Some optimal inequalities on Bochner-Kähler manifolds with Casorati curvatures

C.W. Lee, J.W. Lee

Abstract. The main purpose of this article is to construct optimal inequalities on some submanifolds in a Bochner-Kähler manifold involving Casorati curvatures.

M.S.C. 2010: 53A07, 49K35.
Key words: Bochner tensor; generalized normalized δ-Casorati curvature; Bochner-Kähler manifold; Einstein; slant; invariant; anti-invariant.

1 Introduction

The Bochner tensor was introduced by S. Bochner in Kähler manifolds analogue of the Weyl conformal curvature tensor [1]. The Bochner tensor is equal to the fourth order Chern-Moser curvature tensor of CR-manifolds by Webster [19]. Webster showed that a Bochner-Kähler surface is nothing but a self-dual Kähler surface in Penrose's thoery. A Kähler manifold is said to be Bochner-Kähler if its Bochner curvature tensor vanishes. Bochner-Kähler manifolds with constant scalar curvature are classified in [15]. Moreover, Chen and Dillen investigated goemetric characterizations of Bochner-Kähler and Einstein-Kähler spaces of complex space forms by using the δ-invariants $\delta(n_1, n_2, \cdots, n_k)$ and $\widehat{\delta}(n_1, n_2, \cdots, n_k)$ in [4]. On the other hand, it is well known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the lengh of the second fundamental form, introduced by F. Casorati in [2, 9]. Moreover, there are very interesting optimal inequalities involving Casorati curvatures in [5, 6, 7, 8, 10, 11, 12, 13, 14, 17, 20, 21] for several submanifolds in some space forms with various connections. In our paper, we establish optimal inequalities involving the generalized normalized δ-Casorati curvaures for some submanifolds in a Bochner-Kähler manifold and also characterize theose submanifolds for which the equalities hold.

2 Preliminaries

This section gives several basic definitions and notations for our framework based mainly.

Balkan Journal of Geometry and Its Applications, Vol.23, No.2, 2018, pp. 16-24.

Let M^n be an n-dimensional Riemannian submanifold of a Riemannian manifold $(\bar{M}, \bar{g})$ with the Riemannian metric $\bar{g}$. Let $K(\pi)$ be the sectional curvature of M associated with a plane section $\pi \subset T_pM,\ p \in M$. Assume that $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{n+1}, ..., e_m\}$ is an orthonormal basis of the normal space $T_p^{\perp}M$. Then the scalar curvature τ at p is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$$

and we also set

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha),\ i, j \in \{1, ..., n\},\ \alpha \in \{n+1, ..., m\}.$$

Then it is well-known that the squared mean curvature of the submanifold M in $\bar{M}$ is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$, called *the Casorati curvature* of the submanifold M. Therefore we have

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha} \right)^2.$$

The submanifold M is called *invariantly quasi-umbilical* if there exists $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, ..., \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same.

Suppose now that L is an s-dimensional subspace of T_pM, $s \geq 2$ and let $\{e_1, ..., e_s\}$ be an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the s-plane section L is given by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta).$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{s} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{s} \left(h_{ij}^{\alpha} \right)^2.$$

The generalized normalized δ-Casorati curvatures $\delta_C(t;n-1)$ and $\widehat{\delta}_C(t;n-1)$ of the submanifold M^n are defined for any positive real number $r \neq n(n-1)$ as

$$[\delta_C(t;n-1)]_p = t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < t < n^2 - n$, and

$$\left[\widehat{\delta}_C(t;n-1)\right]_p = t\mathcal{C}_p - \frac{(n-1)(n+t)(t-n^2+n)}{nt}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $t > n^2 - n$.

