A class of Finsler metrics with almost vanishing H-curvature

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Abstract. In this paper, we study a class of Finsler metrics with orthogonal invariance. We find an *equation* that characterizes these Finsler metrics of almost vanishing H-curvature. As a consequence, we show that all orthogonally invariant Finsler metrics of almost vanishing H-curvature are of almost vanishing Ξ-curvature and corresponding one forms are *exact*, generalizing a result previously only known in the case of metrics with vanishing H-curvature.

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1 Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics [2]. There are several non-Riemannian quantities in Finsler geometry, such as the Cartan torsion, the S-curvature, the Ξ-curvature and the H-curvature. The Ξ-curvature is obtained from the S-curvature (see (2.1) below) and the H-curvature is determined by the Ξ-curvature. In fact, we have the following [13, Lemma 2.1]

$$H_{ij} = \frac{1}{4}\left(\Xi_{i\cdot j} + \Xi_{j\cdot i}\right), \tag{1.2}$$

where $\mathbf{\Xi} := \Xi_j dx^j$ and $\mathbf{H} := H_{ij} dx^i \otimes dx^j$ denote the Ξ-curvature and the H-curvature of F respectively, "$\cdot$" denotes the vertical covariant derivative. These quantities vanish for Riemannian metrics, hence they are said to be *non-Riemannian*. The H-curvature gives a measure of failure of a Finsler metric of scalar curvature to be of constant flag curvature. Thus the quantity H deserves further investigation.

One of the important problems in Finsler geometry is to understand geometric meaning of non-Riemannian curvature. Many Finslerian geometers have studied

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Finsler metrics with special curvature properties. See [1, 8, 9, 14, 15, 13]. By (1.2), one can see that the H-curvature almost vanishes, i.e.

$$H_{ij} = \frac{n+1}{2} \theta F_{y^i y^j} \tag{1.3}$$

if the Ξ-curvature almost vanishes, i.e.

$$\Xi_j = -(n+1)F^2 \left(\frac{\theta}{F} \right)_{y^j}, \tag{1.4}$$

where θ is a 1-form on M and $n = dimM$. However, the converse might not be true. Recently, Shen, Xia and Tang have showed that (1.3) is equivalent to (1.4) for Randers metrics [1, 14, 15, 13]. For example, the following Randers metric on $\mathbb{B}^n(\nu)$

$$F = \sqrt{f(|x|)|y|^2 + \kappa^2 f^2(|x|)\langle x, y\rangle^2} + \kappa f(|x|)\langle x, y\rangle$$

has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, where f is any positive differentiable function, κ is a constant and [3, Theorem 1.2]

$$c = \frac{\kappa}{4} \frac{2f(|x|) + |x| f_r(|x|)}{1 + \kappa^2 |x|^2 f(|x|)}.$$

Thus F satisfies the following properties [1, 14, 15, 13]:

(a) (almost vanishing H-curvature)

$$H_{ij} = \frac{n+1}{2} \theta F_{y^i y^j},$$

(b) (almost vanishing Ξ-curvature and exact 1-form)

$$\Xi_j = -(n+1)F^2 \left(\frac{\theta}{F} \right)_{y^j}, \quad \theta = dc,$$

(c) (orthogonal invariance)

$$F(Ax, Ay) = F(x, y), \tag{1.5}$$

where $x \in \mathbb{B}^n(\nu)$, $y \in T_x\mathbb{B}^n(\nu)$ and $A \in O(n)$. Orthogonally invariant (*spherically symmetric* Finsler metrics form, in an alternative terminology (see [5, 4, 11]), a rich class of Finsler metrics. The above example leads to the study orthogonally invariant Finsler metrics of almost vanishing H-curvature. In this paper, we obtain the following main result:

Theorem 1.1. *On $\mathbb{B}^n(\nu)$, any spherically symmetric Finsler metric $F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)$ has almost vanishing H-curvature, i.e.,*

$$H_{ij} = \frac{n+1}{2} \theta F_{y^i y^j}, \quad \theta = \theta_j(x) y^j$$

if and only if

$$us\left[(n+1)\frac{\partial R_1}{\partial s}+3(r^2-s^2)\frac{\partial R_2}{\partial s}+2(n+1)R_4\right]=3(n+1)\theta(\phi-s\phi_s),\quad \theta=\theta_j(x)y^j, \tag{1.6}$$

where R_1, R_2 and R_4 are given in (2.2), (2.3) and (2.5) respectively, and

$$u:=|y|,\quad r:=|x|,\quad s:=\frac{\langle x,\,y\rangle}{|y|}.$$

The proof of Theorem 1.1 is given in Section 4. As an application of Theorem 1.1, we prove that (a) and (c) implies (b).

