Characterizations of contact CR-warped product submanifolds of nearly Sasakian manifolds

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Abstract. In this paper, we study warped product contact CR-submanifolds of a nearly Sasakian manifold. We work out the characterizations in terms of tensor fields under which a contact CR-submanifold of a nearly Sasakian manifold reduces to a warped product submanifold.

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1 Introduction

The geometry of warped product manifolds provide an excellent setting to model space time near black holes and bodies with large gravitational fields. B. Y. Chen [5] initiated the study of CR-warped product submanifold in a Kaehler manifold. Many geometers studied geometric properties in terms canonical structure tensors T and F. B.Y.Chen [4] obtained characterization in terms T, i.e., a CR-subamnifold of a Kaehler manifold is a CR-product if and only if $\nabla T = 0$. Motivated by Chen's paper, the result was extended for warped product submanifolds in almost contact setting by M. I. Munteanu [11]. Recently, V. A. Khan [10] studied contact CR-warped product submanifold of Kenmotsu manifold in terms tensor fields. Therefore its natural to see that how the non-triviality of covariant derivatives of T and F gives rise to a warped product submanifold. Moreover, I. Hesigawa and I. Mihai [6] worked out the necessary and sufficient conditions involving the shape operator of a contact CR-submanifold into a Sasakian manifold under which a submanifold is reduced to contact CR-warped product submanifold. In this paper, we study contact CR-submanifold in brief not in details as our aim is to discuss the warped products. We prove some existence results of contact CR-warped product of a nearly Sasakian manifold by its characterizations in terms of tensor fields T and F. We obtain some initial results on contact CR-submanifold of a nearly Sasakian manifold.

The paper is organized as follows: in Section 2, we review and collect some necessary results. In Section 3, we define a contact CR-submanifold in a nearly Sasakian

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manifolds and derive the integrability conditions and totally geodesic foliation of involving distributions. In Section 4, we obtain some results on characterizations of warped product submanifolds in terms of endomorphisms T and F.

2 Preliminaries

An odd $(2n+1)-$dimensional manifold $\widetilde{M}$ is said to be an *almost contact metric manifold* if it admits an endomorphism φ of its tangent bundle $T\bar{M}$, a vector field ξ (called *structure vector field*), and η (*the dual 1-form*), satisfying the following:

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \varphi(\xi) = 0,\ \eta \circ \varphi = 0,$$
$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V),\ \eta(U) = g(U, \xi), \tag{2.1}$$

for any U, V tangent to $\widetilde{M}$. Similarly, an almost contact metric manifold $\widetilde{M}$ is called *Sasakian structure* if

$$(\widetilde{\nabla}_U \varphi)V = g(U, V)\xi - \eta(V)U \quad \text{and} \quad \widetilde{\nabla}_U \xi = -\varphi U.$$

Furthermore, an almost contact metric manifold is known to be a nearly Sasakian manifold if [16]

$$(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 2g(U, V)\xi - \eta(V)U - \eta(U)V, \tag{2.2}$$

for any vector fields U, V on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

Now let M be a submanifold of $\widetilde{M}$. Then we will denote by ∇ is a induced Riemannian connection on M and g is a Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Let TM and $T^{\perp}M$ are Lie algebras of vector fields tangent to M and normal to M, respectively and $\nabla^{\perp}$ the induced connection on $T^{\perp}M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.3}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.4}$$

for any $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to a normal vector field N) respectively, for an immersion of M into $\widetilde{M}$. They are related as:

$$g(h(U, V), N) = g(A_N U, V). \tag{2.5}$$

Now for any $U \in \Gamma(TM)$, we write

$$\varphi U = TU + FU, \tag{2.6}$$

where TU and FU are the tangential and the normal components of φU, respectively. Similarly, for any $N \in \Gamma(T^{\perp}M)$, we have $\varphi N = tN + fN$, where tN (resp. fN)

are the tangential (resp. the normal) component of φN, respectively. Its easy to observe that for each $U, V \in \Gamma(TM)$, we have $g(TU, V) = -g(U, TV)$. The covariant derivatives of the endomorphisms φ, T and F are defined respectively as:

$$(\widetilde{\nabla}_U \varphi)V = \widetilde{\nabla}_U \varphi V - \varphi \bar{\nabla}_U V, \tag{2.7}$$

