# Some notes on Riemannian extensions

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**Abstract.** In the present paper we study paraholomorphy property of the Riemannian extension by using almost paracomplex structure on the cotangent bundle. Then we investigate locally decomposable Golden structure on the cotangent bundle which related to this almost paracomplex structure.

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## 1 Introduction

Let $M$ be an n-dimensional $C^{\infty}$- manifold with torsion free connection $\nabla$, $T^*M$ be its cotangent bundle and $\pi$ the bundle projection $T^*M \to M$. Let $F(M)\,(F(T^*M))$ be the ring of real-valued $C^{\infty}$ functions on $M(T^*M)$. Then, we denote by $\Im_s^r(M)$ $(\Im_s^r(T^*M))$ be the module over $F(M)\,(F(T^*M))$ of $C^{\infty}$ tensor fields of type (r,s) on $M(T^*M)$.

The local coordinates $\left(U, x^i\right), i = 1, ..., n$ on $M$ introduces on $T^*M$ a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i\right)$, $\bar{i} = n+1, ...2n$, where $x^{\bar{i}} = p_i$ are the components of the covector $p$ in each cotangent space $T_x^*M, x \in U$ with respect to the natural coframe $\left\{dx^i\right\}, i = 1, ..., n$.

The vector and covector (1-form) field $Y \in \Im_0^1(M)$ and $\nu \in \Im_1^0(M)$ have the local expressions $Y = Y^i \frac{\partial}{\partial x^i}$ and $\nu = \nu_i dx^i$ in $U \subset M$, respectively. Then the complete and horizontal lifts ${}^CY, {}^HY \in \Im_0^1(T^*M)$ of $Y \in \Im_0^1(M)$ and the vertical lift ${}^V\nu \in \Im_0^1(T^*M)$ of $\nu \in \Im_1^0(M)$ are given, respectively, by

$$ {}^CY = Y^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i Y^h \frac{\partial}{\partial x^{\bar{i}}} \tag{1.1} $$

$$ {}^HY = Y^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h Y^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2} $$

$$ {}^V\nu = \sum_i \nu_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.3} $$

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with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where $\Gamma_{ij}^h$ are the components of the Levi-Civita connection $\nabla_g$ on $M$ (see [25] for more details).

The Riemannian extension ${}^R\nabla \in \Im_2^0(T^*M)$ which is a pseudo-Riemannian metric is defined by

$$ {}^R\nabla\left({}^C X, {}^C Y\right) = -\gamma\left(\nabla_X Y + \nabla_Y X\right) $$

for any $X, Y \in \Im_0^1(M)$, where $\gamma(\nabla_X Y + \nabla_Y X) = p_m\left(X^j \nabla_j Y^m + Y^j \nabla_j X^m\right)$(see [25, p. 268]). The components of the Riemannian extension is given by the form

$$ {}^R\nabla = \left({}^R\nabla_{IJ}\right) = \begin{pmatrix} -2p_k\Gamma_{ij}^k & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} $$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where $\delta_i^j$ denote the Kronecker delta.

The Riemannian extension ${}^R\nabla \in \Im_2^0(T^*M)$ is completely determined by its action of ${}^H Y$ and ${}^V\nu$ on $T^*M$. Then the Riemann extension ${}^R\nabla$ is given by

$$ \begin{aligned} {}^R\nabla\left({}^V\nu, {}^V\omega\right) &= {}^R\nabla\left({}^H X, {}^H Y\right) = 0, \\ {}^R\nabla\left({}^V\nu, {}^H Y\right) &= {}^V\left(\nu\left(Y\right)\right) = \nu\left(Y\right) \circ \pi \end{aligned} \tag{1.4} $$

for any $X, Y \in \Im_0^1(M)$ and $\nu, \omega \in \Im_1^0(M)$[25].

The Riemannian extensions were defined by Patterson and Walker [19] and intensively studied for the cotangent bundle [1, 2, 3, 4, 5, 6, 9, 11, 16, 20, 22].

## 2 The paraholomorphy properties of the Riemannian extensions

An almost product structure $F \in \Im_1^1\left(M^{2m}\right)$ is defined by $F^2 = I$ . The pair $\left(M^{2m}, F\right)$ is called an almost product manifold . An almost paracomplex manifold is an almost product manifold $\left(M^{2m}, F\right)$, such that the two eigenbundles $T^+M^{2m}$ and $T^-M^{2m}$ associated to the two eigenvalues +1 and -1 of $F$, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure $F$, we obtain the set $\{I, F\}$ on $M$, which is an isomorphic representation of the algebra of order 2, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j), j^2 = 1$ [8].