Corollary 1.2. *Let F be a orthogonally invariant Finsler metric on $\mathbb{B}^n(\nu)$. Then the H-curvature almost vanishes given by (1.3) if and only if the Ξ-curvature almost vanishes given by (1.4). In this case, the corresponding 1-form θ is an exact form.*

See Section 4 for the proof of Corollary 1.2. As a consequence of Corollary 1.2, for $\theta=0$, we get the following result

Corollary 1.3. [12] *Let F an orthogonally invariant Finsler metric on $\mathbb{B}^n(\nu)$. Then the H-curvature vanishes if and only if the Ξ-curvature vanishes.*

A Finsler metric is said to be *R-quadratic* if its Riemann curvature R_y is quadratic in $y\in T_xM$ [3, 9]. In [9], author showed that all of R-quadratic Finsler metrics have vanishing H-curvature. Together with Corollary 1.3, we have the following:

Corollary 1.4. *Let F an orthogonally invariant Finsler metric on $\mathbb{B}^n(\nu)$. Suppose that F is R-quadratic, then F has vanishing Ξ-curvature.*

For recent results of $(\alpha,\,\beta)$-metrics of almost vanishing H-curvature, we refer the reader to [17].

2 Preliminaries

Let $F=F(x,\,y)$ be a Finsler metric on a manifold M. Let $\gamma(t)$ be the geodesic with $\gamma(0)=x$ and $\dot\gamma(0)=y$. Let

$$\mathbf{S}(x,\,y)=\frac{d}{dt}\left[\tau(\gamma(t),\,\dot\gamma(t)\right]_{t=0},$$

where $\tau(x,\,y)$ is the distortion of F. $\mathbf{S}(x,\,y)$ is called the *S-curvature* [1, 3, 13]. We consider the following non-Riemannian quantity, $\mathbf{\Xi}=\Xi_j dx^j$, on the tangent bundle TM:

$$\Xi_j:=\mathbf{S}_{\cdot j|i}y^i-\mathbf{S}_{|j}, \tag{2.1}$$

where "$|$" denotes the horizontal covariant derivative. $\mathbf{\Xi}$ is called the Ξ-*curvature of* F [13] (χ-*curvature* in an alternative terminology in [1]).

The *H-curvature* $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined in (1.2). Let F be a Finsler metric on $\mathbb{B}^n(\nu) := \{x \in \mathbb{R}^n\,;\, |x| < \nu\}$. F is said to be *spherically symmetric* if it satisfies $F(Ax,\, Ay) = F(x,\, y)$ for all $x \in \mathbb{B}^n(\nu)$, $y \in T_x\mathbb{B}^n(\nu)$ and $A \in O(n)$. Let $|,\,|$ and $\langle,\,\rangle$ be the standard Euclidean norm and inner product on $\mathbb{R}^n$. In [5], Huang-Mo showed the following:

Lemma 2.1. *A Finsler metric F on $\mathbb{B}^n(\nu)$ is orthogonally invariant if and only if there is a function $\phi : [0,\, \nu) \times \mathbb{R} \to \mathbb{R}$ such that*

$$F(x,\, y) = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right),$$

where $(x,\, y) \in \mathcal{T}\mathbb{B}^n(\nu) := T\mathbb{B}^n(\nu)\backslash\{0\}$.

Let us recall a formula for the Riemann curvature of an orthogonally invariant Finsler metric $F = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$.

Let

$$R_1 := P^2 - \frac{1}{r}(sP_r + rP_s) + 2Q[1 + sP + (r^2 - s^2)P_s] \tag{2.2}$$

$$R_2 := 2Q(2Q - sQ_s) + \frac{1}{r}(2Q_r - sQ_{rs} - rQ_{ss}) + (r^2 - s^2)(2QQ_{ss} - Q_s^2) \tag{2.3}$$

$$R_3 := -sR_2 \tag{2.4}$$

$$R_4 := \frac{2}{r}P_r - Q_s - P_{ss} - \frac{s}{r}P_{rs} + 2Q(P - sP_s) + 2(r^2 - s^2)QP_{ss} - sPQ_s - (r^2 - s^2)P_sQ_s - PP_s \tag{2.5}$$