$$(\widetilde{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \tag{2.8}$$

$$(\widetilde{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.9}$$

for all $U, V \in \Gamma(T\bar{M})$ and $N \in \Gamma(T^{\perp}M)$. A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *totally umbilical* if

$$h(U, V) = g(U, V)H, \tag{2.10}$$

where H is the mean curvature vector of M. Furthermore, if $h(U, V) = 0$, $\forall U, V \in \Gamma(TM)$, then M is said to be *totally geodesic* and if $H = 0$, them M is called *minimal* in $\widetilde{M}$. Let us denote tangential and normal parts of $(\widetilde{\nabla}_U \varphi)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$, respectively. Then it can be decomposed as:

$$(\widetilde{\nabla}_U \varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V.$$

$$\mathcal{P}_U V = (\widetilde{\nabla}_U T)V - A_{FV}U - th(U, V). \tag{2.11}$$

$$\mathcal{Q}_U V = (\widetilde{\nabla}_U F)V + h(U, TV) - fh(U, V). \tag{2.12}$$

Similarly, let us denote the tangential and normal parts of $(\widetilde{\nabla}_U \phi)N$ by $\mathcal{P}_U N$ and $\mathcal{Q}_U N$, respectively. Thus we have decomposed as:

$$(\widetilde{\nabla}_U \varphi)N = \mathcal{P}_U N + \mathcal{Q}_U N,$$

for any $N \in \Gamma(T^{\perp}M)$, where $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ are defined as:

$$\mathcal{P}_U N = (\widetilde{\nabla}_U t)N + TA_N X - A_{fN}U,$$

$$\mathcal{Q}_U N = (\widetilde{\nabla}_U f)N + h(tN, U) + FA_N U.$$

For a *nearly Sasakian* structure we have

$$(a)\ \mathcal{P}_U V + \mathcal{P}_V U = 2g(U, V)\xi - \eta(V)U - \eta(U)V, \quad (b)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0, \tag{2.13}$$

for each $U, V \in \Gamma(TM)$. It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which will be further used:

$$\left.\begin{array}{ll}
(i) & \mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W,\\
(ii) & \mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W,\\
(iii) & \mathcal{P}_U(W+Z) = \mathcal{P}_U W + \mathcal{P}_U Z,\\
(iv) & \mathcal{Q}_U(W+Z) = \mathcal{Q}_U W + \mathcal{Q}_U Z,\\
(v) & g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W),\\
(vi) & g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N),\\
(vii) & \mathcal{P}_U \varphi V + \mathcal{Q}_U \varphi V = -\varphi(\mathcal{P}_U V + \mathcal{Q}_U V).
\end{array}\right\} \tag{2.14}$$

3 Contact CR-submanifolds

A submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $\widetilde{M}$ is said to be *invariant*, if $\varphi(T_xM) \subseteq (T_xM)$, and *anti-invariant* if $\varphi(T_xM) \subset (T_x^{\perp}M)$ for each $x \in M$.

Definition 3.1. A submanifold M tangent to structure vector field ξ of an almost contact metric manifold $\widetilde{M}$ is said to be a *contact CR-submanifold*, if there exist a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that:

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus < \xi >$, *where* $< \xi >$ *is an 1-dimensional distribution spanned by* ξ,

(ii) $\mathcal{D}$ *is invariant, i.e.,* $\varphi(\mathcal{D}) \subseteq \mathcal{D}$,

(iii) $\mathcal{D}^{\perp}$ *is anti-invariant, i.e.,* $\varphi(\mathcal{D}^{\perp}) \subset (T^{\perp}M)$.

Let d_1 and d_2 be, respectively, the dimensions of the invariant distribution $\mathcal{D}$ and of the anti-invariant distribution $\mathcal{D}^{\perp}$ of a contact CR-submanifold of a given almost contact metric manifold $\widetilde{M}$. Then M is *invariant* if $d_2 = 0$, and *anti-invariant* if $d_1 = 0$. It is called *proper contact CR-subamnifold* if neither $d_1 = 0$ nor $d_2 = 0$. Moreover, if ν is an invariant subspace under φ of the normal bundle $T^{\perp}M$, then in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = F\mathcal{D}^{\perp} \oplus \nu$. Let us denote the orthogonal projections on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then, for any $U \in \Gamma(TM)$, we define

$$U = P_1U + P_2U + \eta(U)\xi, \tag{3.1}$$

where $P_1U \in \Gamma(\mathcal{D})$ and $P_2U \in \Gamma(\mathcal{D}^{\perp}$. From (2.5), (2.6) and (3.1), we have

$$TU = \varphi P_1U, \quad FU = \varphi P_2U.$$

It is straightforward to observe that we obtain:

$$(i)\ TP_2 = 0, \quad (ii)\ FP_1 = 0, \quad (iii)\ t(T^{\perp}M) \subseteq \mathcal{D}^{\perp}, \quad (iv)\ f(T^{\perp}M) \subset \nu.$$

