati tangcnti di varieta riemanniane*, Boll. U.M.I. 6(2A)(1983), 55-63.

[25] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker Inc., New York (1973).

Author's address:

Hichem El hendi
Department of Mathematics,
University of Bechar,
PO Box 417, 08000, Bechar, Algeria.
E-mail: elhendihichem@yahoo.fr

Lakehal Belarbi
Department of Mathematics,
Laboratory of Pure and Applied Mathematics,
University of Mostaganem (U.M.A.B.),
B.P.227,27000, Mostaganem, Algeria.
E-mail: lakehalbelarbi@gmail.com