If $\overline{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and ∇ is the covariant differentiation induced on M, then the Gauss and Weingarten formulas are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \forall X, Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^{\perp})$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N. If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇, then we have the Gauss equation:

$$\begin{aligned} \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ \bar{g}(h(X,W),h(Y,Z)) \\ &- \bar{g}(h(X,Z),h(Y,W)), \end{aligned} \tag{2.1}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

Assume now that $(\bar{M}^m, \bar{g}, J)$ is an almost Hermitian with an almost complex structure J and a Riemannian mmetric $\bar{g}$ satisfying for

$$\bar{g}(J\cdot, J\cdot) = \bar{g}(\cdot,\cdot) \quad \text{and} \quad J^2 = -\mathrm{Id},$$

where Id denotes the identity tensor field of type (1, 1) on $\bar{M}$. Moreover, if the almost complex structure J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\bar{g}$, then $(\bar{M}, \bar{g}, J)$ is said to be a Kähler manifold.

The Bochner curvature tensor on a Kähler manifold is defined by [18]

$$\begin{aligned} B(X,Y,Z,W) = \overline{R}(X,Y,Z,W) &- L(Y,Z)\bar{g}(X,W) \\ &+ L(X,Z)\bar{g}(Y,W) - L(X,W)\bar{g}(Y,Z) \\ &+ L(Y,W)\bar{g}(X,Z) - L(JY,Z)\bar{g}(JX,W) \\ &+ L(JX,Z)\bar{g}(JY,W) - L(JX,W)\bar{g}(JY,Z) \\ &+ L(JY,W)\bar{g}(JX,Z) + 2L(JX,Y)\bar{g}(JZ,W) \\ &+ 2L(JZ,W)\bar{g}(JX,Y), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} L(X,Y) &= \frac{1}{2n+4}Ric(X,Y) - \frac{\tau}{8(n+1)(n+2)}\bar{g}(X,Y) \\ L(X,Y) &= L(Y,X), \quad L(JX,Y) = -L(X,JY), \end{aligned} \tag{2.3}$$

for all $X, Y, Z, W \in \Gamma(T\bar{M})$.

Let $(\bar{M}, \bar{g}, J)$ be a Kähler manifold. If the Bochner tensor B on $\bar{M}$ vanishes identically, $(\bar{M}, \bar{g}, J)$ is called a Bochner-Kähler manifold. From (2.2), the curvature tensor $\overline{R}$ of a Bochner-Kähler manifold is given by

$$
\begin{aligned}
\overline{R}(X,Y,Z,W) &= L(Y,Z)\bar{g}(X,W) - L(X,Z)\bar{g}(Y,W) \\
&+ L(X,W)\bar{g}(Y,Z) - L(Y,W)\bar{g}(X,Z) \\
&+ L(JY,Z)\bar{g}(JX,W) - L(JX,Z)\bar{g}(JY,W) \\
&+ L(JX,W)\bar{g}(JY,Z) - L(JY,W)\bar{g}(JX,Z) \\
&- 2L(JX,Y)\bar{g}(JZ,W) - 2L(JZ,W)\bar{g}(JX,Y).
\end{aligned} \tag{2.4}
$$

As a generalization of CR-submanifolds, B.-Y. Chen introduced the notion of slant submanifolds. We introduce the definition of slant submanifolds of Bochner-Kähler manifolds as follows:

Definition 2.1. A submanifold M of a Bochner-Kähler manifold $(\bar{M}, \bar{g}, J)$ is said to be *slant* if for any $p \in M$, the angle θ between JX and T_pM is constant. In other words, the angle does not depend on the choice of $p \in M$ and $X \in T_pM$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of M in $\bar{M}$.
If $\theta = 0$ $\left(\theta = \frac{\pi}{2}\right)$, M is called an invariant (anti-invariant) submanifold of $\bar{M}$, respectively. If $0 < \theta < \frac{\pi}{2}$, M is called a proper slant submanifold of $\bar{M}$.

The following lemma plays a key role in the proof of our theorems.