Lemma 3.1. *Let M be a contact CR-submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then $\mathcal{D} \oplus \xi$ is integrable if and only if,*

$$2g(\nabla_XY, Z) = g(A_{\varphi Z}X, \varphi Y) + g(A_{\varphi Z}Y, \varphi X),$$

for any $X, Y \in \Gamma(D\oplus < \xi >)$ *and* $Z, \in \Gamma(D^{\perp})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z \in \Gamma(\mathcal{D}^{\perp})$. Then, by using (2.1) and (2.3), we have $g(\nabla_XY, Z) = g(\widetilde{\nabla}_XY, Z) = g(\varphi\widetilde{\nabla}_XY, \varphi Z) + \eta(\widetilde{\nabla}_XY)\eta(Z)$. Since $\eta(Z) = 0$, we get $g(\nabla_XY, Z) = g(\varphi\widetilde{\nabla}_XY, \varphi Z)$. Thus by (2.7), we obtain $g(\nabla_XY, Z) = g(\widetilde{\nabla}_X\varphi Y, \varphi Z) - g((\widetilde{\nabla}_X\varphi)Y, \varphi Z)$. From the *Gauss formula* and the structure equations (2.2), we get $g(\nabla_XY, Z) = g(h(X, \varphi Y), \varphi Z) + g((\widetilde{\nabla}_Y\varphi)X, \varphi Z) - 2g(X, Y)g(\xi, \varphi Z) + \eta(X)g(Y, \varphi Z) + \eta(Y)g(X, \varphi Z)$. Again, by using (2.3), (2.7) and (2.1), we obtain $g(\nabla_XY, Z) = g(h(X, \varphi Y), \varphi Z) + g(h(Y, \varphi X), \varphi Z) - g(\widetilde{\nabla}_YX, Z)$. Then, by a property of the Lie bracket and by the relation between the second fundamental form and the shape operator, we get the desired result. □

Lemma 3.2. *The anti-invariant distribution $\mathcal{D}^\perp$ of the contact CR-submanifold M of a nearly Sasakian manifold $\widetilde{M}$ defines a totally geodesic foliation if and only if*

$$g(h(Z,\varphi X),\varphi W)+g(h(W,\varphi X),\varphi Z)=0,$$

for any $X\in\Gamma(\mathcal{D}\oplus<\xi>)$ and $Z,W\in\Gamma(\mathcal{D}^\perp)$.

Proof. Let $Z,W\in\Gamma(\mathcal{D}^\perp)$ and $X\in\Gamma(\mathcal{D}\oplus<\xi>)$. By using (2.1) and (2.3), we have $g(\nabla_Z W,X)=g(\varphi\bar{\nabla}_Z W,\varphi X)$. From (2.7), we get $g(\nabla_Z W,X)=g(\widetilde{\nabla}_Z\varphi W,\varphi X)-g((\widetilde{\nabla}_Z\varphi)W,\varphi X)$. Thus (2.4) and (2.2) imply that $g(\nabla_Z W,X)=-g(A_{\varphi W}Z,\varphi X)+g((\widetilde{\nabla}_W\varphi)Z,\varphi X)$. Again, by using (2.7) and (2.4), we arrive at $2g(\nabla_Z W,X)=-g(A_{\varphi W}Z,\varphi X)-g(A_{\varphi Z}W,\varphi X)+g([Z,W],X)$, which proves our assertion. □