A tensor field $\vartheta \in \Im_q^0\left(M^{2m}\right)$ is called pure with respect to the paracomplex structure $F$ if

$$ \vartheta\left(FY_1, Y_2, ..., Y_q\right) = \vartheta\left(Y_1, FY_2, ..., Y_q\right) = ... = \vartheta\left(Y_1, Y_2, ..., FY_q\right) $$

for any $Y_1, Y_2, ..., Y_q \in \Im_0^1\left(M^{2m}\right)$ [24].

Using the paracomplex structure $F$ and the pure tensor field $\vartheta$, the operator $\phi_F$ defined by

$$ \begin{aligned} (\phi_F\vartheta)\left(Y, Z_1, ..., Z_q\right) &= \left(FY\right)\left(\vartheta\left(Z_1, ..., Z_q\right)\right) \\ &\quad - Y(\vartheta\left(FZ_1, Z_2, ..., Z_q\right)) + \vartheta\left(\left(L_{Z_1}F\right)Y, Z_2, ..., Z_q\right) \\ &\quad + ... + \vartheta\left(Z_1, Z_2, ..., \left(L_{Z_q}F\right)Y\right), \end{aligned} \tag{2.1} $$

where $L_Y$ denotes the Lie derivative with respect to $Y$ and $\phi_F \vartheta \in \Im^0_{q+1}\left(M^{2m}\right)$ [24].

If $\phi_F \vartheta = 0$ , then $\vartheta$ is said to be an almost paraholomorphic with respect to the paracomplex algebra $R(j)$ ( see [17], [23]).

An almost para-Norden manifold $\left(M^{2m}, F, g\right)$ is given a real differentiable manifold $M^{2m}$ endowed with an almost paracomplex structure $F$ and a pseudo-Riemannian metric $g \in \Im^0_2\left(M^{2m}\right)$ satisfying the Nordenian property (or purity condition)

$$g(FY, Z) = g(Y, FZ) \tag{2.2}$$

for any $Y, Z \in \Im^1_0\left(M^{2m}\right)$. The almost para-Norden manifold $\left(M^{2m}, F, g\right)$ is called a paraholomorphic Norden manifold (or para-Kähler-Norden manifold) such that $\nabla F = 0$, where $\nabla$ is Levi-Civita connection of $g$. We know that $\nabla F = 0$ is equivalent to $\phi_F g = 0$ [23].

Salimov and Agca presented in [21] an almost paracomplex structure on $T^*M$ by

$$\begin{cases} F^H Y = {}^V\tilde{Y}, \\ F^V \nu = {}^H\tilde{\nu} \end{cases} \tag{2.3}$$

for any $Y \in \Im^1_0(M)$ and $\nu \in \Im^0_1(M)$ , where $\tilde{Y} = g \circ Y \in \Im^0_1(M)$, $\tilde{\nu} = g^{-1} \circ \nu \in \Im^1_0(M)$.

We put

$$W\left(\bar{X}, \bar{Y}\right) = {}^R\nabla\left(F\bar{X}, \bar{Y}\right) - {}^R\nabla\left(\bar{X}, F\bar{Y}\right).$$

If $W\left(\bar{X}, \bar{Y}\right) = 0$ for all vector fields $\bar{X}$ and $\bar{Y}$ which are of the form ${}^V\nu, {}^V\omega$ or ${}^HX, {}^HY$, then $W = 0$. From (1.2)-(1.4) and (2.3)

$$\begin{aligned}
&W\left({}^Vv, {}^V\omega\right) = {}^R\nabla\left(F^V\nu, {}^V\omega\right) - {}^R\nabla\left({}^V\nu, F^V\omega\right) \\
&= {}^R\nabla\left({}^H\tilde{\nu}, {}^V\omega\right) - {}^R\nabla\left({}^V\nu, {}^H\tilde{\omega}\right) = {}^V\left(\omega\left(\tilde{\nu}\right)\right) - {}^V\left(\nu\left(\tilde{\omega}\right)\right) = 0, \\
&W\left({}^HY, {}^V\nu\right) = -W\left({}^V\nu, {}^HY\right) = {}^R\nabla\left(F^HY, {}^V\nu\right) - {}^R\nabla\left({}^HX, F^V\nu\right) \\
&= {}^R\nabla\left({}^VY, {}^V\nu\right) - {}^R\nabla\left({}^HX, {}^H\nu\right) = 0, \\
&W\left({}^HX, {}^HY\right) = {}^R\nabla\left(F^HX, {}^HY\right) - {}^R\nabla\left({}^HX, F^HY\right) \\
&= {}^R\nabla\left({}^V\tilde{X}, {}^HY\right) - {}^R\nabla\left({}^HX, {}^V\tilde{Y}\right) = {}^V\left(\tilde{X}(Y)\right) - {}^V\left(\tilde{Y}(X)\right) = 0
\end{aligned}$$

i.e. ${}^R\nabla$ is pure with respect to the almost paracomplex structure $F$. Hence we have the following theorem.