Lemma 2.1. *[16] Let*

$$\Gamma = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = k\}$$

be a hyperplane of $\mathbb{R}^n$*, and* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *a quadratic form given by*

$$f(x_1, x_2, \cdots, x_n) = a\sum_{i=1}^{n-1}(x_i)^2 + b\,(x_n)^2 - 2\sum_{1 \leq i < j \leq n} x_i x_j, \qquad a > 0,\ b > 0.$$

Then, by the constrained extremum problem, f *has the global extreme as follows:*

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{k}{a+1}, \quad x_n = \frac{k}{b+1} = \frac{k(n-1)}{(a+1)b} = (a-n+2)\frac{k}{a+1},$$

provided that

$$b = \frac{n-1}{a-n+2}.$$

3 Inequalities involving Casorati curvatures

Let M be a submanifold of a Bochner-Kähler manifold $(\bar{M}, \bar{g}, J)$. Let $p \in M$ and the set $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_m\}$ be orthonormal bases of T_pM and $T_p^{\perp}M$, respectively. From (2.4), we have

$$
\begin{aligned}
\overline{R}(e_i, e_j, e_j, e_i) &= L(e_j, e_j)\bar{g}(e_i, e_i) + L(e_i, e_i)\bar{g}(e_j, e_j) \\
&+ 6L(e_i, Je_j)\bar{g}(e_i, Je_j) - 2L(e_i, Je_j)\bar{g}(e_i, e_i).
\end{aligned} \tag{3.1}
$$

From (3.1), we have

$$\sum_{i,j=1}^{n} \overline{R}(e_i, e_j, e_j, e_i) = (2n-2)\sum_{i=1}^{n} L(e_i, e_i) + 6\sum_{i,j=1}^{n} L(e_i, Je_j)\bar{g}(e_i, Je_j) \tag{3.2}$$

Combining (2.1) and (3.2), we obtain

$$\begin{aligned} 2\tau &= n^2||H||^2 - ||h||^2 + (2n-2)\sum_{i=1}^{n} L(e_i, e_i) + 6\sum_{i,j=1}^{n} L(e_i, Je_j)\bar{g}(e_i, Je_j) \\ &= n^2||H||^2 - n\mathcal{C} + (2n-2)\sum_{i=1}^{n} L(e_i, e_i) + 6\sum_{i,j=1}^{n} L(e_i, Je_j)\bar{g}(e_i, Je_j) \end{aligned} \tag{3.3}$$

We now consider the following quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) - 2\tau + (2n-2)\sum_{i=1}^{n} L(e_i, e_i) + 6\sum_{i,j=1}^{n} L(e_i, Je_j)\bar{g}(e_i, Je_j),$$

where L is a hyperplane of T_pM. Without loss of generality we can assume that L is spanned by $e_1, ..., e_{n-1}$. Then we derive

$$\begin{aligned} \mathcal{P} = &\sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}\left[\frac{n^2+n(t-1)-2t}{r}\left(h_{ii}^{\alpha}\right)^2 + \frac{2(n+t)}{n}\left(h_{in}^{\alpha}\right)^2\right] \\ &+ \sum_{\alpha=n+1}^{m}\left[\frac{2(n+t)(n-1)}{t}\sum_{1=i<j}^{n-1}\left(h_{ij}^{\alpha}\right)^2 - 2\sum_{i<j=1}^{n} h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{t}{n}\left(h_{nn}^{\alpha}\right)^2\right] \\ \geq &\sum_{\alpha=n+1}^{m}\left[\sum_{i=1}^{n-1}\frac{n^2+n(t-1)-2t}{t}\left(h_{ii}^{\alpha}\right)^2 - 2\sum_{1=i<j}^{n} h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{t}{n}\left(h_{nn}^{\alpha}\right)^2\right]. \end{aligned} \tag{3.4}$$