4 Warped product CR-submanifods

The warped products manifolds were introduced by Bishop and O'Neill [2]. They defined the warped products as follows: let M_1 and M_2 be two Riemannian manifolds with the corresponding Riemannian metrics g_1 and g_2 respectively, and let f be a positive differentiable function on M_1. Consider the product manifold $M_1\times M_2$ with its projections $\pi_1: M_1\times M_2\to M_1$ and $\pi_2: M_1\times M_2\to M_2$. Then their *warped product manifold* $M=M_1\times_f M_2$ is the Riemannian manifold $M_1\times M_2=(M_1\times M_2,g)$ equipped with the Riemannian structure g given by

$$g(X,Y)=g_1(\pi_{1*}X,\pi_{2*}Y)+(f\circ\pi_1)^2 g_2(\pi_{2*}X,\pi_{2*}Y),$$

for any vector fields X,Y tangent to M, where $*$ is the symbol for the tangent maps. A warped product manifold $M=M_1\times_f M_2$ is said to be *trivial* or simply *Riemannian product manifold*, if the warping function f is constant. We recall below several results for warped product manifolds.

Lemma 4.1. *[2] Let $M=M_1\times M_2$ be a warped product manifold with the warping function f. Then*

(i) $\nabla_X Y\in\Gamma(TM_1)$ *is the lift of* $\nabla_X Y$ *on* M_1*,*

(ii) $\nabla_X Z=\nabla_Z X=(X\ln f)Z$*,*

(iii) $\nabla_Z W=\nabla'_Z W-g(Z,W)\nabla\ln f$*,*

for each $X,Y\in\Gamma(TM_1)$ and $Z,W\in\Gamma(TM_2)$, where $\nabla\ln f$ is the gradient of $\ln f$, defined by $g(\nabla\ln f,X)=X\ln f$, where ∇ and ∇' denote the Levi-Civita connections on M and M_2, respectively.

The non-existence of warped product contact CR-submanifolds of the type $M=M_\perp\times_f M_T$ and $M=M_T\times_f M_\perp$ was proved in [16], when the structure vector field ξ is tangent to M_T and to $M_\perp$, respectively.

Lemma 4.2. *[16] Let a CR-warped product submanifold $M_T\times_f M_\perp$ of a nearly Sasakian manifold $\widetilde{M}$ be, such that M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\widetilde{M}$, respectively. Then we have* *(i)* $\xi \ln f = 0$,

(ii) $g(h(X,Y), \varphi Z) = 0$,

(iii) $g(h(X,Z), \varphi Z) = -(\varphi X \ln f) + \eta(X)||Z||^2$,

(iv) $g(h(\xi, Z), \varphi Z) = -||Z||^2$,

for each $X, Y \in \Gamma(TM_T)$ *and* $Z \in \Gamma(TM_\perp)$.

Before stating the characterization theorem, we first discuss several properties of contact CR-warped product submanifolds of a nearly Sasakian manifold. For any $X, Y \in \Gamma(TM_T)$, the properties (2.11) and (2.14)(i) imply that

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2th(X,Y) + 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{4.1}$$

Since, M_T is totally geodesic in M, $(\widetilde{\nabla}_X T)Y$ completely lies on M_T and its second fundamental identically vanishes. Thus, by comparing the component which is tangent to M_T in formula (4.1), we obtain $th(X,Y) = 0$, which implies that $h(X,Y) \in \nu$, and

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{4.2}$$

If we set $Y = \xi$ in the above equation, we derive

$$\begin{aligned}(\widetilde{\nabla}_X T)\xi + (\widetilde{\nabla}_\xi T)X &= 2g(X,\xi)\xi - \eta(X)\xi - \eta(\xi)X \\ &= 2\eta(X)\xi - \eta(X)\xi - X.\end{aligned}$$

From (2.8), it can be easily seen that

$$(\widetilde{\nabla}_\xi T)X = -(\widetilde{\nabla}_X T)\xi + \eta(X)\xi - X$$

$$= T\nabla_X \xi + \eta(X)\xi - X. \tag{4.3}$$

Lemma 4.3. *Let* $M = M_T \times_f M_\perp$ *be a CR-warped product submanifold of a nearly Sasakian manifold* $\widetilde{M}$. *In this case, we have:*

$$\begin{aligned}(\widetilde{\nabla}_Z T)X &= (TX \ln f)Z, \\ (\widetilde{\nabla}_U T)Z &= g(P_2 U, Z)T\nabla \ln f,\end{aligned}$$

for each $U \in \Gamma(TM)$, $X \in \Gamma(TM_T)$ *and* $Z \in \Gamma(TM_\perp)$.