**Theorem 2.1.** *The triple $\left(T^*M, {}^R\nabla, F\right)$ is an almost para-Nordenian manifold.*

In [25, p.238 and p.277], the Lie bracket operation for horizontal and vertical lifts of vector fields on the cotangent bundle is given by the following

$$\begin{aligned}
&\left[{}^HY, {}^HZ\right] = {}^H[Y, Z] + \gamma R(Y, Z) = {}^H[Y, Z] + {}^V\left(pR(Y, Z)\right), \\
&\left[{}^HY, {}^V\nu\right] = {}^V\left(\nabla_Y \nu\right), \\
&\left[{}^V\gamma, {}^V\nu\right] = 0
\end{aligned}$$

for all $Y, Z \in \Im^1_0(M)$ and $\gamma, \nu \in \Im^0_1(M)$.

Now we give the condition for the Riemannian extension to be para-holomorphic with respect to the almost paracomplex structure $F$. From (2.1), (2.3), (2.4) and by using the fact that ${}^V\nu^V f = 0$ and ${}^HY^V f = {}^V(Yf)$, where $f \in \Im^0_0(M)$, we get

$$\begin{aligned}
&\left(\phi_F {}^R\nabla\right)\left({}^HX, {}^HY, {}^V\nu\right) = \left(F^HX\right)\left({}^R\nabla\left({}^HY, {}^V\nu\right)\right) - {}^HX\left({}^R\nabla\left(F^HY, {}^V\nu\right)\right) \\
&+ {}^R\nabla\left(\left(L_{{}^HY}F\right){}^HX, {}^V\nu\right) + {}^R\nabla\left({}^HY, \left(L_{{}^V\nu}F\right){}^HX\right) \\
&= -\left({}^R\nabla\left({}^V\nu, {}^H\left(g^{-1} \circ pR(Y, X)\right)\right)\right) \\
&= -{}^V\left(\nu\left(g^{-1} \circ pR(Y, X)\right)\right) = -{}^V\left(g^{-1}\left(pR(Y, X), \nu\right)\right) \\
&= {}^V\left(pR(X, Y)\tilde{\nu}\right).
\end{aligned}$$

Same calculation we find

$$\begin{aligned}\left(\phi_F{}^R\nabla\right)\left({}^V\omega,{}^HY,{}^HZ\right)&={}^V\left(pR\left(Y,\tilde{\omega}\right)Z+pR\left(Z,\tilde{\omega}\right)Y\right),\\ \left(\phi_F{}^R\nabla\right)\left({}^HX,{}^V\omega,{}^HY\right)&={}^V\left(pR\left(X,Y\right)\tilde{\omega}\right)\end{aligned}$$

and the others are zero. Therefore, we have

**Theorem 2.2.** *The almost para-Norden manifold* $\left(T^*M,{}^R\nabla,F\right)$ *is para-holomorphic if and only if* $M$ *is flat.*

# 3 Locally Decomposable Golden Structures

Golden structure as a polynomial structure [13] on a Riemannian manifold was created by M.Crasmareanu and C. Hretcanu [7, 14, 15]. Let $\psi$ be a (1,1)-tensor field on $M$. If the polynomial $x^2-x-1$ is the minimal polynomial for a structure $\psi$ satisfying $\psi^2-\psi-1=0$, then $\psi$ is defined a Golden structure on $M$ and $(M,\psi)$ is a Golden manifold.

A Golden Riemannian manifold can be defined as a triple $(M,\psi,g)$ which consist of a Riemannian manifold $(M,g)$ endowed with a Golden structure $\psi$ such that

$$g\left(\psi Y,Z\right)=g\left(Y,\psi Z\right)$$

for any $Y,Z\in\Im_0^1\left(M\right)$ [7].

Suppose that $(M,g)$ be a pseudo-Riemannian manifold endowed with the Golden structure $\psi$ such that $\psi$ satisfies the condition

$$g\left(\psi Y,Z\right)=g\left(Y,\psi Z\right) \tag{3.1}$$

for all $Y,Z\in\Im_0^1\left(M\right)$. Hence we say that $(M,\psi,g)$ is a Golden pseudo-Riemannian manifold. Some applications are given in [18].