For $\alpha = n+1, \cdots, m$, let us consider the quadratic form $f_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$f_\alpha\left(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}\right) = \frac{n^2+n(t-1)-2t}{t}\sum_{i=1}^{n-1}\left(h_{ii}^{\alpha}\right)^2 - 2\sum_{i<j=1}^{n} h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{t}{n}\left(h_{nn}^{\alpha}\right)^2, \tag{3.5}$$

and the constrained extremum problem

$$\min f_\alpha$$

$$\text{subject to } F^\alpha : h_{11}^{\alpha} + \cdots + h_{nn}^{\alpha} = c^\alpha,$$

where c^α is a real constant. Comparing (3.5) with the quadratic function in Lemma 2.1, we see that

$$a = \frac{n^2+n(t-1)-2t}{t}, \qquad b = \frac{t}{n}.$$

Therefore, we have the critical point $(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha})$, given by

$$h_{11}^{\alpha} = h_{22}^{\alpha} = \cdots = h_{n-1\ n-1}^{\alpha} = \frac{tc^\alpha}{(n+t)(n-1)}, \qquad h_{nn}^{\alpha} = \frac{nc^\alpha}{n+t},$$

is a global minimum point by Lemma 2.1. Moreover, $f_\alpha\left(h_{11}^\alpha, \cdots, h_{nn}^\alpha\right)=0$. Therefore, we have

$$\mathcal{P} \geq 0, \tag{3.6}$$

which implies

$$\begin{aligned}
2\tau(p) \leq & t\mathcal{C}+\frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) \\
& +(2n-2)\sum_{i=1}^{n} L(e_i,e_i)+6\sum_{i,j=1}^{n} L(e_i,Je_j)\bar{g}(e_i,Je_j) \\
& =t\mathcal{C}+\frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) \\
& +\frac{(2n-2)(3n+4)-6||P||^2}{2(2n+2)(2n+4)}\tau-\frac{6}{2n+4}\sum_{i,j=1}^{n} Ric(e_i,Je_j)\bar{g}(e_i,Je_j),
\end{aligned}$$

where $||P||^2=\sum_{i,j=1}^n g^2(Je_i,e_j)$ for $JX=PX+QX, X\in\Gamma(TM)$ whose PX and QX are the tangential and normal components of JX, respectively.

From (2.3), we derive

$$\begin{aligned}
\frac{5n^2+23n+20+3||P||^2}{4(n+1)(n+2)}\tau \leq & t\mathcal{C}+\frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) \\
& -\frac{3}{n+2}\sum_{i,j=1}^{n} Ric(e_i,Je_j)\bar{g}(e_i,Je_j).
\end{aligned}$$

Therefore, we derive

$$\begin{aligned}
\rho \leq & \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\left(t\mathcal{C}+\frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L)\right) \\
& -\frac{6(n+1)}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\sum_{i,j=1}^{n} Ric(e_i,Je_j)\bar{g}(e_i,Je_j).
\end{aligned}$$

Therefore, we have the following theorem:

Theorem 3.1. *Let M^n be an n-dimensional Riemannian submanifold of a Bochner-Kähler manifold $(\bar{M},\bar{g},J)$. When $0<t<n^2-n$, the generalized normalized δ-Casorati curvature $\delta_C(t,n-1)$ on M^n satisfies*

$$\begin{aligned}
\rho \leq & \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\delta_C(t,n-1) \\
& -\frac{6(n+1)}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\sum_{i,j=1}^{n} Ric(e_i,Je_j)\bar{g}(e_i,Je_j).
\end{aligned}$$

Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in a Bochner-Kähler manifold $(\bar{M},\bar{g},J)$, *such that with respect to suitable orthonormal tangent frame $\{\xi_1, \cdots, \xi_n\}$ and normal orthonormal frame $\{\xi_{n+1}, \cdots, \xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \cdots, m\}$, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & ... & 0 & 0 \\ 0 & a & 0 & ... & 0 & 0 \\ 0 & 0 & a & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a & 0 \\ 0 & 0 & 0 & ... & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \ A_{n+2} = \cdots = A_m = 0.$$