Proof. From Lemma 4.1(ii), we have $\nabla_X Z = \nabla_Z X = (X \ln f)Z$, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$. Using the above equation in (2.8), we obtain

$$(\widetilde{\nabla}_Z T)X = \nabla_Z TX - T\nabla_Z X = (TX \ln f)Z - (X \ln f)TZ = (TX \ln f)Z,$$

which is the first part of the Lemma. On the other hand, from (2.8), for each $Z \in \Gamma(TM_\perp)$ and $U \in \Gamma(TM)$, we derive

$$(\widetilde{\nabla}_U T)Z = \nabla_U TZ - T\nabla_U Z = -T\nabla_U Z.$$

From property (3.1), the above equation can be expressed as:

$$(\widetilde{\nabla}_U T)Z = -T\nabla_{P_1U}Z - T\nabla_{P_2U}Z - \eta(U)T\nabla_\xi Z.$$

Then Lemma 4.1(ii) and $TZ = 0$ yield

$$(\widetilde{\nabla}_U T)Z = (P_1U \ln f)TZ - T\nabla_{P_2U}Z - \eta(U)(\xi \ln f)TZ = -T\nabla_{P_2U}Z.$$

Thus by the property(ii) of Lemma 4.1, we finally get

$$(\widetilde{\nabla}_U T)Z = -T\nabla'_{P_2U}Z + g(P_2U, Z)T\nabla \ln f = g(P_2U, Z)T\nabla \ln f,$$

which is the second part of the Lemma, which completes the proof. □

Theorem 4.4. *Let M be a contact CR-submanifold of a nearly Sasakian manifold $\widetilde{M}$, with both the distributions integrable. Then M is locally isometric to a CR-warped product if and only if*

$$(\widetilde{\nabla}_U T)U = (TP_1U\mu)P_2U + ||P_1U||^2\xi + ||P_2U||^2 T\nabla\mu - \eta(U)P_1U, \tag{4.4}$$

or, equivalently,

$$(\widetilde{\nabla}_U T)V + (\widetilde{\nabla}_V T)U = (TP_1V \ln f)P_2U + (TP_1U \ln f)P_2V + 2g(P_1U, P_1V)\xi$$

$$+ 2g(P_2U, P_2V)T\nabla \ln f - \eta(U)P_1V - \eta(V)P_1U, \tag{4.5}$$

for each $U, V \in \Gamma(TM)$ and for any C^∞-function μ on M, with $Z\mu = 0$ for each $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let M be a contact CR-warped product submanifold of $\widetilde{M}$. Then for any $U \in \Gamma(TM)$ with the property (3.1), we can write:

$$(\widetilde{\nabla}_U T)U = (\widetilde{\nabla}_{P_1U} T)P_1U + (\widetilde{\nabla}_{P_2U} T)P_1U + \eta(U)(\widetilde{\nabla}_\xi T)P_1U$$

$$+(\widetilde{\nabla}_U T)P_2U + \eta(U)(\widetilde{\nabla}_U T)\xi.$$

From (4.2) and (4.3), for the contact CR-warped product case, from Lemma 4.3 we obtain

$$(\widetilde{\nabla}_U T)U = ||P_1U||^2\xi - \eta(P_1U)P_1U + (TP_1U \ln f)P_2U + ||P_2U||^2 T\nabla \ln f$$

$$+\eta(U)T\nabla_{P_1U}\xi - \eta(U)T\nabla_{P_1U}\xi.$$

Since $\ln f = \mu$, we get

$$(\widetilde{\nabla}_U T)U = (TP_1U\mu)P_2U + ||P_1U||^2\xi + ||P_2U||^2 T\nabla\mu - \eta(U)P_1U,$$

which is the desired result (4.4). Furthermore, by replacing U by $U + V$ in the above equation and using the linearity of vector fields, we get (4.5). *Conversely*, suppose that M is a contact CR-submanifold of $\widetilde{M}$, with both distributions integrable on M such that (4.5) holds for a C^∞-function μ on M, with $Z\mu = 0$ for each $Z \in \Gamma(\mathcal{D}^\perp)$.