$$R_5 := -R_1 - sR_4, \tag{2.6}$$

where $P_s := \frac{\partial P}{\partial s}$, $P_r := \frac{\partial P}{\partial r}$, $Q_s := \frac{\partial Q}{\partial s}$, $Q_r := \frac{\partial Q}{\partial r}$, $Q_{ss} := \frac{\partial^2 Q}{\partial s^2}$, $r := |x|$, $s := \frac{\langle x,\, y\rangle}{|y|}$, P and Q are given by

$$Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \qquad P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}\left[s\phi + (r^2 - s^2)\phi_s\right].$$

We have the following [7, 4]

Lemma 2.2. *Let $F = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$ be an orthogonally invariant Finsler metric on $\mathbb{B}^m(\nu)$. Then the Riemann curvature of F is given by*

$$R^i_j = u^2R_1\delta^{ij} + u^2R_2x^ix^j + uR_3x^iy^j + uR_4x^jy^i + R_5y^iy^j, \tag{2.7}$$

where $u = |y|$.

3 Ξ-curvature and H-curvature

In this section, we are going to give expressions of non-Riemannian quantities $\mathbf{H}$ and $\boldsymbol{\Xi}$ of orthogonally invariant Finsler metrics (see (3.15) and (3.16) below).

By (2.4), (2.6) and Lemma 2.2, we can easily get a formula for the Ricci curvature $Ric = \sum_{j=1}^{m} R^j_j$.

$$Ric = nu^2R_1 + u^2|x|^2R_2 + u\langle x,\, y\rangle R_3 + u\langle x,\, y\rangle R_4 + |y|^2R_5 = u^2R, \tag{3.1}$$

where

$$R := (n-1)R_1 + (r^2 - s^2)R_2. \tag{3.2}$$

We have

$$\begin{aligned} \frac{\partial}{\partial y^j} Ric &= \frac{\partial}{\partial y^j}(u^2R) \\ &= \frac{\partial u^2}{\partial y^j}R + u^2\frac{\partial R}{\partial s}s_{y^j} = uR_sx^j + (2R - sR_s)y^j, \end{aligned} \tag{3.3}$$

where $R_s := \frac{\partial R}{\partial s}$ and we have used

$$\frac{\partial u^2}{\partial y^j} = 2y^j, \quad s_{y^j} = \frac{ux^j - sy^j}{u^2}. \tag{3.4}$$

By simple calculations, we have

$$s_{y^k}y^k = 0, \quad s_{y^k}x^k = \frac{r^2 - s^2}{u}. \tag{3.5}$$

We denote $\frac{\partial R_j}{\partial s}$ by R_{js} $j = 1, \cdots, 5$. By using (2.7), we obtain

$$\begin{aligned} \frac{\partial R^i_j}{\partial y^k} &= 2y^kR_1\delta^i_j + u^2R_{1s}s_{y^k}\delta^i_j + 2y^kR_2x^ix^j + u^2R_{2s}s_{y^k}x^ix^j \\ &\quad + \frac{y^k}{u}R_3x^iy^j + uR_{3s}s_{y^k}x^iy^j + uR_3x^i\delta^j_k \\ &\quad + \frac{y^k}{u}R_4x^jy^i + uR_{4s}s_{y^k}x^jy^i + uR_4x^j\delta^i_k \\ &\quad + R_{5s}s_{y^k}y^iy^j + R_5\delta^i_ky^j + R_5y^i\delta^j_k. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_i \frac{\partial R^i_j}{\partial y^i} &= u[R_{1s} + 2sR_2 + (r^2 - s^2)R_{2s} + R_3 + (n+1)R_4]x^j \\ &\quad + [2R_1 - sR_{1s} + sR_3 + (r^2 - s^2)R_{3s} + (n+1)R_5]y^j, \end{aligned} \tag{3.6}$$

where we have used (3.5) and the second equation of (3.4). By (2.4), we have

$$R_{3s} = -R_2 - sR_{2s}.$$

Taking this together with (2.3), (2.5) and (3.6), we obtain

$$\sum_i \frac{\partial R^i_j}{\partial y^i} = u\mathfrak{M}x^j + \mathfrak{N}y^j, \tag{3.7}$$

where

$$\mathfrak{M} := R_{1s} + sR_2 + (r^2 - s^2)R_{2s} + (n+1)R_4, \tag{3.8}$$

and

$$\mathfrak{N} := (1-n)R_1 - sR_{1s} - r^2R_2 - s(r^2 - s^2)R_{2s} - (n+1)sR_4. \tag{3.9}$$