Gezer et. al. presented in [12] a Golden structure on $T^*M$ by

$$\begin{aligned}\psi^HY&=\tfrac{1}{2}\left({}^HY+\sqrt{5}{}^V\tilde{Y}\right)\\ \psi^V\nu&=\tfrac{1}{2}\left({}^V\nu+\sqrt{5}{}^H\tilde{\nu}\right)\end{aligned} \tag{3.2}$$

for any $Y\in\Im_0^1\left(M\right)$ and $\nu\in\Im_1^0\left(M\right)$, where $\tilde{Y}=g\circ Y\in\Im_1^0\left(M\right)$, $\tilde{\nu}=g^{-1}\circ\nu\in\Im_0^1\left(M\right)$. Then we see that this structure related to $F$ defined by (2.3) . Using (1.2)-(1.4), (3.1), (3.2), we calculate

$$A\left(\bar{Y},\bar{Z}\right)={}^R\nabla\left(\psi\bar{Y},\bar{Z}\right)-{}^R\nabla\left(\bar{Y},\psi\bar{Z}\right)$$

for any $\bar{Y},\bar{Z}\in\Im_0^1\left(T^*M\right)$, then we have $A\left(\bar{Y},\bar{Z}\right)=0$. Hence ${}^R\nabla$ is pure with respect to $\psi$ and we have the following theorem.

**Theorem 3.1.** *The triple* $\left(T^*M,{}^R\nabla,\psi\right)$ *is a Golden pseudo-Riemannian manifold.*

In [12], a Golden Riemannian manifold $(M,\psi,g)$ is a locally decomposable Golden Riemannian manifold if and only if $\phi_Fg=0$ (or equivalently $\phi_\psi g=0$ ) where $F$ is the corresponding almost product structure. We put

$$\begin{aligned}\left(\phi_\psi{}^R\nabla\right)\left(\bar{X},\bar{Y},\bar{Z}\right)&=\left(\psi\bar{X}\right)\left({}^R\nabla\left(\bar{Y},\bar{Z}\right)\right)-\bar{X}\left({}^R\nabla\left(\psi\bar{Y},\bar{Z}\right)\right)\\ &+{}^R\nabla\left(\left(L_{\bar{Y}}\psi\right)\bar{X},\bar{Z}\right)+{}^R\nabla\left(\bar{Y},\left(L_{\bar{Z}}\psi\right)\bar{X}\right)\end{aligned}$$

for any $\bar{X}, \bar{Y}, \bar{Z} \in \Im_0^1 (T^*M)$. Then we have

$$\begin{aligned} \left(\phi_\psi {}^R\nabla\right) \left({}^V\omega, {}^HY, {}^HZ\right) &= \tfrac{\sqrt{5}}{2} {}^V \left(pR\left(Y, \tilde{\omega}\right) Z + pR\left(Z, \tilde{\omega}\right) Y\right), \\ \left(\phi_\psi {}^R\nabla\right) \left({}^HX, {}^V\omega, {}^HY\right) &= \tfrac{\sqrt{5}}{2} {}^V \left(pR\left(X, Y\right) \tilde{\omega}\right), \\ \left(\phi_\psi {}^R\nabla\right) \left({}^HX, {}^HY, {}^V\omega\right) &= \tfrac{\sqrt{5}}{2} {}^V \left(pR\left(X, Y\right) \tilde{\omega}\right) \end{aligned}$$

and the others are zero. Therefore, we have

**Theorem 3.2.** . *The triple* $\left(T^*M, {}^R\nabla, \psi\right)$ *is a locally decomposable Golden pseudo-Riemannian manifold if and only if* $M$ *is flat.*

**Remark 3.1.** The horizontal lift ${}^H\varphi \in \Im_1^1 (T^*M)$ is given by

$$\begin{aligned} {}^H\varphi {}^HY &= {}^H \left(\varphi Y\right), \\ {}^H\varphi {}^V\nu &= {}^V \left(\nu \circ \varphi\right) \end{aligned}$$

for any $Y \in \Im_0^1 (M)$ and $\nu \in \Im_1^0 (M)$. Also the followings hold

$$ {}^HI = I, \left({}^H\varphi\right)^2 = {}^H \left(\varphi^2\right) ... \tag{3.3}$$

where $I$ is the unit tensor field of type (1,1) [25].

From (3.3), $\varphi^2 - \varphi - I = 0$ implies $\left({}^H\varphi\right)^2 - {}^H\varphi - I = 0$. Hence, one can say that if $\varphi$ is a Golden structure on $M$, then ${}^H\varphi$ is also a Golden structure on $T^*M$ [10]. In [1], we know that Riemannian extension ${}^R\nabla$ is pure with respect to ${}^H\varphi$. Then we obtain

**Theorem 3.3.** *The triple* $\left(T^*M, {}^R\nabla, {}^H\varphi\right)$ *is a Golden pseudo-Riemannian manifold.*

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