Then for any $X, Y \in \Gamma(\mathcal{D}\oplus < \xi >)$ and the fact that $P_2X = 0$, by using in (4.5) we obtain

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{4.6}$$

Since $\widetilde{M}$ is a nearly Sasakian manifold, then from (2.11) and (2.13)(a) we derive

$$(\widetilde{\nabla}_X T)Y + (\widetilde{\nabla}_Y T)X = 2th(X,Y) + 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{4.7}$$

Thus from (4.6) and (4.7), we obtain $th(X,Y) = 0$. This means that $h(X,Y) \in \nu$. However, $\mathcal{D}\oplus < \xi >$ is integrable, and hence from Lemma 3.1, we infer that $g(\nabla_X Y, Z) = 0$ for each $Z \in \Gamma(\mathcal{D}^\perp)$, which means that $\nabla_X Y \in \Gamma(\mathcal{D}\oplus < \xi >)$, i.e., the leaves of the distribution $\mathcal{D}\oplus < \xi >$ are totally geodesic in M. On the other hand, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$ and from (4.5), we obtain

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = 2g(Z,W)T\nabla\mu. \tag{4.8}$$

From (2.11) and (2.13)(a), we get

$$(\widetilde{\nabla}_Z T)W + (\widetilde{\nabla}_W T)Z = A_{FW}Z + A_{FZ}W + 2th(Z,W) + 2g(Z,W)\xi. \tag{4.9}$$

Thus from (4.8) and (4.9), it follows that

$$A_{FW}Z + A_{FZ}W = 2g(Z,W)T\nabla\mu - 2th(Z,W) - 2g(Z,W)\xi.$$

Taking the inner product with φX and using (2.5) and (2.6), we obtain

$$g(h(Z,\varphi X),\varphi W) + g(h(W,\varphi X),\varphi Z) = 2g(Z,W)g(T\nabla\mu,\varphi X).$$

From (2.3), we find that

$$g(\widetilde{\nabla}_Z \varphi X,\varphi W) + g(\widetilde{\nabla}_W \phi X,\varphi Z) = 2g(Z,W)g(T\nabla\mu,\varphi X).$$

Using the orthogonality of vector fields, we derive

$$g(\widetilde{\nabla}_Z \varphi W,\varphi X) + g(\widetilde{\nabla}_W \phi Z,\varphi X) = -2g(Z,W)g(T\nabla\mu,\varphi X).$$

Then by using the property of covariant derivative (2.7), we find that

$$\begin{aligned}-2g(Z,W)g(\varphi\nabla\mu,\varphi X) &= g((\widetilde{\nabla}_Z\varphi)W + (\widetilde{\nabla}_W\varphi)Z,\varphi X)\\ &\quad + g(\varphi\widetilde{\nabla}_Z W,\varphi X) + g(\varphi\widetilde{\nabla}_W Z,\varphi X).\end{aligned}$$

Applying the property of nearly Sasakian structure in the first term of the right hand side of the last equation and (2.1), we get

$$g(\widetilde{\nabla}_Z W, X) + g(\widetilde{\nabla}_W Z, X) = -2g(Z,W)g(\nabla\mu,X) - 2g(Z,W)\eta(\nabla\mu)\eta(X),$$

which implies that

$$g(\nabla_Z W + \nabla_W Z, X) = -2g(Z,W)g(\nabla\mu,X) - 2g(Z,W)(\xi\ln f)\eta(X).$$

Since $\mathcal{D}^\perp$ is assumed to be integrable and $\xi \ln f = 0$, we obtain

$$g(\nabla_Z W, X) = -g(Z, W) g(\nabla\mu, X). \tag{4.10}$$

Let $M_\perp$ be a leaf of $\mathcal{D}^\perp$ and let $h^\perp$ be the second fundamental form of the immersion of $M_\perp$ into M. Then for any $X \in \Gamma(\mathcal{D} \oplus <\xi>)$, we find that

$$g(h^\perp(Z, W), X) = g(\nabla_Z W, X). \tag{4.11}$$

From (4.10) and (4.11), it can be easily seen that

$$g(h^\perp(Z, W), X) = -g(Z, W) g(\nabla\mu, X),$$

which means that

$$h^\perp(Z, W) = -g(Z, W)\nabla\mu.$$

From the above equation, we conclude that $M_\perp$ is *totally umbilical* in M with the *mean curvature* vector satisfies $H = -\nabla\mu$. Now we can prove that H is parallel corresponding to the normal connection $\mathcal{D}$ of $M_\perp$ in M. To this aim, we consider $Y \in \Gamma(\mathcal{D} \oplus <\xi>)$ and $Z \in \Gamma(\mathcal{D}^\perp)$,

$$\begin{aligned} g(\mathcal{D}_Z \nabla\lambda, Y) &= g(\nabla_Z \nabla\lambda, Y) = Zg(\nabla\lambda, Y) - g(\nabla\lambda, \nabla_Z Y) \\ &= Z(Y(\lambda)) - g(\nabla\lambda, [Z, Y]) - g(\nabla\lambda, \nabla_Y Z) = Y(Z\lambda) + g(\nabla_Y \nabla\lambda, Z) = 0. \end{aligned}$$