The following lemma is well-known [13]:

Lemma 3.1. [9, 13]

$$\Xi_j = -\frac{1}{3}\left(2\sum_i \frac{\partial R^i_j}{\partial y^i} + \frac{\partial}{\partial y^j}Ric\right). \tag{3.10}$$

Plugging (3.3) and (3.7) into (3.10), we obtain

$$\Xi_j = -\frac{1}{3}\left[u(2\mathfrak{M} + R_s)x^j + (2\mathfrak{N} + 2R - sR_s)y^j\right]. \tag{3.11}$$

By using (3.2) we have

$$R_s = (n-1)R_{1s} + (r^2 - s^2)R_{2s} - 2sR_2. \tag{3.12}$$

From which together with (3.8) we have

$$2\mathfrak{M} + R_s = (n+1)R_{1s} + 3(r^2 - s^2)R_{2s} + 2(n+1)R_4 := \kappa. \tag{3.13}$$

By (3.2), (3.9), (3.12) and (3.13),

$$2\mathfrak{N} + 2R - sR_s = -s\kappa. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.11), we obtain the following formula for $\mathbf{\Xi}$:

$$\Xi_j = -\frac{\kappa}{3}(ux^j - sy^j), \tag{3.15}$$

where κ is given in (3.13). Taking this together with (3.4) yields

$$\Xi_{j\cdot i} = -\frac{\kappa_s}{3u^2}(ux^j - sy^j)(ux^i - sy^i) - \frac{\kappa}{3}\left(\frac{x^jy^i - x^iy^j}{u} + \frac{s}{u^2}y^jy^i - s\delta^{ji}\right),$$

where $\kappa_s := \frac{\partial\kappa}{\partial s}$. Plugging this into (1.2) yields

$$\begin{aligned} H_{ij} &= -\frac{\kappa_s}{6u^2}(ux^j - sy^j)(ux^i - sy^i) - \frac{s\kappa}{6}\left(\frac{1}{u^2}y^jy^i - \delta^{ji}\right) \\ &= \frac{1}{6}\left[s\kappa\delta^{ji} - \kappa_s x^ix^j + \frac{s\kappa_s}{u}(x^jy^i + x^iy^j) - \frac{s}{u^2}(\kappa + s\kappa_s)y^jy^i\right]. \end{aligned} \tag{3.16}$$

4 Almost vanishing H-curvature

In this section, we will prove Theorem 1.1 and 1.2. Using (3.4), we obtain

$$u_{y^iy^j} = \frac{u^2\delta^{ij} - y^iy^j}{u^3}, \tag{4.1}$$

$$s_{y^iy^j} = \frac{3sy^iy^j - ux^iy^j - ux^jy^i - su^2\delta_{ij}}{u^4}. \tag{4.2}$$

Proof of Theorem 1.1. F can be rewritten as $F = u\phi(r,s)$, where $u = |y|$, $r = |x|$, $s = \frac{\langle x,\, y\rangle}{|y|}$. It follows that

$$F_{y^j} = u_{y^j}\phi + u\phi_s s_{y^j} \tag{4.3}$$

and

$$F_{y^jy^k} = u_{y^jy^k}\phi + (u_{y^j}s_{y^k} + u_{y^k}s_{y^j})\phi_s + us_{y^j}s_{y^k}\phi_{ss} + us_{y^jy^k}\phi_s. \tag{4.4}$$

Plugging (3.4), (4.1) and (4.2) into (4.4) yields

$$\begin{aligned} u^3F_{y^iy^j} &= (u\delta^{ij} - y^iy^j)\phi + [y^i(ux^j - sy^j) + y^j(ux^i - sy^i)]\phi_s \\ &\quad + (ux^i - sy^i)(ux^j - sy^j)\phi_{ss} \\ &\quad + [3sy^iy^j - u(x^iy^j + x^jy^k) - su^2\delta_{ij}]\phi_s \\ &= u^2(\phi - s\phi_s)\delta_{ij} + u^2\phi_{ss}x^ix^j - u\phi_{ss}(x^iy^j + x^jy^i) \\ &\quad - (\phi - s\phi_s - s^2\phi_{ss})y^iy^j. \end{aligned} \tag{4.5}$$