Corollary 3.2. *Let M^n be an n-dimensional Einstein submanifold of a Bochner-Kähler manifold $(\bar{M}^m, \bar{g}, J)$. Then, for a Ricci curvature λ, we obtain*

$$\rho \leq \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\delta_C(t, n-1) - \frac{6(n+1)||P||^2}{n(n-1)\left(5n^2+23n+20+3||P||^2\right)}\lambda.$$

Moreover, the equality case holds if and only if with respect to a suitable frames $\{e_1, ..., e_n\}$ on M and $\{e_{n+1}, ..., e_m\}$ on $T_p^\perp M$, $p \in M$, the components of h satisfy

$$h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1\ n-1}^\alpha = \tfrac{t}{n(n-1)}h_{nn}^\alpha, \qquad \alpha \in \{n+1, \cdots, m\},$$
$$h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\}(i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.$$

For a slant submanifold of a Bochner-Kähler manifold, we have following corollaries.

Corollary 3.3. *Let M^n be an n-dimensional slant submanifold of a Bochner-Kähler manifold $(\bar{M}^m, \bar{g}, J)$. When $0 < t < n^2 - n$, we obtain*

$$\rho \leq \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+20+3\cos^2\theta\right)}\delta_C(t, n-1) - \frac{6(n+1)}{n(n-1)\left(5n^2+23n+20+3\cos^2\theta\right)}\sum_{i,j=1}^{n} Ric(e_i, Je_j)\cos^2\theta,$$

where θ is a slant function. Moreover, the equality case holds if and only if with respect to a suitable frames $\{e_1, ..., e_n\}$ on M and $\{e_{n+1}, ..., e_m\}$ on $T_p^\perp M$, $p \in M$, the components of h satisfy

$$h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1\ n-1}^\alpha = \tfrac{t}{n(n-1)}h_{nn}^\alpha, \qquad \alpha \in \{n+1, \cdots, m\},$$
$$h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\}(i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.$$

Corollary 3.4. *Let M^n be an n-dimensional invariant submanifold of a Bochner-Kähler manifold $(\bar{M}^m, \bar{g}, J)$. When $0 < t < n^2 - n$, we obtain*

$$\rho \leq \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+23\right)}\delta_C(t, n-1) - \frac{6(n+1)}{n(n-1)\left(5n^2+23n+23\right)}\sum_{i,j=1}^{n} Ric(e_i, Je_j),$$

Moreover, the equality case holds if and only if with respect to a suitable frames $\{e_1, ..., e_n\}$ *on* M *and* $\{e_{n+1}, ..., e_m\}$ *on* $T_p^\perp M$, $p \in M$, *the components of* h *satisfy*

$$h_{11}^\alpha = h_{22}^\alpha = \cdots = h_{n-1\ n-1}^\alpha = \tfrac{t}{n(n-1)} h_{nn}^\alpha, \qquad \alpha \in \{n+1, \cdots, m\},$$
$$h_{ij}^\alpha = 0, \quad i, j \in \{1, 2, \cdots, n\}(i \neq j), \quad \alpha \in \{n+1, \cdots, m\}.$$

Corollary 3.5. *Let* M^n *be an* n*-dimensional anti-invariant submanifold of a Bochner-Kähler manifold* $(\bar{M}^m, \bar{g}, J)$*. When* $0 < t < n^2 - n$*, we obtain*

$$\rho \leq \frac{8(n+1)(n+2)}{n(n-1)\left(5n^2+23n+20\right)} \delta_C(t, n-1),$$

Moreover, the equality case holds if and only if M *is an invariantly quasi-umbilical subamnifold of Bochner-Kähler manifold.*

Remark 3.1. In the case for $t > n^2 - n$, the methods of finding the above inequailities is analogous. Thus, we leave the problems for readers.

Acknowledgements. Chul Woo Lee was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2018R1D1A1B07040576), and Jae Won Lee was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2017R1D1A1B03033978).

References

[1] S. Bochner, *Curvature and Betti numbers II*, Ann. Math. 50 (1949), 77-93.