Since $Z(\lambda) = 0$ for all $Z \in \Gamma(\mathcal{D}^\perp)$, we get that $\nabla_Y \nabla\lambda \in \Gamma(\mathcal{D} \oplus <\xi>)$. This means that the leaves of $\mathcal{D}^\perp$ are *extrinsic* spheres in M. Hence, by a result of Hiepko[7], we conclude that M is a warped product submanifold, which completes the proof of the theorem. □

Lemma 4.5. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then, for all $X, Y \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$, the following hold true:*

(i) $g((\widetilde{\nabla}_X F)Y, \varphi W) = 0$,

(ii) $g((\widetilde{\nabla}_Z F)X, \varphi W) = -X \ln f g(Z, W)$,

(iii) $g((\widetilde{\nabla}_\xi F)Z, \varphi W) = 0$.

Proof. Let M be a contact CR-warped product submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then for any $X, Y \in \Gamma(TM_T)$ and $W \in \Gamma(TM_\perp)$, we have

$$g((\widetilde{\nabla}_X F)Y, \varphi W) = -g(F\nabla_X Y, \varphi W) = -g(\nabla_X Y, W).$$

The first result directly follows, by using the property given by the above equation, and hence M_T is *totally geodesic* in M. Again, for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$, we infer

$$((\widetilde{\nabla}_Z F)X, \varphi W) = -g(F\nabla_Z X, \varphi W).$$

Using Lemma 4.1(ii), we obtain the second result. Now from (2.12), we have

$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = g(\mathcal{Q}_\xi Z + fh(\xi, Z), \varphi W) = g(\mathcal{Q}_\xi Z, \varphi W),$$

for any $Z, W \in \Gamma(TM_\perp)$. Then from property (2.14) (v)-(vii), we yield

$$g((\widetilde{\nabla}_\xi F)Z, \varphi W) = g(\varphi\xi, \mathcal{P}_Z W) = 0,$$

which is the last result of the Lemma, which concludes the proof. □

Theorem 4.6. *Let M be a contact CR-submanifold of a nearly Sasakian manifold $\widetilde{M}$ with its invariant and anti-invariant distributions integrable. Then M is a contact CR-warped product if and only if*

$$g((\widetilde{\nabla}_U F)V + (\widetilde{\nabla}_V F)U, \varphi W) = g(\mathcal{Q}_{P_1 U} P_2 V, \varphi W) + g(\mathcal{Q}_{P_1 V} P_2 U, \varphi W)$$
$$-(P_1 U \mu) g(P_2 V, W) - (P_1 V \mu) g(P_2 U, W),$$

for each $U, V \in \Gamma(TM)$ and $W \in \Gamma(TM_\perp)$, where μ is a C^∞-function on M satisfying $Z\mu = 0$ for each $Z \in \Gamma(TM_\perp)$.

Proof. Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then for any $U, V \in \Gamma(TM)$ and $W \in \Gamma(TM_\perp)$, from the property (3.1), we obtain

$$\begin{aligned} g((\widetilde{\nabla}_U F)V, \varphi W) &= g((\widetilde{\nabla}_{P_1 U} F)P_1 V, \varphi W) + g((\widetilde{\nabla}_{P_2 U} F)P_1 V, \varphi W) \\ &+ \eta(U) g((\widetilde{\nabla}_\xi F)P_1 V, \varphi W) + g((\widetilde{\nabla}_{P_1 U} F)P_2 V, \varphi W) \\ &+ g((\widetilde{\nabla}_{P_2 U} F)P_2 V, \varphi W) + \eta(U) g((\widetilde{\nabla}_\xi F)P_2 V, \varphi W) + \eta(V) g((\widetilde{\nabla}_U F)\xi, \varphi W). \end{aligned} \tag{4.12}$$