By (3.16) and (4.5), (1.3) holds if and only if

$$\begin{aligned} &s\kappa\delta^{ji} - \kappa_s x^ix^j + \tfrac{s\kappa_s}{u}(x^jy^i + x^iy^j) - \tfrac{s}{u^2}(\kappa + s\kappa_s)y^jy^i \\ &= \frac{3(n+1)\theta}{u^3}\left[u^2(\phi - s\phi_s)\delta^{ij} + u^2\phi_{ss}x^ix^j - u\phi_{ss}(x^iy^j + x^jy^i) - (\phi - s\phi_s - s^2\phi_{ss})y^iy^j\right]. \end{aligned} \tag{4.6}$$

It is easy to see that (4.6) holds if and only if

$$s\kappa = \frac{3(n+1)\theta}{u}(\phi - s\phi_s), \tag{4.7}$$

$$-\kappa_s = \frac{3(n+1)\theta}{u}\phi_{ss}, \tag{4.8}$$

$$s\kappa_s = -\frac{3(n+1)\theta}{u}s\phi_{ss}, \tag{4.9}$$

$$s(\kappa + s\kappa_s) = \frac{3(n+1)\theta}{u}(\phi - s\phi_s - s^2\phi_{ss}). \tag{4.10}$$

By using (3.13), we obtain that (4.7) is equivalent to the first equation of (1.6). Hence it is sufficient to show that (4.7) implies (4.8), (4.9) and (4.10). Since F is a Finsler metric, we see that $\phi - s\phi_s > 0$ [11]. Suppose that (4.7) holds. Note that

$$s = \frac{\langle x,\, y\rangle}{u}. \tag{4.11}$$

It follows that the 1-form θ can be expressed by

$$\theta = \frac{\kappa}{3(n+1)(\phi - s\phi_s)}\langle x,\, y\rangle. \tag{4.12}$$

Furthermore, $\dfrac{\kappa}{3(n+1)(\phi - s\phi_s)}$ is independent of y. In fact, it is only dependent of $|x|$. Let

$$\frac{\kappa}{3(n+1)(\phi - s\phi_s)} := \sigma\left(\frac{|x|^2}{2}\right). \tag{4.13}$$

Plugging (4.13) into (4.12) yields

$$\theta = \sigma\left(\frac{|x|^2}{2}\right)\langle x,\, y\rangle. \tag{4.14}$$

Together with (4.11) we have

$$\frac{\theta}{u} = s\sigma\left(\frac{|x|^2}{2}\right). \tag{4.15}$$

By using (4.13) and (4.15), we obtain

$$\begin{aligned} \kappa_s &= \left[3(n+1)\sigma\left(\tfrac{r^2}{2}\right)(\phi - s\phi_s)\right]_s \\ &= -3(n+1)\sigma\left(\tfrac{r^2}{2}\right)s\phi_{ss} = -3(n+1)\tfrac{\theta}{u}\phi_{ss}. \end{aligned}$$

Thus we obtain (4.8). (4.8)$\times(-s)$ yields (4.9). Finally, (4.10) is easy to obtain from (4.7) and (4.8).

Proof of Corollary 1.2. It suffices to show that the Ξ-curvature almost vanishes given by (1.4) if the H-curvature almost vanishes given by (1.3) and in this case corresponding 1-form is exact. Suppose that $F = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$ has almost vanishing H-curvature. Then (4.7), (4.13) and (4.14) hold. By using (4.14), we have

$$d\left[f\left(\frac{|x|^2}{2}\right)\right] = f'\left(\frac{|x|^2}{2}\right)d\left(\frac{|x|^2}{2}\right) = \sigma\left(\frac{|x|^2}{2}\right)\Sigma_j x^j dx^j = \theta,$$

where $f(t) := \int \sigma(t)dt$. Hence θ is an exact form. Plugging (3.4) into (4.3) yields

$$F_{y^j} = \phi_s x^j + \frac{\phi - s\phi_s}{u}y^j.$$

Combining with (4.14) we get

$$\left(\frac{\theta}{F}\right)_{y^j} = \frac{\sigma\left(\frac{|x|^2}{2}\right)}{F^2}(\phi - s\phi_s)(ux^j - sy^j).$$

Together with (3.15) and (4.13) we obtain that the Ξ-curvature almost vanishes given by (1.4).

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