[2] F. Casorati, *Mesure de la courbure des surfaces suivant l'idée commune. Ses rapports avec les mesures de courbure gaussienne et moyenne*, Acta Math., 14, 1 (1890), 95-110.

[3] B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[4] B.-Y. Chen, F. Dillen, *Totally real bisectional curvature, Bochner-Kaehler and Einstein-Kaehler manifolds*, Differ. Geom. Appl. 10 (1999), 145-154.

[5] S. Decu, S. Haesen, L. Verstraelen, *Optimal inequalities involving Casorati curvatures*, Bull. Transilv. Univ. Braşov Ser. B (N.S.) 14, 49 (2007), suppl., 85-93.

[6] S. Decu, S. Haesen, L. Verstraelen, *Optimal inequalities characterising quasi-umbilical submanifolds*, J. Inequal. Pure Appl. Math. 9, 3 (2008), Article 79, 1-7.

[7] V. Ghişoiu, *Inequalities for the Casorati curvatures of slant submanifolds in complex space forms*, Riemannian geometry and applications. Proceedings RIGA 2011, 145-150, Ed. Univ. Bucureşti, Bucharest, 2011 .

[8] G. He, H. Liu, L. Zhang, *Optimal inequalities for the Casorati curvatures of submanifolds in generalized space forms endowed with semi-symmetric non-metric connections*, Symmetry 8, 11 (2016), Art. 113, 10 pp.

[9] D. Kowalczyk, *Casorati curvatures*, Bull. Transilv. Univ. Brasov Ser. III 1, 50 (2008), 209-213.

[10] C.W. Lee, D.W. Yoon, J.W. Lee, *Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections*, J. Inequal. Appl. 2014, 2014:327.

[11] C. W. Lee, J. W. Lee, G.-E. Vîlcu, and D. W. Yoon, *Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, B. Korean Math. Soc. 52, 5 (2015), 1631-1647.

[12] C. W. Lee, J. W. Lee, G. E. Vîlcu, *A new proof for some optimal inequalities involving generalized normalized δ-Casorati curvatures*, J. Inequal. Appl., 2015, 2015:30.

[13] J. W. Lee, C. W. Lee, D. W. Yoon, *Inequalities for generalized δ-Casorati curvatures of submanifolds in real space forms endowed with a semi-symmetric metric connection*, Rev. Union Mat. Argent., 57, 2 (2016), 53-62.

[14] J.W. Lee, G.E. Vîlcu, *Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in quaternionic space forms*, Taiwanese J. Math., 19, 3 (2015), 691-702.

[15] M. Matsumoto, S. Tanno, *Kählerian spaces with parallel or vanishing Bochner curvature tensor*, Tensor (N.S.), 27, (1973), 291-294.

[16] M. M. Tripathi, *Inequalities for algebraic Casorati curvatures and their applications*, Note di Matematica., 37, (2017), 161-186.

[17] V. Slesar, B. Şahin, G.E. Vîlcu, *Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms*, J. Inequal. Appl., 2014, 2014:123.

[18] S. Tachibana, R. Liu, *Notes on Kähler metrics with vanishing Bochner curvature tensor*, Kodai Math. Sem. Rep., 22, (1970), 313-321.

[19] S. M. Webster, *On the pseudo-conformal geometry of a Kähler manifold*, Math. Z., 157, (1977), 265-270.

[20] P. Zhang, L. Zhang, *Casorati inequalities for submanifolds in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection*, Symmetry, 8, 4 (2016), 329-335.

[21] P. Zhang, L. Zhang, *Inequalities for Casorati curvatures of submanifolds in real space forms*, Adv. Geom., 16, 3 (2016), 329-335.

Authors' addresses:

Chul Woo Lee
Department of Mathematics,
Kyungpook National University, Daegu 41566, South Korea.
E-mail: mathisu@knu.ac.kr

Jae Won Lee (Corresponding author)
Department of Mathematics Education RINS,
Gyeongsang National University, Jinju 52828, South Korea.
E-mail: leejaew@gnu.ac.kr