Using (2.12) and Lemma 4.5, we get

$$\begin{aligned} g((\widetilde{\nabla}_U F)V, \varphi W) &= g(\mathcal{Q}_{P_2 U} P_2 V, \varphi W) + g(\mathcal{Q}_{P_1 U} P_2 V, \varphi W) \\ &-(P_1 V \mu) g(P_2 U, W). \end{aligned} \tag{4.13}$$

By the polarization identity, we infer

$$\begin{aligned} g((\widetilde{\nabla}_V F)U, \varphi W) &= g(\mathcal{Q}_{P_2 V} P_2 U, \varphi W) + g(\mathcal{Q}_{P_1 V} P_2 U, \varphi W) \\ &-(P_1 U \mu) g(P_2 V, W). \end{aligned} \tag{4.14}$$

Thus from (4.13), (4.14) and (2.14)(ii) we get the desired result. Now for the converse, suppose that M be a CR-submanifold of a nearly Sasakian manifold $\widetilde{M}$ with integrable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$. Then, for any $X, Y \in \Gamma(\mathcal{D} \oplus < \xi >)$, we obtain

$$g((\widetilde{\nabla}_X F)Y + (\widetilde{\nabla}_Y F)X, \varphi W) = 0,$$

Using the fact that $P_2 X = P_2 Y = 0$ in (4.12), from (2.10), we get

$$2g(fh(X,Y), \varphi W) - g(h(X,TY) + h(Y,TX), \varphi W) = 0,$$

which implies that

$$g(h(X,TY), \varphi W) + g(h(Y,TX), \varphi W) = 0.$$

By applying (2.3) and (2.6), we obtain

$$g(\widetilde{\nabla}_X \phi Y, \varphi W) + g(\widetilde{\nabla}_Y \varphi X, \varphi W) = 0.$$

From the covariant derivative property (2.7) and (2.3), it is easily seen that

$$g((\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X, \varphi W) + g(\nabla_X Y + \nabla_Y X, W) = 0.$$

Taking account of this observation, from the nearly Sasakian manifold property, we obtain $g(\nabla_X Y + \nabla_Y X, W) = 0$, which implies that $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D}\oplus < \xi >)$. Since $\mathcal{D}\oplus < \xi >$ is assumed to be a integrable, then $\nabla_X Y \in \Gamma(\mathcal{D}\oplus < \xi >)$, i.e., $\mathcal{D}\oplus < \xi >$ is parallel. In other words, the leaves of $\mathcal{D}\oplus < \xi >$ are totally geodesic in M. Now, for any $X \in \Gamma(\mathcal{D}\oplus < \xi >)$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$, it follows from (4.12) that

$$g((\widetilde{\nabla}_X F)Z, \varphi W) + g((\widetilde{\nabla}_Z F)X, \varphi W) = g(\mathcal{Q}_X Z, \varphi W) - (X\mu)g(Z, W).$$

From (2.12) and (2.9), we obtain

$$g(\mathcal{Q}_X Z + fh(X, Z), \varphi W) - g(F\nabla_Z X, \varphi W) = g(\mathcal{Q}_X Z, \varphi W) - (X\mu)g(Z, W),$$

which implies that $g(F\nabla_Z X, \varphi W) = (X\mu)g(Z, W)$. After simplifications, we get $g(\nabla_Z W, X) = -(X\mu)g(Z, W)$. Since $\mathcal{D}^\perp$ is assumed to be integrable, consider a leaf $M_\perp$ of $\mathcal{D}^\perp$. If ∇' denotes the induced Riemannian connection on $M_\perp$, and $h^\perp$ is the second fundamental form of the immersion $M_\perp$ into M, then in view of (2.3), the last equation can be written as:

$$g(h^\perp(Z, W), X) = -(X\mu)g(Z, W).$$

Using the property of the gradient function, we get

$$g(h^\perp(Z, W), X) = -g(Z, W)g(\nabla\mu, X),$$

which implies that $h^\perp(Z, W) = -g(Z, W)\nabla\mu$. This means that $M_\perp$ is *totally umbilical* in M with *mean curvature* vector $H = -\nabla\mu$. Now we can easily prove that H is parallel corresponding to the normal connection $\mathcal{D}$ of $M_\perp$ in M similar to Theorem 4.4, which means that the leaves of $\mathcal{D}^\perp$ are *extrinsic spheres* in M. Then, by the result of Hiepko [7], M is a warped product submanifold, which completes the proof of the Theorem. □

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